Exercise sheet # 09
Algebraic Geometry SS 2018

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Exercise 35 (4 P)

Theorem 3.3.3 states that for a projective variety Y we have $\mathcal{O}(Y) \cong k$. Does this statement remain true if

1. one considers quasi-projective varieties instead?
2. one considers a closed subset Y of $\mathbb{P}^n$ which is not necessarily irreducible?
3. one takes the field to be $\mathbb{R}$ and $Y = \mathbb{P}^1_{\mathbb{R}}$?

Background on morphisms

Let X, Y be varieties. A *morphism* from X to Y is a continuous map $\phi : X \to Y$ such that for ever open set $V \subset Y$ and every regular function $f : V \to k$ also the function $\phi^* f := f \circ \phi|_U : U \to k$, with $U := \phi^{-1}(V) \subset X$, is regular.
Accordingly, an *isomorphism* between two varieties X and Y is a bijection $\phi : X \to Y$ such that ϕ and ϕ^{-1} are morphisms in the above sense.

Exercise 36 (8 P)

1. Show that the coordinate charts $\varphi_i : U_i \to \mathbb{A}^n$ of $\mathbb{P}^n$ are isomorphisms of varieties.
2. Show that for a morphism $\phi : X \to Y$, precomposing with ϕ induces an algebra homomorphism $\phi_p^* : \mathcal{O}_{\phi(p),Y} \to \mathcal{O}_{p,X}$.
3. Show that isomorphic varieties have isomorphic rings of regular functions, isomorphic local rings and isomorphic function fields.
4. For affine varieties X, Y, in section 1.5 we considered polynomial maps from X to Y. Show that a map $f : X \to Y$ is a polynomial map iff it is a morphism of varieties in the above sense.

   Remark: This shows in particular that the notion of isomorphism for affine varieties defined in section 1.5 agrees with notion defined above.
5. Let X, Y be varieties and let $f : X \to k$ and $\phi : X \to Y$ be maps. Let $\{U_\alpha\}$ be a open cover of X.

   Show that f is regular iff $f|_{U_\alpha}$ is regular for all α. Show that ϕ is a morphism iff $\phi|_{U_\alpha}$ is a morphism for all α.

Please turn over.

Exercise 37 (4 P)

Let X, Y be varieties and $\phi : X \to Y$ a morphism. Show that the following are equivalent:

1. ϕ is an isomorphism.
2. ϕ is an homeomorphism, and for all $p \in X$ the algebra homomorphism $\phi_p^* : \mathcal{O}_{\phi(p),Y} \to \mathcal{O}_{p,X}$ from exercise 35 part 2 is an isomorphism.

Exercise 38 (8 P)

Consider the map $F : \mathbb{P}^n \to \mathbb{P}^N$ from exercise 34.

1. Show that F is a homeomorphism onto its image.
2. Show that F is an isomorphism of varieties onto its image.
3. Consider the map F for $n = 1$, $d = 2$, i.e. $F : \mathbb{P}^1 \to \mathbb{P}^2$, $(p_0 : p_1) \mapsto (p_0^2 : p_0p_1 : p_1^2)$. Let $X = \mathbb{P}^1$ and $Y = \mathrm{im}(F)$. Show that $S(X) \not\cong S(Y)$.

   Remark: Since by part 2, X and Y are isomorphic, this provides an example that the homogeneous coordinate ring is not invariant under isomorphisms of projective varieties.