Exercise sheet # 01
Algebraic Geometry SS 2018

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Exercise 1 (Algebras and rings) [11 P]

1. Consider the following two definitions of an algebra (associative and with 1) over a field K: A *algebra over* K (or K*-Algebra* for short) is ...

   (a) ... a K-vector space A together with a bilinear map $\cdot : A \times A \to A$ and an element $1 \in A$ such that for all $a, b, c \in A$: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ and $a \cdot 1 = a = 1 \cdot a$.

   (b) ... a ring A with 1 together with a ring homomorphism $\iota : K \to Z(A)$ (of rings with 1), where $Z(A)$ denotes the center of A.

   In which sense are these definitions equivalent? Why is ι automatically injective if $A \neq \{0\}$? What happens if in (b) you only demand that $\iota : K \to A$ instead of $\iota : K \to Z(A)$?

   Hint: One way to formalise this question is this: Fix a set A and consider the two sets $M_{(\mathrm{a})}$ and $M_{(\mathrm{b})}$, where $M_{(\mathrm{a})}$ is the set of all K-algebra structures on A in the sense of definition (a), and dito for $M_{(\mathrm{b})}$. Find a bijection.

2. Defininitions (a) and (b) result in different looking notions of algebra homomorphisms: Let A and A' be K-Algebren. An *algebra homomorphism* $A \to A'$ is ...

   (a) ... a K-linear map $f : A \to A'$ with $f(1) = 1$, s.t. for all $a, b \in A$ we have $f(a \cdot b) = f(a) \cdot f(b)$.

   (b) ... a homomorphism $f : A \to A'$ of rings with 1, s.t. $f \circ \iota = \iota'$.

   Check that the equivalence from part 1 is compatible with these two notions of homomorphisms.

Please turn over.

Exercise 2 (Ideals) [7 P]

Let R be a commutative ring with 1 (we will omit "with 1" in the following).

1. An ideal $J \subset R$ is called *maximal*, if $J \neq R$ and if for every ideal I with $J \subset I \subset R$ we have that $I = J$ or $I = R$. („One cannot squeeze another ideal between J and R.“).

   Show that an ideal $J \subset R$ is maximal if and only if R/J is a field.

2. An ideal $J \subset R$ is called *prime ideal*, if $J \neq R$ and if for all $a, b \in R$ we have: $ab \in J \Rightarrow a \in J$ or $b \in J$.

   Show that an ideal $J \subset R$ is prime if and only if R/J is an integral domain. (An integral domain is a commutative ring without non-zero zero divisors and with $1 \neq 0$.)

3. Find a maximal ideal in $\mathbb{Z}[X]$, as well as a prime ideal which is not maximal. Can you find a maximal ideal which is not prime?

4. (Supplementary problem with 0P – not hard but tedious to write down. Please read and understand how it works.)
   One can define ideals also for commutative algebras.

   (a) In definition (a) from exercise 1 one would define an ideal as a sub-vector space J of A such that for all $a \in A$ and $j \in J$ one has $a \cdot j \in J$. In definition (b) one would say: "A subset $J \subset A$ is an ideal if it is an ideal of the underlying ring.“ Why is this the same?

   (b) Why is in descriptions (a) and (b) of a commutative K-algebra A and of an ideal J the quotient A/J again a k-algebra in a canonical way?

Exercise 3 (An algebraic set) [6 P]

Consider the following subset of $\mathbb{C}^3$,

$$M = \{(t^3, t^4, t^5) \in \mathbb{C}^3 \,|\, t \in \mathbb{C}\} \ ,$$

as well as the following subsets of $\mathbb{C}[x, y, z]$,

$$T_{\text{wrong}} = \{y^3 - x^4, z^3 - x^5\} \quad , \quad T_{\text{correct}} = \{xz - y^2, yz - x^3, z^2 - x^2 y\} \ .$$

1. Show that $M \subsetneq Z(T_{\text{wrong}})$ and $M = Z(T_{\text{correct}})$.

2. (Supplementary problem with 0P) What happens if we take an arbitrary algebraically closed field instead of $\mathbb{C}$? (The case of characteristic 3 demands special attention.)