# Exercise sheet # 07
# Algebraic Geometry SS 2018

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**Exercise 25** (5 P)
Let $R$ be a commutative graded ring and let $I \subset R$ be an ideal. Denote by $\pi : R \to R/I$ the canonical projection. Show that the following statements are equivalent.

1. $I$ is homogeneous.
2. If $i \in I$ can be written as $i = \sum_{d\in\mathbb{Z}} u_d$ with $u_d \in R_d$ (and with only finitely many $u_d$ non-zero), then also $u_d \in I$ for all $d \in \mathbb{Z}$.
3. $I$ is generated by homogeneous elements.
4. $R/I = \bigoplus_{d\in\mathbb{Z}} \pi(R_d)$.

**Exercise 26** (7 P)
Let $R$ be a graded commutative ring and let $I \subset R$ be a homogeneous ideal. Show:

1. $I$ is prime if and only if for all homogeneous elements $f, g \in R$ we have: $fg \in I \Rightarrow f \in I \vee g \in I$.
2. $\sqrt{I}$ is homogeneous
3. Suppose $(I_\alpha)_{\alpha\in M}$ is a family of homogeneous ideals. Then also $\langle I_\alpha | \alpha \in M\rangle$ and $\bigcap_{\alpha\in M} I_\alpha$ are homogeneous ideals.

What about the converse implications in 2 and 3?

**Exercise 27** (9 P)
Show:

1. The ideal of a cone is proper (i.e. not all of the polynomial ring) and homogeneous, and homogeneous proper ideals describe cones.
2. Let $M \subset \mathbb{P}^n$ be a non-empty subset. Then $I_a(C(M)) = I_p(M)$.
3. Let $J \subsetneq S$ be a homogeneous ideal. Then $C(Z_p(J)) = Z_a(J)$ and $P(Z_a(J)) = Z_p(J)$.
4. The bijections $P$ and $C$ between subset of $\mathbb{P}^n$ and cones in $\mathbb{A}^{n+1}$ restrict to bijections
   (a) between algebraic sets in $\mathbb{P}^n$ and closed cones in $\mathbb{A}^{n+1}$,
   (b) between projective varieties and affine varieties which are cones.

**Please turn over.**

**Exercise 28** (3 P)

Let $m, n \in \mathbb{Z}_{>0}$ and $N = (m+1)(n+1)-1$. Label the coordinates of a point $z \in k^N$ as $z_{i,j}$ with $0 \leq i \leq m$ and $0 \leq j \leq n$. (In other words, pick a permutation to identify $\{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$ with $\{0, 1, \ldots, N\}$.) Consider the map $f : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$,

$$((p_0 : \cdots : p_m), (q_0 : \cdots : q_n)) \mapsto (z_{1,1} : \cdots : z_{m,n})$$

with $z_{i,j} = p_i q_j$.

Show that $f$ is well-defined and injective, and that the image of $f$ is the algebraic set given by the common zeros of

$$T = \{Z_{i,j} Z_{k,l} - Z_{i,l} Z_{k,j} \,|\, 0 \leq i, k \leq m \ , \ 0 \leq j, l \leq n\} \ .$$