Exercise sheet # 10
Algebraic Geometry SS 2018

(Ingo Runkel)

Exercise 39 (5 P)

Let $X \subset \mathbb{P}^n$ be a hypersurface (i.e. the zero set of an irreducible homogeneous polynomial). Show that $\mathbb{P}^n \setminus X$ is isomorphic to an affine variety. What happens if we drop the irreducibility condition from X?

Hint: Suppose $X = Z(\{f\})$, where f has degree d. Use the corresponding embedding $\mathbb{P}^n \to \mathbb{P}^N$ from exercises 34 and 38.

Exercise 40 (4 P)

Show that $U := \mathbb{A}^2 \setminus \{0\}$ is a quasi-affine variety which is not isomorphic to an affine variety.

Hint: Show that $\mathcal{O}(U) \cong k[X, Y]$ and use the relation between morphisms and algebra homomorphisms from Section 3.4.

Exercise 41 (8 P)

Notation:

- An *automorphism* of a variety X is an isomorphism of varieties $\phi : X \to X$.
- Let $\phi : \mathbb{A}^n \to \mathbb{A}^n$ be a morphism (i.e. a polynomial map by exercise 36). Its *Jacobian* is $J(\phi) := \det(D) : \mathbb{A}^n \to k$, where D is the $n \times n$ matrix with polynomial entries given by the formal derivatives $\partial\phi_i/\partial X_j$.

1. Give all automorphism of $\mathbb{A}^1$.
2. Let $M \in GL(n, k)$ and $v \in k^n$. Show that $\phi(p) = Mp + v$ is an automorphism of $\mathbb{A}^n$.
3. Let $\phi : \mathbb{A}^n \to \mathbb{A}^n$ be of the form $\phi_i(p) = p_i + f_i(p_1, \ldots, p_{i-1})$, where we take f_1 to be a constant. Show that ϕ is an automorphism.
4. Let ϕ be an automorphism of $\mathbb{A}^n$. Show that $J(\phi) : \mathbb{A}^n \to k$ is non-zero and constant.

   Remark: In characteristic p the converse statement is false (take $n = 1$ and $X^p - X$). In characteristic zero this is an open question (even for $k = \mathbb{C}$), called the Jacobian Conjecture.

Please turn over.

Exercise 42 (7 P)

Let $Y \subset \mathbb{A}^m$ and $Z \subset \mathbb{A}^n$ be affine varieties.

1. Let $M \subset \mathbb{A}^m$ and $N \subset \mathbb{A}^n$ be closed. Show that $M \times N \subset \mathbb{A}^{m+n}$ is closed.
2. Show that $Y \times Z \subset \mathbb{A}^{m+n}$ is irreducible with respect to the induced topology (which, recall, is different to the product topology).

   Hint: Write $W := Y \times Z$ and suppose $W = W_1 \cup W_2$ with $W_{1,2}$ closed. Consider $Y_j = \{y \in Y | \{y\} \times Z \subset W_j\}$. Show that $Y = Y_1 \cup Y_2$ and that $Y_{1,2}$ are closed. Complete these observations to arrive at a contradiction.
3. Show that $k[Y \times Z] \cong k[Y] \otimes_k k[Z]$.