On the LS-Category of Euclidean Configuration Spaces

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Abstract

We calculate the Lusternik-Schnirelmann-category of the n-th ordered configuration spaces $\widetilde{C}^n(\mathbb{R}^m)$ of $\mathbb{R}^m$ and give bounds for the category of the corresponding unordered configuration spaces $C^n(\mathbb{R}^m)$. Almost all of these bounds are also bounds for the sectional category $\mathrm{secat}(\pi_m^n)$ of the fibration $\pi_m^n : \widetilde{C}^n(\mathbb{R}^m) \to C^n(\mathbb{R}^m)$. In many cases, e.g. if n is a power of 2, we determine $\mathrm{cat}(C^n(\mathbb{R}^m))$ and $\mathrm{secat}(\pi_m^n)$ precisely [Rot05].

Main Computational results

Theorem A (ordered case)
For all $m \geq 2$, i.e. as long as $\widetilde{C}^n(\mathbb{R}^m)$ is connected, we have

$$\mathrm{cat}(\widetilde{C}^n(\mathbb{R}^m)) = n - 1.$$

Theorem B (unordered case)

Let k_n be the number of 1's in the dyadic expansion of n. Then we have for all $n, m \in \mathbb{N}$

$$(n-k_n)\cdot(m-1) \leq \mathrm{secat}(\pi_m^n) \leq \mathrm{cat}(C^n(\mathbb{R}^m)) \leq (n-1)\cdot(m-1).$$

If (i) n is a power of two or $n = 3$
or (ii) n is a prime and m is odd
or (iii) $m \leq 2$,
we have

$$\mathrm{cat}(C^n(\mathbb{R}^m)) = (n-1)\cdot(m-1).$$

There are several slight improvements of this, e.g. if $n = p$ is a prime and m is even, or using Corollary E.

Qualitative results

Proposition C
Let X be an n-dimensional CW-Complex such that $X - X^{(k-1)}$ is connected, where $X^{(r)}$ denotes the r-skeleton of X. Then we have

$$\mathrm{cat}(X - X^{(k-1)}) \leq n - k.$$

Proposition D
If $n = n_1 + \cdots + n_l$, then we have the inequality

$$\mathrm{cat}\left(C^n(\mathbb{R}^m)\right) \geq \mathrm{cat}\left(C^{n_1}(\mathbb{R}^m) \times \cdots \times C^{n_l}(\mathbb{R}^m)\right).$$

Corollary E

$$\mathrm{cat}(C^n(\mathbb{R}^m)) \geq \mathrm{cat}(C^{n-1}(\mathbb{R}^m)).$$

Sketch of Proofs

To derive **Theorem A**, we bound $\mathrm{cat}(\widetilde{C}^n(\mathbb{R}^m))$ from below by the integral cup-length and from above by a standard argument using that there is an $(m-2)$-connected $(n-1)\cdot(m-1)$-dimensional CW-model for $\widetilde{C}^n(\mathbb{R}^m)$. As the lower bound, this model is obtained through cohomological input due to Fred Cohen.

The upper bound in **Theorem B** follows from Proposition C using a CW-model for the 1-point-compactification $C^n(\mathbb{R}^m)_\infty$ of $C^n(\mathbb{R}^m)$ ([Vas92]).
The lower bound for $\mathrm{secat}(C^n(\mathbb{R}^m))$ in Theorem B is obtained by finding a class of category weight $(n-k_n)\cdot(m-1)$ in the cohomology of $C^n(\mathbb{R}^m)$ with $\mathbb{Z}/2\mathbb{Z}$-coefficients. The existence of such a class follows from Vassiliev's observation, that the obvious morphism $H^*(\Sigma_n;\mathbb{Z}/2\mathbb{Z}) \to H^*(C^n(\mathbb{R}^m);\mathbb{Z}/2\mathbb{Z})$ is surjective (see [Vas92]) and the determination of the cohomological dimension $\mathrm{cohdim}_{\mathbb{Z}/2\mathbb{Z}}$ of the unordered configuration space $C^n(\mathbb{R}^m)$. Again this follows by examination of Vassiliev's CW-decomposition of $C^n(\mathbb{R}^m)_\infty$.
The sharpenings of this lower bound in Theorem B in case when $n = p$ is a prime and m is odd is due to an analogous observation when we replace $\mathbb{Z}/2\mathbb{Z}$- by $\mathbb{Z}/p\mathbb{Z}$-coefficients. Here again, we use results due to Fred Cohen [Coh73], [CLM76] and Erich Ossa [Oss96]. For $n = p = 3$ this sharpening can be generalized to even m using $\mathrm{cat}(C^3(\mathbb{R}^{m+1})) \leq \mathrm{cat}(C^3(\mathbb{R}^m)) + 2$ (argue geometrically). For $m = 2$ the exact determination of $\mathrm{cat}(C^n(\mathbb{R}^2))$ follows with Theorem A since $n - 1 = \mathrm{cat}(\widetilde{C}^n(\mathbb{R}^2)) \leq \mathrm{cat}(C^n(\mathbb{R}^2))$ (covering argument).

Proposition C is proved very similarly as one shows, that each CW-subcomplex is a deformation retract of some open neighborhood. The notion of subspace category is useful.

Proposition D follows from the diagram

$$C^n(\mathbb{R}^m) \leftarrow C^{n_1,\ldots,n_l}(\mathbb{R}^m) \to C^{n_1}(\mathbb{R}^m) \times \cdots \times C^{n_l}(\mathbb{R}^m)$$

where $C^{n_1,\ldots,n_l} := C^n(\mathbb{R}^m)/\Sigma_{n_1} \times \cdots \times \Sigma_{n_l}$, the first map is the obvious covering and the second the inclusion. Observe that the inclusion has a homotopy section.

References

[Coh73] Cohen, Fred: Cohomology of braid spaces. Bull. Amer. Math. Soc. 79 (1973), 763–766.

[CLM76] Cohen, Fred; Lada, Thomas J.; May, J. Peter: The homology of iterated loop spaces. Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976.

[CLOT03] Cornea, Octav; Lupton, Gregory; Oprea, John; Tanré, Daniel: Lusternik-Schnirelmann category. Mathematical Surveys and Monographs, 103. American Mathematical Society, Providence, RI, 2003. xviii+330 pp.

[Oss96] Ossa, Erich: On the cohomology of configuration spaces. Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994), 353–361, Progr. Math., 136, Birkhäuser, Basel, 1996.

[Rot05] Roth, Fridolin: Zur Lusternik-Schnirelmann-Kategorie Euklidischer Konfigurationsräume. Diplomarbeit, University of Bonn, 2005. With an english summary. Available from the author on demand.

[Vas92] Vassiliev, V. A.: Complements of discriminants of smooth maps: topology and applications. Translated from the Russian by B. Goldfarb. Translations of Mathematical Monographs, 98. American Mathematical Society, Providence, RI, 1992. vi+208 pp.