Exercise sheet # 02
Algebraic Geometry SS 2018

(Ingo Runkel)

Exercise 4 (Maximal ideals and continuous functions) [6 P]

Let $C([0,1])$ denote the $\mathbb{C}$-algebra consisting of continuous functions from the closed interval $[0,1] \subset \mathbb{R}$ to $\mathbb{C}$.

1. Show that $C([0,1])$ is not noetherian.

2. Let $J \subset C([0,1])$ be an ideal. Show that J is maximal if and only if there is a $p \in [0,1]$ such that $J = \{f \in C([0,1]) | f(p) = 0\}$.

   (Supplementary problem with 0P) Can you make your argument work for more general topological spaces than $[0,1]$?

Exercise 5 [2 P]

Let R be a noetherian ring and $J \subsetneq R$ an ideal. Show that there is a maximal ideal $\mathfrak{m}$ of R such that $J \subset \mathfrak{m}$.

Exercise 6 [4 P]

A commutative ring (or a commutative algebra) is called *reduced* if it does not contain nonzero nilpotent elements (i.e. $x^n = 0$ for some $n > 0$ implies $x = 0$).

1. What is the relation to integral domains? Is there a reduced ring with $1 \neq 0$ which is not an integral domain?

2. Let R be a commutative ring and let $J \subset R$ be an ideal. Show that J is a radical ideal if and only R/J is reduced.

Please turn over.

Exercise 7 (Coordinate rings of algebraic sets) [8 P]

Let $M \subset \mathbb{A}^n$ be an algebraic set. The *affine coordinate ring* $k[M]$ of M is defined as the following subalgebra of the algebra of functions on M:

$$k[M] := \{f : M \to k | \text{ there is a polynomial } F \in A \text{ such that } f = F|_M\} .$$

In words, $k[M]$ consists of all functions which are restrictions of polynomials to M. Note that several distinct polynomials may define the same function on M.

1. Show that $k[M]$ is isomorphic to $A/I(M)$ as a k-algebra.

   Hint: For $f \in k[M]$ choose a $P \in A$ such that $f = P|_M$ and try to define the isomorphism by mapping f to $P + I(M)$.

2. Show that $k[M]$ is a finitely generated reduced k-algebra.

3. Let B be a finitely generated reduced k-algebra. Show that there is $n > 0$ and $M \subset \mathbb{A}^n$ algebraic such that $B \cong k[M]$ as k-algebras.

4. Let $M \subset \mathbb{A}^2$, $M = Z(\{Y^2 - X\})$ be a parabola in k^2. Show that $k[M] \cong k[\mathbb{A}^1]$, i.e. that the affine coordinate rings of M and of one-dimensional affine space are isomorphic.

5. Show that for $M = Z(\{XY\}) \subset \mathbb{A}^2$, $k[M]$ is not isomorphic to $k[\mathbb{A}^1]$.

Exercise 8 (Unions and intersections of algebraic sets) [4 P]

Show that a finite union of algebraic sets is an algebraic set. Show that an arbitrary intersection of algebraic sets is an algebraic set. What about arbitrary unions of algebraic sets?