Exercise sheet # 11
Algebraic Geometry SS 2018

(Ingo Runkel)

Exercise 43 (4 P)

Give an example of a regular function f on a quasi-affine variety U such that f cannot be written as $f = g/h$ for polynomials which are defined on all of U.

Hint: Consider $V \subset \mathbb{A}^4$ given by the zero set of $X_1X_4 - X_2X_3$. Let H be the hyperplane $X_2 = X_4 = 0$. Note that $H \subset V$. Define $U = V \setminus H$. Consider the function $f : U \to k$ which is equal to X_1/X_2 on the open subset $\{X_2 \neq 0\}$ and equal to X_3/X_4 on the open subset $\{X_4 \neq 0\}$. (Why is this even well-defined? Why is f regular? Why can it not be written as g/h on all of U?).

Exercise 44 (2 P)

Consider the function $(a_0, a_1, a_2) \mapsto (a_1a_2, a_0a_2, a_0a_1)$. Turn this into a rational function $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Compute ϕ^2 and conclude that ϕ is birational. Give open subsets where ϕ restricts to an isomorphism.

Exercise 45 (3 P)

From the proof of Theorem 3.5.6: Let

$$\tilde{\alpha} : \{\text{dominant rational maps } X \dashrightarrow Y\} \to \mathrm{Hom}_{\mathrm{alg}}(k(Y), k(X))$$

be the map $\phi \mapsto \phi^*$ and let $\tilde{\beta}$ be the map in the opposite direction constructed in the lecture. Show that $\tilde{\alpha}$ and $\tilde{\beta}$ are inverse to each other.

Exercise 46 (4 P)

Let $M \subset \mathbb{P}^n$ be a closed set.

1. Show that there is some $d_0 > 0$ such that for each $d \geq d_0$ one can find homogeneous polynomials f_α, all of which have degree d, such that $M = Z(\{f_\alpha | \alpha\})$.
2. Give an example of an irreducible M such that $I(M)$ cannot be generated by homogeneous polynomials of the same degree.

Please turn over.

Exercise 47 (3 P)

Let $f, g \in k[X_1, \ldots, X_n]$ be homogeneous polynomials of degrees $\deg(f) = d$, $\deg(g) = d-1$. Suppose that f, g are coprime. Let M be the zero set of $f - X_0 g$ in $\mathbb{P}^n$. Show that M is a projective variety which is birational to $\mathbb{P}^{n-1}$.

Exercise 48 (8 P)

Recall the injective map $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$ with $N = (m+1)(n+1) - 1$ from exercise 28 (and which is called the *Segre embedding*). Denote the corresponding bijection to the image by ζ.

1. By exercise 28, the image of ζ is a closed subset of $\mathbb{P}^N$. Show that it is irreducible, i.e. that it is a projective variety.

We use ζ to identify $\mathbb{P}^m \times \mathbb{P}^n$ with its image. In this way, $\mathbb{P}^m \times \mathbb{P}^n$ becomes a projective variety. E.g. a subset $M \subset \mathbb{P}^m \times \mathbb{P}^n$ is closed iff $\zeta(M)$ is closed, a map $F : \mathbb{P}^m \times \mathbb{P}^n \to X$ for some variety X is a morphism iff $F \circ \zeta^{-1}$ is a morphism, etc. Show:

2. The closed sets in $\mathbb{P}^m \times \mathbb{P}^n$ are common zero sets of polynomials $f \in k[X_0, \ldots, X_m, Y_0, \ldots, Y_n]$ which are homogeneous of degree d_1 in the X_i and of degree d_2 in the Y_j. That is, writing f as a sum of monomials

$$X_0^{a_0} \cdots X_m^{a_m} Y_0^{b_0} \cdots Y_n^{b_n}$$

we have $\sum a_i = d_1$ and $\sum b_j = d_2$ for each summand of f.

3. If $M \subset \mathbb{P}^m$ and $N \subset \mathbb{P}^n$ are closed subsets, then $M \times N$ is closed in $\mathbb{P}^m \times \mathbb{P}^n$.

4. The projections $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m$ and $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n$ are morphisms.

5. If $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ are quasi-projective varieties, then $X \times Y$ is a quasi-projective variety in $\mathbb{P}^m \times \mathbb{P}^n$, i.e. $\zeta(X \times Y)$ is quasi-projective in $\mathbb{P}^N$. In particular $X \times Y$ is irreducible. If X and Y are projective, so is $X \times Y$.