1 Introduction

During my PhD the goal is to study questions related to conformal and topological field theories (CFT resp. TFT), motivated by ideas in physics and especially 3D quantum gravity in the context of Anti-de-Sitter/Conformal field theory correspondence (AdS/CFT duality). More concretely, the question of what is a well-defined and sensible gravity partition function in three dimensions leads to the study of mapping class group representations as well as theories that are suitable for such a gravitational interpretation.

The path integral in 3D gravity contains contributions of geometries which approach asymptotically the torus. The contribution of thermal AdS_3, which is topologically a solid torus, is called the vacuum seed. In [MW] it has been argued that the saddle-point approximation consists of geometries, which are obtained by gluing the torus boundary on the solid torus via an element in $\mathrm{SL}(2,\mathbb{Z})$. In [CGHMV], they expressed the torus partition function of the Ising CFT which has central charge $c = \frac{1}{2}$ as an orbit sum of the vacuum seed with respect to the action of the mapping class group which is present in the given theory. This gravity/CFT correspondence on the torus was then extended to include partition functions on any genus g surface [JLLSW]. For the proof, it was crucial that the mapping class group action on an given closed surface given by the TFT has finite image (property F), which makes the sum a finite sum, and furthermore the action is irreducible, which highly constraints the subspace of mapping class group invariants on the double.

Mathematically, 2D rational conformal field theories (RCFTs) have been analysed and studied as the boundary theories of 3D TQFTs with defects. Information about the chiral theory is encoded in the representation category of the respective chiral algebra, which is a semisimple modular tensor category $\mathcal{C}$ (modular fusion category or MFC for short). The bulk TQFT is the Reshetikhin-Turaev TQFT associated to this MFC and symmetric special Frobenius algebras, which we can think of as surface defects, provide the construction of the full CFT in terms of correlators. Hence, the mathematical ingredients we use are modular fusion categories, module categories (which arise in the analysis of defects) and mapping class group representations, which are encoded by the restriction of the TQFT on the invertible bordisms.

2 Previous Projects

Mapping class group representations and Morita classes of algebras [RR]
The question of irreducibility of mapping class group representations is motivated by the more physics oriented project, which is listed below. Let us introduce briefly the relevant notions. Given a MFC $\mathcal{C}$ and a closed surface Σ, the TQFT assigns the state space $V^{\mathcal{C}}(\Sigma)$ which carries an action of the mapping class group Mod_Σ. In this project, we studied theories where $V^{\mathcal{C}}(\Sigma)$ are irreducible representations and we found the following result:

Theorem 2.1. Let $\mathcal{C}$ be a modular fusion category over an algebraically closed field of characteristic zero. If the projective mapping class group representations $V_g^{\mathcal{C}}$ are irreducible for all $g \geq 0$, then every simple non-degenerate algebra in $\mathcal{C}$ is Morita-equivalent to the tensor unit.

Here, a non-degenerate algebra means an algebra with non-degenerate trace pairing, which in particular carries a natural structure of symmetric Frobenius algebra. Note also that this theorem is in relation to the result of [AF], where they give a reducibility criterion which we restate as: Suppose there exists $g \geq 1$ with $V^{\mathcal{C}}(\Sigma_g)$ irreducible, then any non-degenerate algebra A has full centre $Z(A)$ isomorphic as object to the full centre of the trivial unit algebra $Z(\mathbb{1})$. Hence, our result uses a stronger hypothesis but provides an algebra isomorphism of the full centres, which is equivalent to providing a Morita equivalence of the respective algebras [KR]. The proof of Theorem 2.1 uses a contruction which assigns to the full centres $Z(A)$ and $Z(\mathbb{1})$ mapping class group invariants on the double. Using irreducibility one obtains a symmetric 2-cocycle of the universal grading group describing the difference of their multiplication constants, whose cohomology class is trivial, thus proving an algebra isomorphism between the full centres.

Perhaps, more interesting is the statement implication of Theorem 2.1 in terms of module categories, which states that any semisimple indecomposable $\mathcal{C}$-module category with trace is equivalent to $\mathcal{C}$. For $\mathcal{C}$ pseudo-unitary, one can drop the trace requirement. In the context of TQFT, this implies that the theory has no non-trivial topological defects, i.e. we have only the transparent defect.

RCFT correlators as mapping class group averages [RR-prep]
In this project we extend the results of [CGHMV, JLLSW] by giving a relation between RCFT correlators and mapping class group averages for theories with certain irreducibility and finiteness properties. Recall in the construction of the full RCFT, the input is the MFC $\mathcal{C}$, which encodes the chiral information needed, and a symmetric special Frobenius algebra A. This leads to the conformal correlators $\mathrm{Cor}_A^{\mathcal{C}}(\Sigma)$, where Σ is a surface with insertions, and they live in the space of full conformal blocks $V^{\mathcal{C}}(\hat{\Sigma})$. The surface $\hat{\Sigma}$ is the double of Σ which for surfaces without boundary can be seen as the the disjoint union of the surface and its orientation reversal. Then, we have the following:

Theorem 2.2. Let Σ be a surface with insertions such that the projective representation $V^{\mathcal{C}}(\Sigma)$ is irreducible and $V^{\mathcal{C}}(\mathrm{Mod}(\Sigma)) \subset \mathrm{End}(V^{\mathcal{C}}(\Sigma))$ is finite and let $x \in V^{\mathcal{C}}(\hat{\Sigma})$ be an element such that $d_\Sigma(x) \neq 0$. Then, there exists $\lambda_\Sigma \in \Bbbk^\times$ such that

$$\langle x \rangle_\Sigma = \lambda_\Sigma \, \mathrm{Cor}_A^{\mathcal{C}}(\Sigma)$$

for any symmetric special Frobenius algebra A.

There are 16 inequivalent modular fusion categories of Ising-type, i.e. with the Ising fusion rules. We show that all such Ising modular categories satisfy the irreducibility and finiteness conditions of Theorem 2.2 for any surface with insertions. This extends the result of [JLLSW] in two ways: it includes the non-unitary Ising categories and includes insertions on the surfaces.

3 Current Projects

Absence of global symmetries
In [HO] it is conjectured that gravitational theories should have no global symmetries. In

the construction of [RR-prep] the corresponding bulk theory is a three-dimensional TQFT with associated modular fusion category being the double $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$. In TQFT global symmetries are represented by invertible surface defects. A surface defect M is called invertible if there is a surface defect N such that $M \otimes N \cong \mathbb{1} \cong N \otimes M$ with respect to the fusion of defects and $\mathbb{1}$ being the transparent defect. Surface defects in Reshetikhin-Turaev TQFTs have been analysed in [FSV]. For the TQFT associated to the modular fusion category $\mathcal{D}$ surface defects correspond to $\mathcal{D}$-module categories. Hence, in our case where $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ global symmetries correspond to invertible $\mathcal{C}$-$\mathcal{C}$-bimodule categories. Therefore, I am investigating the following: Let $\mathcal{C}$ be a non-degenerate braided fusion category with no non-trivial indecomposable left module categories. Then, $\mathcal{C}$ has no non-trivial invertible bimodule categories. This statement is true in all of the examples we have seen so far.

Modular fusion categories with irreducible mapping class group representations
The property of irreducibility is not studied extensively in the literature and therefore there are not many examples. The examples I am aware of are Ising modular fusion categories and for the modular fusion category $\mathcal{C}(sl_2, k)$ associated to the affine Lie algebra $\hat{sl}_2$ at certain levels k. In this project we try to extend this list of examples. More precisely, it has been shown that for a prime p and $k + 2 = p$ the mapping class group representations on closed surfaces are irreducible. However, it is expected but still open that for $k + 2 = p^2$ mapping class group representations are irreducible. In this project, I will try to expand on this list as well as extend known results to higher genera and insertions.

4 Potential Future Projects

Gauging the boundary theory
Orbifold CFTs are interesting and appear in a lot of AdS/CFT considertations. In this spirit we want to study what happens to the mathematical properties, we investigated before, when gauging a theory which carries a gauge group action. More precisely, let G be a finite group acting on a braided fusion category $\mathcal{C}$. Given a braided G-crossed extension of $\mathcal{C}$ and passing to the equivariantisation, one arrives to the orbifold or gauge theory $\mathcal{C}/G$ [CGPW]. Using this algebraic description of an orbifold theory, we can study questions such as the finiteness condition. Namely, in [RR-prep] we used that mapping class group images are finite in the chiral theory associated to $\mathcal{C}$ and we may ask if this is true for the orbifold theory $\mathcal{C}/G$. This question has an independent application to the property F conjecture. The property F conjecture refers to the conjecture that a modular fusion category is weakly integral if and only if it has the finiteness property with respect to the mapping class group representations. Weakly integral categories are conjectured to be weakly group-theoretical. However, weakly group-theoretical categories can be obtained from gauging either a pointed fusion category or the Deligne product of a pointed fusion category with an Ising category [N]. Therefore, showing that property F is preserved by gauging would prove a significant part of the conjecture[1].

More concretely, one can look at the cases where G is the permutation group $G = S_N$,

[1] Originally, the conjecture considers braided fusion categories and their braid group representations for which it has been shown that weakly group-theoretical fusion categories have property F [GN].

where this includes permutation orbifolds, such as bilayer phases. Bilayer phases in particular have been studied explicitly in my master thesis where the associated modular fusion category comes from the $\mathbb{Z}_2$-gauging of $\mathcal{C} \boxtimes \mathcal{C}$. This may lead to some interesting applications to topological phases of matter using the bulk-boundary correspondence.

Regularisation of Averages

Property F is used to give a well-defined notion of mapping class group average in the theory. However, it would be helpful to include more theories and have a regularisation procedure. Regularisation problems of such mapping class group sums may be studied in the spirit of Geometric Recursion [ABO] using hyperbolic geometry and in particular Teichmüller theory. At first glance, a regularisation procedure for mapping class group averages seems out of reach as mapping class groups are non-ameanable, and therefore have no "good" measure for such an average. However, we are only interested in the mapping class group action and thus may find that the representation image, even if infinite, is ameanable. Constructing such an average will allow for a much broader study of our previous established relations.

Sum over Geometries

The gravitational partition function is expressed as a sum over contributions of geometries. In the semi-classical regime, it was argued that the contributions of $SL(2, \mathbb{Z})$ black holes dominate the sum and therefore one obtains the semi-classical approximation to be a mapping class group average. However, in the strongly-coupled regime this serves only as an ansatz and it is natural to ask how other solutions contribute to the sum. In such a project, we would try to understand these new contributions with the help of TQFT. This would refine the irreducibility case we have already considered and possibly provide a duality statement for more general theories.

Averages over an Ensemble of CFT

Using the property of irreducibility, we are able to relate a gravity partition function to a single CFT partition function. In recent years, the idea of having instead an ensemble of CFTs on the boundary of a 3D gravitational theory has been studied in physics literature. This calls for a mathematical formulation of such an ensemble even for RCFTs and a correspondence with gravity.

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