# Exercise sheet # 04
# Algebraic Geometry SS 2018

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**Exercise 14** [2 P]

Let $J$ be an ideal in $A = k[\mathbb{A}^n] = k[X_1, \dots, X_n]$. From Hilbert's Nullstellensatz we learned that $I(Z(J)) = \sqrt{J}$. Show that for a subset $S$ of $\mathbb{A}^n$ (not necessarily an algebraic set) one has $Z(I(S)) = \overline{S}$, the closure of $S$ in the Zariski topology.

**Exercise 15** [9 P]

1. Let $X, Y$ be topological spaces and let $f : X \to Y$ be continuous. Show that if $X$ is irreducible, then so is the image of $f$. What about the converse statement?

2. Let $X$ be a topological space, let $U \subset X$ be a subset and denote by $\overline{U} \subset X$ the closure of $U$. Show that $U$ is irreducible iff $\bar{U}$ is irreducible.

3. Let $M, N$ be algebraic sets and let $f : M \to N$ a polynomial map. Show that if $M$ is an algebraic variety, then so is the closure of $f(M)$ in $N$. Why does one need to take the closure?

4. Show that the set $M \subset \mathbb{A}^3$ from exercise 3 is an algebraic variety.

**Please turn over.**

**Exercise 16** (Zariski topology for rings) [13 P]

Let $R$ be a commutative ring and let $\mathrm{Spec}(R)$ be the set of all prime ideals in $R$. This is called the *spectrum of $R$* or the *prime spectrum of $R$*. For an ideal $I \subset R$ define the subset $V(I) \subset \mathrm{Spec}(R)$ as

$$V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \,|\, I \subset \mathfrak{p}\} \ .$$

1. Show that taking closed sets to be the $V(I)$, where $I$ runs over all ideals of $R$, defines a topology on $\mathrm{Spec}(R)$. This is called the *Zariski topology on* $\mathrm{Spec}(R)$.

Let $\mathrm{Max}(R) \subset \mathrm{Spec}(R)$ be the subset of all maximal ideals in $R$. Equip $\mathrm{Max}(R)$ with the subset topology (i.e. the closed sets in $\mathrm{Max}(R)$ are precisely the sets $V(I) \cap \mathrm{Max}(R)$, where $I$ runs over all ideals in $R$).

2. *(Extra problem with 0 points.)*
   Is $\mathrm{Max}(R)$ always/sometimes/never closed in $\mathrm{Spec}(R)$? Or open?

3. Let $S$ be another commutative ring and $\phi : R \to S$ a ring homomorphism. Explain why the map $\phi^{-1}(-)$ of taking preimages, which takes subsets of $S$ to subsets of $R$, restricts to a map $\hat{\phi} : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$, but in general not to a map $\mathrm{Max}(S) \to \mathrm{Max}(R)$.
   (This is why the prime spectrum is sometimes nicer to consider than the maximal spectrum.)

4. Show that $\hat{\phi} : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is continuous.

Let us now apply the above general discussion to the setting treated in the lecture: Let $M \subset \mathbb{A}^n$ be an algebraic set and let $k[M]$ be its coordinate ring. Recall the bijection $\psi : M \to \mathrm{Max}(k[M])$ from exercise 9.

5. On $M$ we defined the Zariski topology in the lecture by taking algebraic sets as closed sets. On $\mathrm{Max}(k[M])$ we defined a topology above (also called Zariski topology). Show that $\psi$ is a homeomorphism (i.e. continuous with continuous inverse).

6. *(Extra problem with 0 points.)*
   Do the points in $\mathrm{Spec}(k[M])$ which are not in $\mathrm{Max}(k[M])$ have a geometric interpretation in $M$?

   Does the problem from part 3 also occur when we consider polynomial maps $f : M \to N$ between algebraic sets? Or, equivalently (by Theorem 1.5.11), algebra homomorphisms $\phi : k[N] \to k[M]$?