Exercise sheet # 03
Algebraic Geometry SS 2018

(Ingo Runkel)

Exercise 9 (Maximal ideals and points of an algebraic set) [3 P]

Let $M \subset \mathbb{A}^n$ be an algebraic set. Let $\tilde{M}$ be the set of all maximal ideals in A which contain $I(M)$. Show that the map $\phi : M \to \tilde{M}$, $p \mapsto I(\{p\})$ does indeed land in $\tilde{M}$ and is a bijection.

Exercise 10 [3 P]

Show Lemma 1.4.9: Let $T \subset S \subset R$ be commutative rings.

1. If S is finite over T and R is finite over S then R is finite over T.
2. Let $a \in R$. If there is a normalised polynomial $f \in S[X]$ such that $f(a) = 0$, then $S[a]$ is finite over S.

Interlude on polynomial algebras:
Let L be a field. The following properties of polynomial algebras are assumed from an algebra class and can be used without proof.

1. $L[X]$ is a principal ideal domain.
   This means that for every ideal $I \subset L[X]$ one can find $f \in L[X]$ such that $I = \langle f \rangle$, which is a simple consequence of polynomial division and the Euclidean algorithm.
2. $B := L[X_1, \dots, X_n]$ is a unique factorisation domain (aka. factorial ring).
   Showing this is more work. To explain what this even means, we need two more notions: Let $f \in B$ be a non-constant polynomial.
   - f is called *prime* if $f|gh$ implies $f|g$ or $f|h$. (Equivalently, $\langle f \rangle$ is a prime ideal. (Why?))
   - f is called *irreducible* if $f = gh$ implies that either g or h is equal to f up to a constant factor.

   In a unique factorisation domain, an element is prime iff it is irreducible, and every non-zero element can be written as a product of prime elements in a unique way up to order and units (= invertible elements of B = non-zero constant polynomials in our case).

See e.g. Lang, Algebra, Sections II.5 and IV.2 (in particular Cor. 2.4).

Please turn over.

Exercise 11 [11 P]

1. Does any part of the equivalence in the weak Nullstellensatz (Thm. 1.4.5) remain valid if k is not algebraically closed?
2. Is there a statement of the type "If the weak Nullstellensatz holds then the field is algebraically closed."?
3. Show that $\langle X^2+Y^2 \rangle$ and $\langle X, Y \rangle$ are distinct radical ideals in $\mathbb{R}[X, Y]$. What does that tell you about Hilbert's Nullstellensatz (Thm. 1.4.1)?

Exercise 12 [5 P]

Let J be an ideal in $\mathbb{R}[X_1, \dots, X_n]$. Show that there is $f \in \mathbb{R}[X_1, \dots, X_n]$ such that $Z(J) = Z(\{f\})$. Why does this in general not work over $\mathbb{C}$? Does it work for some n?

Exercise 13 (Example of Noether normalisation) [2 P]

Let K be an infinite field. Consider the algebra $B = K[X_1, X_2]/J$ with $J = \langle (X_2)^2 + (X_1)^3 + (X_1)^4 \rangle$. Find $Y_1, \dots, Y_m \in B$ which are algebraically independent over K, such that B is finite over $K[Y_1, \dots, Y_m]$.