Exercise sheet # 08
Algebraic Geometry SS 2018

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Exercise 29 (4 P)

Let $Y \subset \mathbb{P}^n$ be a quasi-projective variety and let $f : Y \to k$ be a function. Recall the standard charts $U_i \subset \mathbb{P}^n$ and $\varphi_i : U_i \to \mathbb{A}^n$. Write $Y_i := \varphi_i(Y \cap U_i) \subset \mathbb{A}^n$ and

$$f_i = \left[Y_i \xrightarrow{\varphi_i^{-1}} U_i \xrightarrow{f} k\right] .$$

1. Show that Y_i is either empty or a quasi-affine variety.
2. Show that the following are equivalent:
   (a) f is regular on Y.
   (b) f_i is regular on Y_i for all i.

Exercise 30 (6 P)

Let R be a commutative ring and $S \subset R$ a multiplicatively closed subset.

1. Write $j : R \to S^{-1}R$ for the ring homomorphism $j(r) = \frac{r}{1}$. Let T be a commutative ring and let $\varphi : R \to T$ be a ring homomorphism such that $\varphi(s)$ is invertible in T for all $s \in S$. Show that there is a unique ring homomorphism $\tilde{\varphi} : S^{-1}R \to T$ such that $\tilde{\varphi} \circ j \; \varphi$. Use this to describe localisation by a universal property.
2. In the setting of part 1: Show that if φ is injective, then so is $\tilde{\varphi}$. What about the converse statement? What about surjectivity?

Exercise 31 (3 P)

Let R be a commutative ring and let $S, S' \subset R$ be two multiplicatively closed subsets such that $S \subset S'$. Make a compatibility statement about iterated localisations. How are $\mathrm{Quot}(R)$ and $\mathrm{Quot}(S^{-1}R)$ related (and what are the conditions on R and S for this question to make sense)?

Please turn over.

Exercise 32 (2 P)

Show: In a local ring the maximal ideal is formed by all non-invertible elements. What can you say if, conversely, all non-invertible elements form an ideal?

Exercise 33 (3 P)

Let R be an integral domain and let $\mathrm{Max}(R)$ the set of maximal ideals. For $\mathfrak{m} \in \mathrm{Max}(R)$, think of the localisation $R_{\mathfrak{m}}$ as a subset of $\mathrm{Quot}(R)$. Show that

$$\bigcap_{\mathfrak{m} \in \mathrm{Max}(R)} R_{\mathfrak{m}} = R \ .$$

Hint: Suppose z lies in the intersection but not in R. Consider the ideal I of R given by all $r \in R$ such that $rz \in R$. It will have to lie in some maximal ideal.

Exercise 34 (6 P)

Let $n, d \geq 1$. Write $M_0, \ldots, M_N$ for the monomials of degree d in the $n+1$ variables $X_0, X_1, \ldots, X_n$ (there are $N + 1 = \binom{n+d}{n}$ of these). Define the map $F : \mathbb{P}^n \to \mathbb{P}^N$ by $(p_0 : \ldots, : p_n) \mapsto (M_0(p) : \cdots : M_N(p))$.

1. Why is this well-defined? Write out the map in the example $n = 1$, $d = 2$.
2. Let $\theta : k[Y_0, \ldots, Y_N] \to k[X_0, \ldots, X_n]$ be the homomorphism which sends Y_a to $M_a \in k[X_0, \ldots, X_n]$. Show that $\mathfrak{p} = \ker(\theta)$ is a homogeneous prime ideal. (And so $Z(\mathfrak{p})$ is a projective variety.)
3. Show that $F(\mathbb{P}^n) = Z(\mathfrak{p})$.