Exercise sheet # 06
Algebraic Geometry SS 2018

(Ingo Runkel)

Exercise 21 (4 P)

Let K be a field and let $H \subset K^{n+1}$ an n-dimensional sub-vector space. Give a bijection between the disjoint union $H \cup \mathbb{P}(H)$ and $\mathbb{P}^n_K$. Does a similar decomposition statement hold for $\mathbb{P}(V)$ (instead of $\mathbb{P}^n_K$) when V is an infinte-dimensional K-vector space?

Exercise 22 (Complex projective space in metric topology) (4 P)

Recall the subsets $U_i \subset \mathbb{P}^n_{\mathbb{C}}$, $i = 0, \dots, n$ and the bijections $\varphi_i : U_i \to \mathbb{C}^n$. Think of the φ_i as coordinate charts which endow $\mathbb{P}^n_{\mathbb{C}}$ with a topology induced from the metric topology on $\mathbb{C}^n$. In particular, a map $f : X \to \mathbb{P}^n_{\mathbb{C}}$, for X a topological space, is continuous iff $\varphi_i \circ f$ is continuous on its domain of definition (for the metric topology on $\mathbb{C}^n$) for all i.

Let $S = \{z \in \mathbb{C}^{n+1} | |z| = 1\}$ and consider the map $f : S \to \mathbb{P}^n_{\mathbb{C}}$, $z \mapsto \mathbb{C}z$, the one-dimensional subspace spanned by z. Show that f is a continuous surjection and conclude that $\mathbb{P}^n_{\mathbb{C}}$ is compact.

Exercise 23 (Proof of Lemma 2.2.3) (4 P)

Let k be an algebraically closed field as usual and write $\mathbb{P}^n = \mathbb{P}^n_k$. Recall the coordinate charts $\varphi_i : U_i \to \mathbb{A}^n$.

1. Let $N \subset \mathbb{P}^n$ be closed. Show that $\varphi_i(U_i \cap N) \subset \mathbb{A}^n$ is closed.

2. Let $M \subset \mathbb{A}^n$ be closed. Show that there exists a closed $N \subset \mathbb{P}^n$ such that $\varphi_i(U_i \cap N) = M$.

Please turn over.

Exercise 24 (12 P)

Consider the homeomorphism $\varphi_0 : U_0 \to \mathbb{A}^n$ for $U_0 \subset \mathbb{P}^n$ one of the standard coordinate charts. Let $\mathcal{P}_0$ be the set of all projective varieties in $\mathbb{P}^n$ which are not contained in the hyperplane $H_0 = Z(\{X_0\})$. Let $\mathcal{A}$ be the set of all affine varieties in $\mathbb{A}^n$.

1. Let $P \in \mathcal{P}_0$. Why does $\overline{P \cap U_0} = P$ hold? Does this hold for all projective varieties in $\mathbb{P}^n$ or does it use the condition $P \not\subset H_0$?
2. Show that the assignment $\varphi_* : \mathcal{P}_0 \to \mathcal{A}$, $P \mapsto \varphi_0(P \cap U_0)$ is well-defined and is in fact a bijection.
3. Does φ_* remain a bijection if in the above definition of $\mathcal{P}_0$ and $\mathcal{A}$ we replace "projective variety" and "affine variety" by "algebraic set"?
4. Show that U_0 is dense in $\mathbb{P}^n$ (in the Zariski topology). Show that $\mathbb{P}^n$ is irreducible.
5. *(Extra problem with 0 points.)*
   Consider the algebraic set $V = Z(\{X_1^2 - X_2^2 - 1\}) \subset \mathbb{A}^2$. Show that V is irreducible. What are the points in $\varphi_*^{-1}(V)$ outside of U_0?