r-spin TQFTs & the Arf invariant

1) Spin in arbitrary dimension $d \in \mathbb{Z}_+$

1.1) Spin groups

Def $Spin(d) \longrightarrow SO(d)$ connected double cover

Prop if $d \geq 3$ $Spin(d)$ is the universal cover

[Fr: Ch 1.4] p16 — if $d=2$ $Spin(2) \simeq U(1)$: not simply conn.

Def $r \in \mathbb{Z}_+$ $Spin^r(2) \longrightarrow SO(2)$ r-fold cover

$$Spin^r(2) = \mathbb{R}/r\mathbb{Z}, \qquad x \longmapsto e^{2\pi i x}$$

$Spin^0(2) := \mathbb{R}$: universal cover

1.2) Spin structures

M : d-dim ~~connected~~ oriented Riemannian mfd

$F \longrightarrow M$: oriented orthonormal frame bundle : $SO(d)$-principal b.
↳ agrees w/ orientation of M.

Def r-spin structure on M is a pair (P, p)

p : r 2-fold covering
P : $Spin^r(d)$ 2-principal bundle

$$\begin{tikzcd} P \arrow[rr, "p"] \arrow[dr] & & F \arrow[dl] \\ & M & \end{tikzcd}$$

s.t. p intertwines $Spin^r(d)$ 2 & $SO(d)$ 2 actions

(for $d=2$)

[Say: P is a connected cover of F (c.f. Spin(d) p.b.)
Existence: if Stiefel-Whitney class vanishes.]

Prop For $d=2$ $\exists$ r-spin str on M closed iff $\chi(M) \equiv 0 \ (r)$
closed ($\leftarrow$ = compact w/o bdry)
↑ Euler char.
[

1.3) morphisms of spin structures & spin manifolds

Def isomorphism of r-spin structures is a morphism of $Spin^r(d)$ 2 p.b's

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "p"'] & & P' \arrow[dl, "p'"] \\ & F & \end{tikzcd}$$

Def a spin manifold is a manifold together with a spin str on it.

a morphism of spin mfds (r-spin surface) is a morphism of $Spin(d)$ p.b.'s (r-spin surface) : $Spin^r(d)$ p.b's
(morphism of r-spin surfaces)

$$\begin{tikzcd} P \arrow[r, "\tilde{f}"] \arrow[d] & P' \arrow[d] \\ F \arrow[r] \arrow[d] & F' \arrow[d] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

Rem • An iso of spin strs is a morph. of spin mfd's w/ $\bar{f}=\mathrm{id}$

• A mor. of spin surfaces is an iso of spin strs $\bar{f}^* P' \leftarrow P$.
(pull back)

• # of iso classes of spin strs on a torus $= 4$

# of iso classes of spin tori $= 2$

[Say: one can get rid of the metric by considering $GL^+(d)$ pb's etc]

2) Iso classes of r-spin str's & of r-spin surfaces. $(d=2)$

- Let Σ_g be a closed surface of genus g [Say works for surf. w/ bdries too]
- Take a simple closed curve on Σ_g and write $\tilde{\gamma}$ for a unique up to homotopy lift to $F \to \Sigma$. [Rem: only works in 2d]
- $P \xrightarrow{f} F$ is a $\mathbb{Z}/r$-principal bundle [so can consider holonomies of lifted curves]
- write $\xi = (P,p)$.

Def $q_\xi(\gamma) := \mathrm{Hol}(\hat{\gamma}) \in \mathbb{Z}/r$
lift of γ to P.

Rem A (+ or) curve bounding a disk lifts to $P \Rightarrow \mathrm{Hol} = 1$.

Prop [Ra, prop 2.4] Assume $I_r(\Sigma_g) \neq \emptyset$. The map $I_r(\Sigma_g) := \{\text{iso classes of r-spin strs on } \Sigma\} \longrightarrow (\mathbb{Z}/r)^{2g}$ is a bijection.

$$[\xi] \mapsto (q_\xi(a_1), q_\xi(b_1), \dots, q_\xi(a_g), q_\xi(b_g))$$

⚠ ↑ is not a group hom. ($I_r(\Sigma_g)$ has no nat. grp. str.)

To calculate iso classes of r-spin surfaces one needs to consider the induced action of the MCG & calculate orbits.

Example $D_{a_i}(s_1,t_1,\dots,s_g,t_g) = (s_1,t_1,\dots,s_i,t_i+s_i,\dots,s_g,t_g) \in (\mathbb{Z}/r)^{2g}$
↑ Dehn twist around a_i.

Prop [GG, prop 5] [Ra, thm 2.9] Assume that there exist r-spin strs on Σ_g. Then
• if $g=1$, then for every divisor of r there is precisely 1 orbit,
• if $g \geq 2$, then if r is odd there is precisely 1 orbit,
if r is even there are precisely 2 orbits.

[Say: [Ra] does ↑ w/ boundaries. ~~For the~~ To show that the 2 orbits are indeed distinct one needs the Arf invariant.]

3) The Arf invariant ($d=2$)

3.1) $r=2$

Def let $(V, \langle , \rangle)$ be a symplectic vector space over $\mathbb{F}_2 = \mathbb{Z}/2$.
A quadratic form on V is a map $Q: V \to \mathbb{Z}/2$ s.t.

$$Q(a+b) = Q(a) + Q(b) + \langle a,b \rangle .$$

Def let $(a_i, b_i)_{i=1}^{\dim V/2}$ denote a symplectic basis of V. The Arf invariant of Q

$$A(Q) := \sum_{i=1}^{\dim V/2} Q(a_i)\, Q(b_i) \quad (\text{mod } 2).$$

Prop [Jo, lem3] $A(Q)$ is independent of the choice of basis and it is invariant under symplectic transformations.

Prop [Ra, prop2.4] the assignment
$$I_2(\Sigma) \to \{\text{q.f.'s on } H_1(\Sigma, \mathbb{Z}/2)\} \text{ is a bijection}$$
$$[\xi] \longmapsto q_\xi$$

Prop [the action of the MCG on $H_1(\Sigma, \mathbb{Z}_2)$ ~~[illegible]~~ is symplectic.

Def The Arf inv of a spin str. ξ is $A(q_\xi)$.

Prop $A(q_\xi)$ is inv. under the MCG action.

3.2. r even

Def $A(q_\xi) = \sum_{i=1}^{g} q_\xi(\tilde a_i)\, q_\xi(\tilde b_i)$ (mod 2) generalized Arf inv.

Prop [Ra, prop2.8, thm2.9]
- A is invariant under MCG action
- A distinguishes MCG orbits for $g \geq 2$.

4) r-spin TQFTs

Fact: Arf invariant is the value of a spin TQFT.
[GK sec6.1] [MS, sec2.6] [BT, sec4.2]

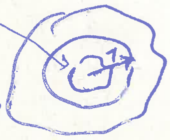
4.1) r-spin bordisms

Prop There are r iso classes of r-spin strs on $\mathbb{C}^\times$: $\mathbb{C}^\lambda$ $\lambda \in \mathbb{Z}/r$

Def r-spin surface w/ parametrized bdry is an r-spin surf Σ together with

$$\varphi_{in}: \bigsqcup_{b:\text{ in bdry}} \mathcal{C}^{\lambda_b}_{\geq 1} \hookrightarrow \Sigma \hookleftarrow \bigsqcup_{c:\text{ out bdry}} \mathcal{C}^{\lambda_c}_{\leq 1} : \varphi_{out}$$

where $\mathcal{C}^{\lambda}_{\geq 1}$, $\mathcal{C}^{\lambda}_{\leq 1}$ are collars:



and φ_{in} & φ_{out} are r-spin surface embeddings & or. preserving

s.t. $\operatorname{im}\varphi_{in}|_{\sqcup S_1} - \operatorname{im}\varphi_{out}|_{\sqcup S_1} = \partial\Sigma$

Def • Bord_2^r: cat of r-spin bordisms   obj: $\bigsqcup_j (S_1, \lambda_j)$

mor: r-spin surf. w/ param bdries

• r-spin TQFT: $Z: \mathrm{Bord}_2^r \to \mathcal{J}$ symm. mon. functor

$\mathcal{J}$ symm. mon. cat.

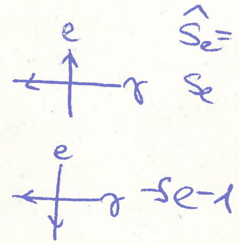
4.2 Combinatorial model of r-spin surfaces

Def Σ surface w/ PLCW decomposition (some sort of cell dec.)

- oriented edges
- every face has a single marked edge
- every edge has a label $s_e \in \mathbb{Z}_r$

s.t. the assignment is admissible: $\sum \hat{s}_e \equiv D - N + 1$ @ inner vert.

$\sum \hat{s}_e \equiv D - N + \delta_b(1-\lambda_b)$ @ bdry v.



Prop [SN, thm 4.18] There is a 1-1 correspondence between iso classes of r-spin str's & edge labels up to an equivalence relation.

Rem Computation of MCG orbits can be recovered from the comb. model

[IR, LS] **Lem** $\mathrm{Hol}(\tilde{\gamma}) = \sum \hat{s}_e + \hat{\delta}$.

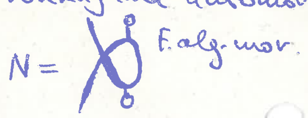
+ Arf inv from TQFT calc.

4.3 Lattice r-spin TQFT $\mathcal{J}$: symm mon cat.

input: • comb. r-spin surf.

• FA A $\in \mathcal{J}$ s.t. $N^r = \mathrm{id}$, and $\mu \circ \Delta = \mathrm{id}$. ($\Delta$-sep.)

Nakayama automorph.



$\to$ assign n-fold pairing to each n-gon

$\to$ connect (s_e+1) to each edge e

$\to$ add N^{-s_e-1} @ in boundaries

$\Rightarrow$ cylinder w/ $\mathcal{C}^{\lambda}$ bdry $\to P_\lambda =$ $(1-\lambda)$ idempot $P_\lambda = \iota_\lambda \circ \pi_\lambda$ split idempot $A \rightleftarrows Z_\lambda$ state space

[SN Sec 5] **Prop:** get an r-spin TQFT Z_A (in part. indep of comb model)

Example: $Z_A(\Sigma_{g,b}) = \varepsilon \circ \prod_{i=1}^{g} \varphi(s_i, t_i) \circ \eta$

4.4. r-spin TQFT computing the Arf inv. (assume r even) [IR, LS]

$A = k\mathbb{Z}/2 \in \mathrm{SVect}$ $(2 \nmid \mathrm{char}\, k)$ $\varepsilon(g^m) = \delta_{m,0}$ $\leadsto$ $Z_\lambda = k g^\lambda$ $\lambda \in \mathbb{Z}/2$

$Z_A(\Sigma_g) = \frac{1}{2^{g-1}} (-1)^{\sum_{i=1}^{g} (s_i+1)(t_i+1)}$ Decompose Σ_g into $(4g)$-gon w/ edge labels s_i, t_i.

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