Exercise sheet # 05
Algebraic Geometry SS 2018

(Ingo Runkel)

Exercise 17 (4 P)

Show that every affine variety is compact in the Zariski topology. Can you make this a general statement about appropriate topological spaces so that the previous claim becomes a special case?

Exercise 18 (7 P)

1. In the lecture we considered $V = Z(Y^2 - X^3) \subset \mathbb{A}^2$ and we proved that the polynomial map $f : \mathbb{A}^1 \to V$, $t \mapsto (t^2, t^3)$ is bijective but not an isomorphism. Show that in fact there is no isomorphism between $\mathbb{A}^1$ and V at all.

2. Let k be of characteristic p, e.g. $k = \overline{\mathbb{F}}_p$. Consider the polynomial map $f : \mathbb{A}^n \to \mathbb{A}^n$, $(u_1, \dots, u_n) \mapsto ((u_1)^p, \dots, (u_n)^p)$. Is it a bijection of sets? Is it an isomorphism of algebraic sets?

Exercise 19 (6 P)

Let $T = \{XZ - Y^2, YZ - X^3\}$ and $N = Z(T) \subset \mathbb{A}^3$ (recall exercise 3). Give the unique decomposition of N into irreducible components (and prove that your components are indeed irreducible, and that you have found all of them).

Hint: Consider the z-axis.

Exercise 20 (7 P)

Let X be a topological space. An (at first sight somewhat funny) notion of dimension for X is as follows. The *dimension* of X is the supremum over all n such that there is a chain

$$X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n$$

of irreducible closed subset X_i of X.

1. What is the dimension of a Hausdorff topological space?

2. Show that the dimension of $\mathbb{A}^1$ is one. Show that the dimension of $\mathbb{A}^n$ is at least n. (It is actually exactly n, but this requires tools we do not yet have.)

3. Call a chain as above *maximal* if it cannot be made longer by adding new X_i at the beginning or end or somewhere in the middle. Can you think of an example where there are two maximal chains of different length?

   Hint: Try an algebraic set which consists of irreducible components of different dimensions.

4. In the case of algebraic sets in affine space, what is the algebraic counterpart of the above notion of dimension?