M.Sc. Thesis

Permutation Orbifolds in Reshetikhin-Turaev TQFT

prepared by

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1 Introduction and Summary

Defects in field theories have gained considerable attention in recent years, providing a model of introducing embedded manifolds of lower dimension and studying the interplay of the theory along these. These notions have been studied extensively in 3-dimensional Topological Field Theories [FRS02, KS10, FSV13, DKR11]. In [CMS16], they gave a functorial definition of such defect TQFTs in dimension 3 extending the notion of ordinary TQFTs of Atiyah-Segal. This was further generalized for dimension n in [CRS19]. In [CRS17], they constructed the so-called Reshetikhin-Turaev (RT) TQFT with defects, based on the well-known TQFT of Reshetikhin and Turaev [RT90, RT91] and motivated by the work of [FSV13], which considers surface defects, which separate regions of possibly different RT theories. In the RT with defects constructed in [CRS17], every region is governed by the same RT type theory, surface defects are labelled by symmetric Δ-separable Frobenius algebras and line defects are certain multi-modules over algebras, which label incident surface defects.

A useful tool in the study of TQFTs with defects is that of orbifolds [DPR, FFRS09]. In the context of TQFTs with defects as in [CRS19], orbifolds are a specific choice of defect labels, called *orbifold datum*, which enable the construction of an ordinary TQFT, called the *orbifold theory*. Orbifold data in the RT TQFT with defects have been studied in [CRS18]. They consist of a Frobenius algebra A, an A-$(A \otimes A)$ bimodule T, bimodule maps α and $\overline{\alpha}$, a certain algebra endomorphism ψ and a scalar ϕ subject to certain conditions. A more geometric approach on orbifolds in 3-dimensional TQFTs and even (extended) 3-2-1-TQFTs was established in [SW18]. There are interesting examples of such orbifolds based on the work on G-crossed extension theory [DGNO09, ENOM09, BN13] with possible applications in condensed matter physics and topological quantum computing [KK12, BJQ13, BBCW14]. Let $\mathcal{D}$ be a G-crossed extension of $\mathcal{C}$. On the one hand, [CRS18] defines an orbifold datum for $\mathcal{C}$. On the other hand, one can pass to the *equivariantization* of $\mathcal{D}^G$ and consider the corresponding TQFT, called *gauge theory* of $\mathcal{C}$. In [CRS18], the authors conjecture that the two approaches are the same, i.e. the orbifold theory is isomorphic to the gauge theory. A particular case of such orbifolds is that of permutation orbifolds of a topological bilayer phase $\mathcal{C}\boxtimes\mathcal{C}$ [BJQ13, FS14], which makes use of the $\mathbb{Z}_2$-crossed extension determined in [BS11]. The physical realization of such models by, for instance, lattice dislocations and their potential advantages in universal quantum computing are discussed in [BJQ13].

The first result of this work is giving the permutation orbifold datum of $\mathcal{C} \boxtimes \mathcal{C}$, which makes use of the $\mathbb{Z}_2$-crossed extension in [BS11]. For that, we prove:

Theorem 1. *The datum $\mathcal{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ described in section 3.2 forms an orbifold datum for the bilayer system $\mathcal{C} \boxtimes \mathcal{C}$.*

Moreover, by following the construction of the orbifold theory, we compute invariants of the spaces $\mathbb{S}^3$, $\mathbb{S}^1 \times \mathbb{S}^2$ and the lens space $L(-2, 1)$.

The second part is providing, for a given G-crossed extension $\mathcal{D}$, new G-crossed extensions $\mathcal{D}^{\omega,\sigma,\nu}$ in Proposition 4.1. This family of G-crossed extensions gives rise to a family of orbifold data for the neutral component. For $G = \mathbb{Z}_2$, we prove:

Theorem 2. *The datum $\mathcal{A}^{j,k} = (A^{j,k}, T^{j,k}, \alpha^{j,k}, \overline{\alpha}^{j,k}, \psi^{j,k}, \phi)$ (see Sec. 4.2) for any $j \in \{0, 1, 2, 3\}$ and $k \in \{0, 1\}$ forms an orbifold datum for $\mathcal{C}$. In particular, the orbifold datum $\mathcal{A} \equiv \mathcal{A}^{0,0}$ is part of a family of eight orbifold data.*

We then compare these orbifold data and finally, we find that the lens space $L(-2,1)$ detects the cocycle ω, which modifies the associator.

The thesis is organized as follows:

- In Section 2, we introduce some of the mathematical prerequisites needed. Section 2.1 briefly introduces the relevant notions of the theory of tensor categories, where we also fix the conventions used here (taken mainly from [EGNO15, BK01, TV17]). In Section 2.2, one can find a review of the algebraic data, which appear in the RT TQFT with defects. Finally, the theory of G-crossed extensions and related notions is given in Section 2.3 along with some interesting examples.

- In Section 3.1, we recall the construction of RT TQFTs with defects and their associated orbifold data [CRS17, CRS18] and in Section 3.2, the topological bilayer phase and the corresponding permutation defects [BJQ13, FS14]. We also determine the permutation orbifold data and Section 3.3 includes the orbifold invariants.

- Section 4 provides a family of G-crossed extensions. The corresponding orbifold data for the $G = \mathbb{Z}_2$ case are presented in Section 4.2. Finally, we compare the different orbifold data and their orbifold theories.

We also include appendices A and B for doing graphical calculus with partitions respectively compare a convention difference for the $\mathbb{Z}_2$-crossed extensions.

The main new results in this thesis is the explicit computation of the permutation orbifold data of $\mathcal{C} \boxtimes \mathcal{C}$ obtained by its $\mathbb{Z}_2$-extension, the study of a family of eight orbifold data and the comparison of invariants of the lens space $L(-2,1)$ for their orbifold theories.

There are several directions one can go beyond this work. An interesting problem is to prove the conjecture of [CRS18], that the gauge theory [CGPW16, BN13] and the orbifold theory are isomorphic. One can also try to address this for the $\mathbb{Z}_2$-crossed category $\mathcal{C} \boxtimes \mathcal{C} \oplus \mathcal{C}$ by using the results of this work, and the work of [EJP18] on the fusion rules of the corresponding equivariantization. Furthermore, one can try to generalize the results to multi-layer phases, i.e. on $\mathcal{C}^{\boxtimes N}$ (see [Pas18]).

Acknowledgments

I would like to thank my supervisor Prof. Dr. Ingo Runkel for his guidance, remarks and comments during the preparation of this thesis. Moreover, I would like to thank Prof. Dr. Christoph Schweigert for being the second referee. Special thanks to Vincentas Mulevicius and my classmates Eilind Karlsson and Merlin Christ for various helpful discussions.

2 Preliminaries

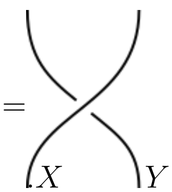
2.1 Conventions

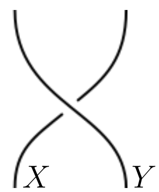
We briefly recall some notions in the theory of tensor categories and fix conventions used in this text. We adopt the definitions introduced in [EGNO15] and the graphical calculus as introduced in [BK01, Chapter 2].

Throughout this thesis, the field $\mathbb{K}$ will be algebraically closed and of characteristic 0.

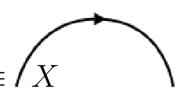
For a monoidal category $\mathcal{C}$, we will write (unless otherwise indicated) $\otimes$ for the monoidal product, $\mathbb{1}$ for the unit object, $a_{X,Y,Z}$ for the associator of objects X, Y and Z. The monoidal category $\mathcal{C}^{\mathrm{mop}}$ will denote the monoidal category with the opposite monoidal product $\otimes^{\mathrm{op}}$.

For the braiding of two objects X and Y we will write $c_{X,Y} : X \otimes Y \to Y \otimes X$. Graphically, we write

$c_{X,Y} =$ 

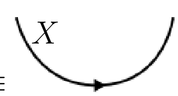
$c^{-1}_{Y,X} =$ 

For a braided category $\mathcal{C}$, we write $\mathcal{C}^{\mathrm{rev}}$ for the braided category with the same underlying monoidal category, but equipped with the opposite braiding, i.e. $c^{\mathrm{rev}}_{X,Y} := c^{-1}_{Y,X}$. A *rigid category* is a monoidal category, for which every object X has a left dual ${}^*X$ and a right dual X^*. We adopt the convention used in [EGNO15] (which is the opposite to [BK01] and [TV17]), i.e. a right dual comes with the (right) evaluation map

$\overrightarrow{ev}_X \equiv$ 

$: X \otimes X^* \to \mathbb{1}$

and coevaluation map

$\overrightarrow{coev}_X \equiv$ 

$: \mathbb{1} \to X^* \otimes X$

which are subject to the snake equations. Similarly, the left dual ${}^*X$ of X comes with left evaluation map $\overleftarrow{ev}_X : {}^*X \otimes X \to \mathbb{1}$ and coevaluation map $\overleftarrow{coev}_X : \mathbb{1} \to X \otimes {}^*X$. This convention is chosen such that the graphical representation is actually directed to the right, respectively left. For the definition of pivotal, spherical and ribbon categories see [EGNO15]. A twist of an object X will be denoted by θ_X. In a ribbon category $\mathcal{C}$, we will not distinguish left from right dualities and therefore we will always write X^* and f^* for the dual of an object, respectively morphism. Let $\mathcal{C}$ be a ribbon category. The ribbon category $\mathcal{C}^{\mathrm{rev}}$ is the category $\mathcal{C}$ with opposite braidings and opposite twists, i.e. $\theta^{\mathrm{rev}}_X = \theta^{-1}_X$.

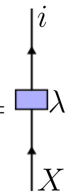
Recall the notion of tensor categories and fusion categories as in [EGNO15, Def. 4.1.1]. Given a fusion category $\mathcal{C}$, we will write I for the set of representatives of isomorphism classes of simple objects in $\mathcal{C}$ and choose $0 \equiv \mathbb{1}$ as the representative of $[\mathbb{1}]$. The duality of $\mathcal{C}$ induces an involution map $\overline{()} : I \to I$, since for any $i \in I$ there exists $\bar{i}$ such that $\bar{i} \cong i^*$ [BK01].

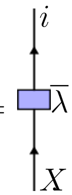
Let X be an object in $\mathcal{C}$ and $i \in I$. We write N_X^i for the dimension of the vector space $\mathcal{C}(i, X)$. An *i-partition* of X [TV17, Chapter 4] consists of a basis $\{p_\lambda^{(i)}\}_{\lambda=1,\ldots,N_X^i}$ of the vector space $\mathcal{C}(X, i)$ and a basis $\{q_\lambda^{(i)}\}_{\lambda=1,\ldots,N_X^i}$ of vector space $\mathcal{C}(i, X)$ such that

$$p_\lambda^{(i)} \circ q_\mu^{(i)} = \delta_{\lambda,\mu} \mathrm{id}_i.$$

A union of i-partitions of X for each $i \in I$, gives an I-partition such that

$$\sum_i \sum_\lambda q_\lambda^{(i)} \circ p_\lambda^{(i)} = \mathrm{id}_X.$$

Graphically, we will write $p_\lambda =$ 

and $q_\lambda =$ 

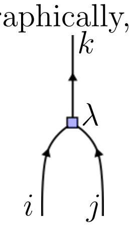
The *fusion coefficients* are defined by

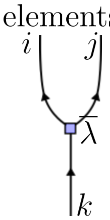
$$N_{ij}^k := \dim \mathcal{C}(k, i \otimes j)$$

and they satisfy

$$N_{ij}^k = N_{ji}^k = N_{i\bar{k}}^{\bar{j}} = N_{\bar{i}\bar{j}}^{\bar{k}} \tag{2.1}$$

A k-partition of $X = i \otimes j$ for $i, j \in I$ describes the fusion (and the split) of i and j. Graphically, fusion basis elements are



and


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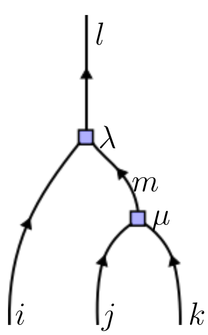
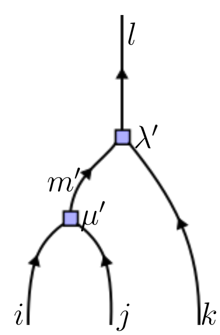
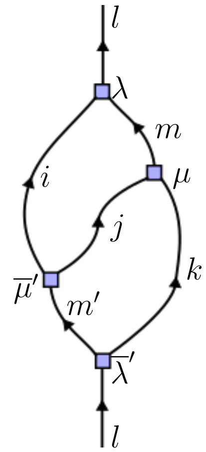
The associativity of the fusion is described by the so called F-matrix. It is defined by



$$= \sum_{m',\lambda',\mu} F_{\mu\mu'}^{\lambda\lambda'}$$


By the definition axioms of partitions, we deduce

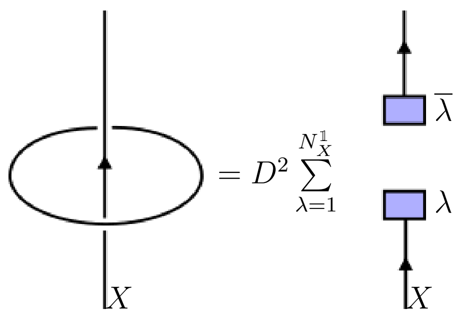
$$F^{\lambda\lambda'}_{\mu\mu'} =$$



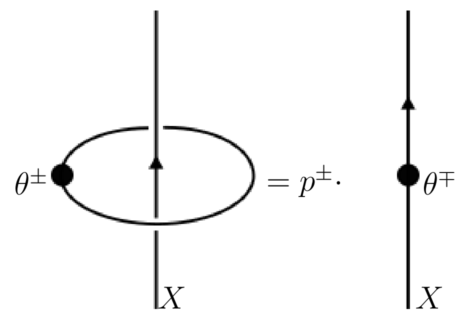
A *modular tensor category* (MTC) is a ribbon fusion category $\mathcal{C}$ such that the s-matrix, see [BK01], is non-degenerate. Equivalently, its symmetric center (or Müger center) $(\mathcal{C})'$[1] is trivial, i.e. every object in the symmetric center is just a direct sum of $\mathbb{1}$'s, or even that $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \simeq Z(\mathcal{C})$, where $Z(\mathcal{C})$ is the *Drinfeld center* of $\mathcal{C}$.

For graphical calculus in modular tensor categories, we adopt the convention used in [BK01], where uncolored links are interpreted as taking the sum of all colorings by the set of representatives of isomrphism classes of simple objects I with a weight of the dimension of each simple object.

Let $\mathcal{C}$ be a modular tensor category. Then, we have the following identities



and



where $p^{\pm} = \sum_{i \in I} \theta_i^{\pm} d_i^2$ and $D = \sqrt{\sum_i d_i^2} = \sqrt{p^+ p^-}$ (see [BK01, Chapter 3]).

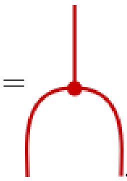
2.2 Algebraic Data

In this section, we briefly recall the algebraic objects, which play a central role in the construction of Reshetikhin-Turaev TQFT with defects (see [TV17], [FRS02]). When dealing with string diagrams of an algebra or a certain module, we often omit putting labels and instead color them by red resp. green.

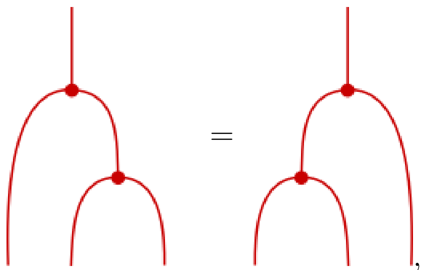
Definition 2.1. Let $\mathcal{C}$ be a monoidal category.

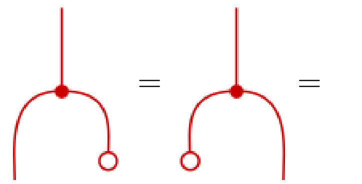
1. An *algebra* in $\mathcal{C}$ is a tuple $A \equiv (A, \mu, \eta)$ consisting of an object A and morphisms $\mu : A \otimes A \to A$ and $\eta : \mathbb{1} \to A$ represented graphically:

[1] This consists of all objects $U \in \mathcal{C}$ such that $c_{X,U} \circ c_{U,X} = \mathrm{id}_{U \otimes X}$ for all objects X.

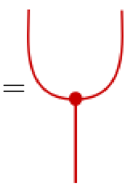
$\mu =$ , $\eta =$ .

These morphisms are subject to the associativity and unitality conditions:

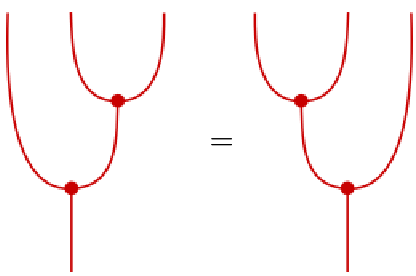


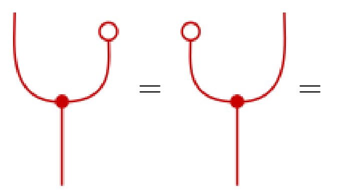


2. Dually, a *coalgebra* in $\mathcal{C}$ is a tuple $C \equiv (C, \Delta, \epsilon)$ consisting of an object C and morphisms

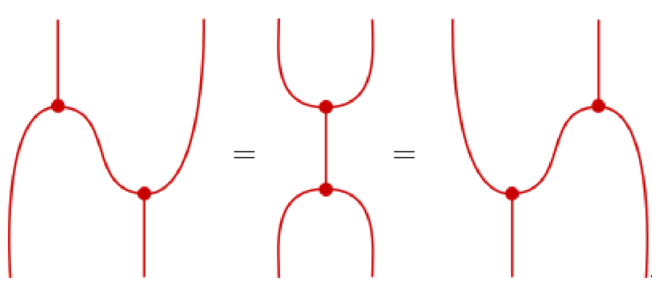
   $\Delta =$  and $\epsilon =$ .

   They are subject to the coassociativity and counitality conditions:





3. A *Frobenius* algebra is a list $A \equiv (A, \mu, \eta, \Delta, \epsilon)$ such that (A, μ, η) forms an algebra, (A, Δ, ϵ) forms a coalgebra and the algebra and coalgebra structures are compatible in the following way



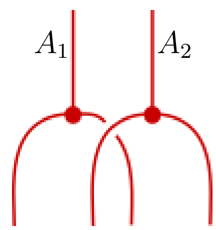
Let $\mathcal{C}$ be a braided category. For an algebra $A = (A, \mu, \eta)$, we define the *opposite algebra* $A^{\mathrm{op}} = (A, \mu^{\mathrm{op}}, \eta)$ with multiplication

$$\mu^{\mathrm{op}} := \mu \circ c_{A,A}.$$

The choice of the braiding instead of the opposite braiding is a convention. If $\mu^{op} = \mu$, the algebra is called *commutative*. Similarly, for a coalgebra (C, Δ, ϵ), the opposite coalgebra $C^{\mathrm{op}} = (C, \Delta^{\mathrm{op}}, \epsilon)$ has comultiplication

$$\Delta^{\mathrm{op}} := c^{-1}_{C,C} \circ \Delta.$$

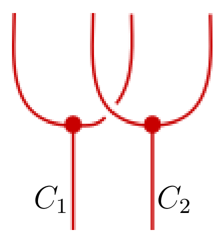
In a braided category, one can define an algebra structure on their monoidal product. Namely, for algebras A_1 and A_2 the monoidal product $A_1 \otimes A_2$ inherits an algebra structure with multiplication



and unit

$$\eta_{A_1 \otimes A_2} = \eta_1 \otimes \eta_2.$$

Similarly, the product of coalgebras C_1 and C_2 has comultiplication



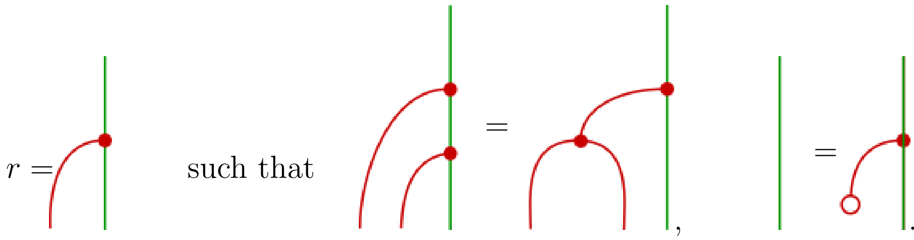
and counit

$$\epsilon_{C_1 \otimes C_2} = \epsilon_1 \otimes \epsilon_2.$$

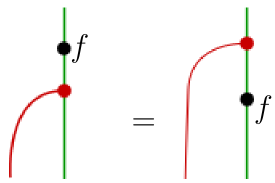
As before, the choice of these braidings is a convention. If A_1 and A_2 are Frobenius algebras, then $A_1 \otimes A_2$ with the algebra and coalgebra structures described above is a Frobenius algebra.

Let $\mathcal{C}$ be rigid and (A, μ, η) be an algebra. Then, its dual A^* admits a coalgebra structure with comultiplication $\Delta := \mu^*$ and counit $\epsilon := \eta^*$. Similarly, the dual C^* of a coalgebra C forms an algebra. If A is a Frobenius algebra, then A^* is also a Frobenius algebra.

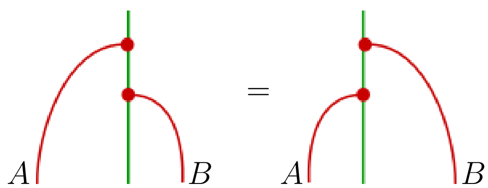
Definition 2.2. 1. Let A be an algebra in a monoidal category $\mathcal{C}$. A *left A-module* is a pair $(M, r) \equiv {}_AM$ where M is an object in $\mathcal{C}$ and r is a left action of A on M, i.e. a morphism



A *right* A-module is a left A-module in $\mathcal{C}^{\mathrm{mop}}$. A module morphism $f : (M, r_M) \to (N, r_N)$ is a morphism $f : M \to N$ such that

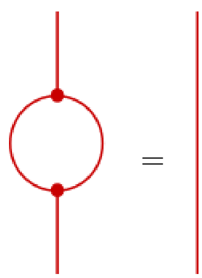


2. Let A and B be two algebras. An A-B-*bimodule* T carries a left A-action $r_A : A \otimes T \to T$ and a right B-action $r_B : T \otimes B \to B$ which are compatible, i.e.



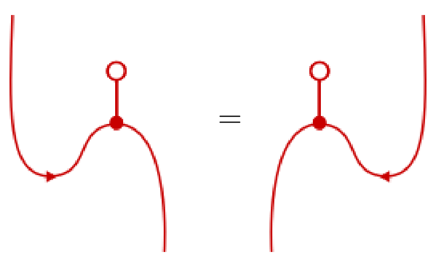
   We will write ${}_A T_B$ to indicate that T carries an A-B-bimodule structure. Any algebra A has an A-A-bimodule structure with actions given by its multiplication.

3. We say that a Frobenius algebra A is *separable* if its multiplication μ splits as an A-A-bimodule map. If its right inverse is the comultiplication, i.e.



   then we say that A is Δ-*separable*.

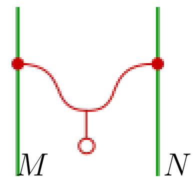
4. Let $\mathcal{C}$ be a ribbon category. A Frobenius algebra A is called *symmetric* if



(2.2)

Remark 2.1.
- The Frobenius property in Definition 2.1 is equivalent to Δ being an A-A-bimodule map.
- In Definition 2.2 (4), the morphism $A \to A^*$ is even an isomorphism of Frobenius algebras.

Let A be a Frobenius algebra, (M, r_M) a right A-module and (N, r_N) a left A-module. The projector $p_{M,N} \in \mathrm{End}(M \otimes N)$ is defined by

$p_{M,N} =$ 

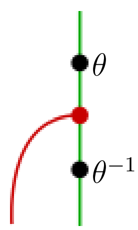
Its image is the balanced tensor product $(M \otimes_A N)$.[2] One can show for any Frobenius algebras A, A' and corresponding right (left) modules M, M' (N, N') that

$$\mathrm{Hom}(M \otimes_A N, M' \otimes_{A'} N')$$

consists of the morphisms $f : M \otimes N \to M' \otimes N'$ such that

$$f \circ p_{M,N} = f = p_{M',N'} \circ f.$$

Definition 2.3. Let (M, r) be an A-module in some ribbon category $\mathcal{C}$. The *twist* of M is the A-module $M^{\mathrm{tw}} = (M, r^{\mathrm{tw}})$ with action

$r^{\mathrm{tw}} :=$ 

Let $A_1, \ldots, A_n$ be algebras. A *multi-module* ${}_{A_1,\ldots,A_n}M$ is an $(A_1 \otimes \cdots \otimes A_n)$-module M, or equivalently an object M with A_i-actions r_i such that

$$r_i \circ (\mathrm{id}_{A_i} \otimes r_j) = r_j \circ (\mathrm{id}_{A_j} \otimes r_i) \circ (c^{-1}_{A_i,A_j} \otimes \mathrm{id}_M)$$

for all $i < j$. Let ${}_{A_1,\ldots,A_n}M$ be a multi-module with actions r_i. Define the multi-module ${}_{A_1,\ldots,A_{j+1},A_1\ldots,A_n}M^{\mathrm{tw}^j}$ by actions r_i for $i \geq j+1$ and r_i^{tw} for $i \leq j$.

Remark 2.2. If A is a commutative symmetric Frobenius algebra, then a left module M is *local* or *dyslectic* if and only if $M^{tw} = M$ [FFRS06].

Definition 2.4. Let $(A_1, \ldots, A_n)$ be a list of algebras and $k \in \mathbb{Z}_+$ be minimal such that $A_i = A_{(i+k) \bmod n}$. A (maximally) cyclic $(A_1, \ldots, A_n)$-multi-module is an $(A_1, \ldots, A_n)$-multi-module together with module isomorphism $\phi : M^{\mathrm{tw}^k} \to M$ such that $\phi^{n/k} = \theta_M^{-1}$.

The definition of cyclic multi-modules coincides with that of $\mathbb{Z}_{n/k}$-equivariant objects as in [EGNO15, Sec. 2.7]. By an $((A_1, \epsilon_1), \ldots, (A_n, \epsilon_n))$-multi-module (where $\epsilon_i = \pm$) one means an $(A_1^{\epsilon_1}, \ldots, A_n^{\epsilon_n})$-multi-module, where $A^+ := A$ and $A^- := A^{\mathrm{op}}$.

[2] It is the coequalizer of $r_M \otimes \mathrm{id}_N$ and $\mathrm{id}_M \otimes r_N$.

2.3 G-crossed categories

We recall some of the notions in the theory of G-crossed categories, which can be found in [EGNO15, Kir04, Tur00], from which we deviate by using a right action convention[3]. Without loss of generality, we restrict our attention to monoidal categories with trivial unitality constraints.

Let G be a finite group. We will write $\underline{G}$ for the monoidal category whose objects are precisely the group elements with only identity morphisms and the monoidal product is given by the group multiplication. Let $\mathrm{Aut}(\mathcal{C})$ be the category with autoequivalences on $\mathcal{C}$ as objects and natural isomorphisms as morphisms.

Definition 2.5. Let $\mathcal{C}$ be a category. A (right) *action* of G on $\mathcal{C}$ is a monoidal functor

$$R : \underline{G}^{mop} \to \mathrm{Aut}(\mathcal{C});\ g \mapsto R_g,$$

such that $R_1 = \mathrm{id}_{\mathcal{C}}$.

Similarly, a G-action on a braided category $\mathcal{C}$ is a monoidal functor $R : \underline{G}^{mop} \to \mathrm{Aut}^{\mathrm{br}}(\mathcal{C})$ such that $R_1 = \mathrm{id}_{\mathcal{C}}$, where $\mathrm{Aut}^{\mathrm{br}}(\mathcal{C})$ now stands for the category of braided autoequivalences and braided natural isomorphisms.

Let $\mathcal{D}$ be a tensor category. A *G-grading* is a decomposition

$$\mathcal{D} = \bigoplus_{g\in G} \mathcal{C}_g$$

which is compatible with the monoidal structure, i.e. for objects $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ we have $X \otimes Y \in \mathcal{C}_{gh}$. In particular, it follows that $\mathbb{1} \in \mathcal{C}_1$ and $X^* \in \mathcal{C}_{g^{-1}}$, when $X \in \mathcal{C}_g$. For $X \in \mathcal{C}_g$, we refer to $g \in G$ as the *degree* of X. A G-grading is called *faithful*, if $\mathcal{C}_g \neq 0$ for all $g \in G$.

Definition 2.6. A *G-crossed category* $\mathcal{D}$ is a tensor category together with:

1. A G-grading: $\mathcal{D} = \bigoplus_{g\in G} \mathcal{C}_g$.
2. An action R of the group G on $\mathcal{D}$, i.e. a monoidal functor $R : \underline{G}^{\mathrm{mop}} \to \mathrm{Aut}(\mathcal{D})$ such that $R_1 = \mathrm{id}_{\mathcal{D}}$, such that for all $g, h \in G$ we have $R_g(\mathcal{C}_h) \subset \mathcal{C}_{g^{-1}hg}$.

For a G-crossed category $\mathcal{D} = \bigoplus_g \mathcal{C}_g$, we will call $\mathcal{C}_1$ the *neutral component* (or neutral sector) and non-trivial components $\mathcal{C}_g$ will be called *(g-)twisted components* (sectors). Accordingly, an object in $\mathcal{C}_1$ will be called a neutral object, while objects in $\mathcal{C}_g$ (for a non-trivial g) will be called (g-)twisted objects. From now on, we will write $R_g(X) \equiv X.g$ for the action of g on the object X and similarly $R_g(f) \equiv f.g$ for the action of g on the morphism f.

Remark 2.3. The monoidal structure on R is a family of monoidal natural isomorphisms $\rho_{g,h} : R_h \circ R_g \xrightarrow{\sim} R_{gh}$ for all $g, h \in G$ which satisfy

$$\rho_{g,hk} \circ (\rho_{h,k} \star \mathrm{id}_{R_g}) = \rho_{gh,k} \circ (\mathrm{id}_{R_k} \star \rho_{g,h})$$

[3] We compare this to the convention of a left action in Appendix B.

for all $g, h, k \in G$. In particular, we have isomorphisms $\rho_{g,h}(X) : X.gh \cong X.g.h$. Here $\circ$ and $\star$ denote the vertical resp. horizontal composition law of natural homomorphisms.

The monoidal structure on the functor R_g is given by a natural isomorphism[4] $\beta_g^R : \otimes \circ (R_g \times R_g) \to R_g \circ \otimes$ such that

$$\begin{tikzcd}
(X.g \otimes Y.g) \otimes Z.g \arrow[r, "a_{X.g,Y.g,Z.g}"] \arrow[d, "\beta_g^R(X,Y)\otimes \mathrm{id}_{Z.g}"'] & X.g \otimes (Y.g \otimes Z.g) \arrow[d, "\mathrm{id}_{X.g}\otimes \beta_g^R(Y,Z)"] \\
(X \otimes Y).g \otimes Z.g \arrow[d, "\beta_g^R(X\otimes Y,Z)"'] & X.g \otimes (Y \otimes Z).g \arrow[d, "\beta_g^R(X,Y\otimes Z)"] \\
((X \otimes Y) \otimes Z).g \arrow[r, "(a_{X,Y,Z}).g"] & (X \otimes (Y \otimes Z)).g
\end{tikzcd}$$

commutes and $\beta_g^R(\mathbb{1}, X) = \mathrm{id}_X = \beta_g^R(X, \mathbb{1})$ for all X.

We will refer to the isomorphisms $\rho_{g,h}$ and β_g^R as the (action) *coherence isomorphisms.* A *G-crossed fusion* category is a fusion category with a G-crossed tensor structure.

Example 2.1.
- Any tensor category $\mathcal{C}$ admits a trivial G-crossed tensor structure with trivial components $\mathcal{C}_g = 0$ for all $g \neq 1$ and action functor $R_g = \mathrm{id}_{\mathcal{C}}$.
- Let $\mathcal{C}$ be a pointed fusion category, i.e. every simple object is invertible. Then, the set of isomorphism classes of simple objects forms a finite group G. This gives a natural G-structure, with the obvious grading and the *conjugation action* $(X).g := i^* \otimes X \otimes i$, where $i \in I$ is the representative of $g \in G$ (for details see [DGNO09, Prop. 4.61]).
- As a special case of the previous example, consider the fusion category $\mathcal{D} = \mathrm{Vect}_G^\omega$. This is the category of G-graded vector spaces with associators defined via a 3-cocycle $\omega \in H^3(G, \mathbb{K}^\times)$. The action on the simple objects is action $\mathbb{K}_h.g := \mathbb{K}_{g^{-1}hg}$ for $g, h \in G$.

Definition 2.7. A *G-crossed braided fusion category* is a G-crossed fusion category equipped with natural isomorphisms $c^R_{X,Y} : X \otimes Y \to Y \otimes X.h$ for all $X \in \mathcal{C}_g$ and $Y \in C_h$ such that the diagrams

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "c^R_{X\otimes Y,Z}"] \arrow[dd, "a_{X,Y,Z}"'] & Z \otimes (X \otimes Y).k \arrow[r, "\mathrm{id}_Z\otimes(\beta_k^R)^{-1}"] & Z \otimes (X.k \otimes Y.k) \arrow[dr, "a^{-1}_{Z,X.k,Y.k}"] & \\
& & & (Z \otimes X.k) \otimes Y.k \\
X \otimes (Y \otimes Z) \arrow[r, "\mathrm{id}_X\otimes c^R_{Y,Z}"] & X \otimes (Z \otimes Y.k) \arrow[r, "a^{-1}_{X,Z,Y.k}"] & (X \otimes Z) \otimes Y.k \arrow[ur, "c^R_{X,Z}\otimes \mathrm{id}_{Y.k}"'] &
\end{tikzcd}$$

[4]We write the superscript R to keep in mind that we deal with a right action. In Appendix B we will compare this to the convention of a left action.

and

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "c^R_{X,Y\otimes Z}"] \arrow[dd, "a^{-1}_{X,Y,Z}"'] & (Y \otimes Z) \otimes X.hk \arrow[r, "\mathrm{id}_{Y\otimes Z}\otimes \rho^{-1}_{g,h}"] & (Y \otimes Z) \otimes X.g.h \arrow[dr, "a_{Y,Z,X.g.h}"] & \\
 & & & Y \otimes (Z \otimes X.g.h) \\
(X \otimes Y) \otimes Z \arrow[r, "c^R_{X,Y}\otimes \mathrm{id}_Z"] & (Y \otimes X.h) \otimes Z \arrow[r, "a_{Y,X.h,Z}"] & Y \otimes (X.h \otimes Z) \arrow[ur, "\mathrm{id}_Y \otimes c^R_{X.h,Z}"] &
\end{tikzcd}$$

commute and $c^R_{\mathbb{1},X} = c^R_{X,\mathbb{1}} = \mathrm{id}_X$.

Remark 2.4. The above diagrams are the G-crossed analogue of the ordinary hexagon axioms. The neutral component $\mathcal{C}_1$ in a G-crossed braided fusion category is, in particular, a braided fusion category.

Example 2.2.
1. Any braided fusion category is trivially a G-crossed braided fusion category as in Example 2.1.
2. Let $\mathcal{D}$ be a pointed braided fusion category. Then, the group G of isomorphism classes of simple objects is abelian and $\mathcal{D}$ is a G-crossed braided category. In particular, let G be a finite abelian group. Then, Vect^ω_G is naturally a G-crossed braided fusion category.

Following [Kir04], we define a *twist* in a G-crossed braided category $\mathcal{D}$ as a family of natural isomorphisms $\theta^R_X : X \to X.g$ for any $X \in \mathcal{C}_g$, such that

1. $\theta^R_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$
2. $(\theta^R_X).h = \theta^R_{X.h}$
3. $\theta^R_{X\otimes Y} = c^R_{Y.h,X.g.h} \circ c^R_{X.g,Y.h} \circ (\theta^R_X \otimes \theta^R_Y)$,

where we omit writing the obvious structure maps of the action functor.

If $\mathcal{D}$ is a G-crossed fusion category equipped with a twist θ such that

$$\theta^R_{X^*} = g^{-1}.(\theta^R_X)^*$$

we say that $\mathcal{D}$ is a *G-crossed ribbon category.* If we restrict our attention to the neutral sector $\mathcal{C}_1$, we notice that it has the structure of an ordinary ribbon category.

Graphical calculus for G-crossed categories is well established [Tur10a]. To distinguish from the ordinary string diagrams in the non-crossed setting, string diagrams for the G-crossed case will be coloured blue. For instance, the braiding $c^R_{X,Y}$ and the morphism $\tilde{c}^R_{X,Y} := (c^R_{Y.g^{-1},X})^{-1} \circ (\mathrm{id}_X \otimes \rho^{-1}_{g,g^{-1}}) : X \otimes Y \to Y.g^{-1} \otimes X$ are graphically represented in Figure 2. The dotted arrows indicate the degree of the corresponding strand. Hence, every strand which passes over gets twisted by the degree (or its inverse) of the other strand.

Let $\mathcal{D} = \bigoplus \mathcal{C}_g$ be a G-crossed braided fusion category. Then, we say that $\mathcal{D}$ is a G-crossed extension of $\mathcal{C} \equiv \mathcal{C}_1$. Similarly, we have the notion of G-crossed ribbon extensions of ribbon fusion categories.

We recall the notion of equivariantization of a category with a group action [EGNO15, Def. 2.7.2]

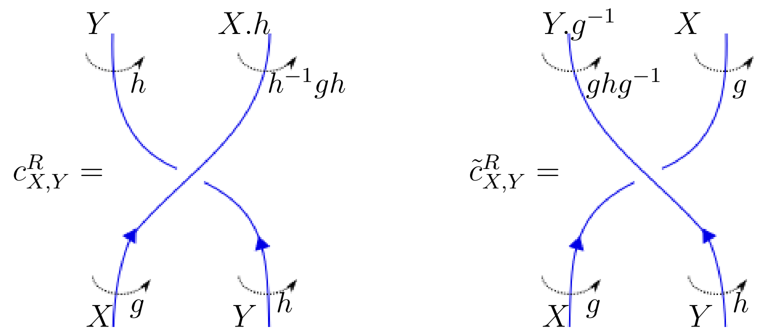


Figure 2: G-crossed braiding graphically.

Definition 2.8. Let $\mathcal{D}$ be a category with a G-action R. The *equivariantization* of $\mathcal{D}$ is a category $\mathcal{D}^G$. The objects are pairs $(X, \{\gamma^X_g\}_{g\in G})$ where $X \in \mathcal{D}$ and $\gamma^X_g : X.g \cong X$ such that

$$\begin{array}{ccc} X.g.h & \xrightarrow{\rho_{g,h}(X)} & X.gh \\ {\scriptstyle (\gamma_g).h}\downarrow & & \downarrow{\scriptstyle \gamma_{gh}} \\ X.h & \xrightarrow{\gamma_h} & X \end{array}$$

commutes. A morphism $(X, \{\gamma^X_g\}_{g\in G}) \to (Y, \{\gamma^Y_g\}_{g\in G})$ is a morphism $f : X \to Y$ in $\mathcal{D}$ such that

$$\gamma^Y_g \circ f.g = f \circ \gamma^X_g$$

Let $\mathcal{D}$ be a G-crossed braided fusion category. Then, its equivariantization is a braided fusion category. In fact, there is a dual construction called deequariantization, which establishes a bijection between G-crossed braided fusion categories and braided fusion categories containing $\mathrm{Rep}(G)$ [EGNO15, Theorem 8.24.3].

Example 2.3. 1. Let $\mathcal{D}$ be a pointed braided fusion category [BN17] with the G-crossed braided structure via conjugation as described in Example 2.1. Then, by the results of [DGNO09, Prop. 4.61], we have a braided equivalence $(\mathcal{D})^G \simeq Z(\mathcal{D})$.

2. In particular, for $\mathcal{D} = \mathrm{Vect}^\omega_G$ in our previous example, we have $(\mathrm{Vect}^\omega_G)^G \simeq Z(\mathrm{Vect}^\omega_G) \simeq \mathrm{Rep}(D^\omega(G))$, where the last equivalence is between the center of Vect^ω_G and the representation category of the twisted Drinfeld double of the group G [NN08].

The simple objects and fusion rules of $\mathcal{D}^G$ were studied in [BN13]. Let X be a simple object in the G-crossed fusion category $\mathcal{D}$ and $\Gamma_X = \{X.g | g \in G\}$ be the orbit of the induced action of G on the set of simples. Moreover, let $G_a := \{g \in G | X.g = X\}$ be the stabilizer of this action. Then, the simple objects in $\mathcal{D}^G$ are in 1-to-1 correspondence with pairs (Γ_X, π), where π is an irreducible projective η-representation for a certain 2-cocycle η. The object associated to the pair (Γ_X, π) is $S_{(\Gamma_X,\pi)} := \pi \otimes \bigoplus_{Y\in\Gamma_X} Y$. If there is a spherical structure on the category, the dimension of this object is given by the formula

$$\dim(S_{(\Gamma_X,\pi)}) = d_X |\Gamma_X| \dim(\pi).$$

This implies

$$D_{\mathcal{D}^G} = |G|^{1/2} D_{\mathcal{D}} = |G| D_{\mathcal{C}} \tag{2.3}$$

3 Defects and Orbifold Data

3.1 Topological Field Theories with defects and orbifold data

The notion of closed n-dimensional TQFTs defined as symmetric monoidal functors on a certain bordism category Bord_n was extended in [CMS16] for $n = 3$ and then in [CRS19] for arbitrary n by introducing the notion of an n-dimensional TQFT with defects (now on a certain source category of decorated bordisms $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$). This is done by enlarging Bord_n into $\mathrm{Bord}_n^{\mathrm{strat}}$ (stratified bordism category) and then labeling strata in this setting with appropriate defect data.

Definition 3.1. 1. A *(closed) n-dimensional stratified manifold* is an n-dimensional closed smooth manifold M together with a filtration $M = F_n \supset \cdots \supset F_0 \supset F_{-1} = \emptyset$ such that:

- $M_j := F_j \backslash F_{j-1}$ is a j-dimensional smooth manifold with a choice of orientation of its connected components M_j^α, also called *j-strata*.
- If $M_i^\alpha \cap \overline{M}_j^\beta \neq \emptyset$, then $M_i^\alpha \subset \overline{M}_j^\beta$.
- The total number of strata is finite.

2. A *morphism* of (closed) stratified manifolds from $(M, \{F_i\})$ to $(M', \{F_i'\})$ is a continuous map $f : M \to M'$ such that $f(M_j) \subset M_j'$ and f restricts to an orientation preserving smooth map on each stratum M_j^α.

One can generalize the above definition to include manifolds with boundary.

Definition 3.2. 1. An n-dimensional stratified manifold with boundary is an n-dimensional smooth manifold M with boundary ∂M together with a filtration $M = F_n \supset \cdots \supset F_0 \supset F_{-1} = \emptyset$ subject to the following conditions:

- The interior M° with filtration $F_j' = M^\circ \cap F_j$ forms a stratified manifold.
- $M_j := F_j \backslash F_{j-1}$ is a smooth submanifold with $\partial M_j \subset \partial M$ and all strata meet ∂M transversally.
- Its boundary ∂M with filtration $F_j'' = \partial M \cap F_{j+1}$ is a closed stratified $(n-1)$-dimensional manifold with orientations on its j-strata given by the induced orientations of the corresponding $(j+1)$-strata in M.

2. A morphism of stratified manifolds with boundary is a continuous map $f : M \to M'$ such that $f(\partial M) \subset \partial M'$, $f(M_j) \subset M_j'$, f restricts to a smooth orientation preserving map on each stratum and $f|_{\partial M}$ is a morphism of (closed) stratified manifolds.

Any smooth manifold (with boundary) can be viewed trivially as a stratified manifold. Let Σ_1, Σ_2 be two closed $(n-1)$-dimensional stratified manifolds. A *bordism* $M : \Sigma_1 \to \Sigma_2$ is a compact n-dimensional stratified manifold M together with an isomorphism (of stratified $(n-1)$-manifolds) $\psi : \partial M \xrightarrow{\sim} \Sigma_1^{\mathrm{rev}} \amalg \Sigma_2$, where Σ_1^{rev} denotes the manifold Σ_1 with reversed orientation and reversed orientation on its strata. Two bordisms $M_1, M_2 : \Sigma_1 \to \Sigma_2$ are called *equivalent* if there exists an isomorphism (of stratified manifolds) $f : M_1 \xrightarrow{\sim} M_2$ compatible with the boundary parametrisations.

Finally, the category $\mathrm{Bord}_n^{\mathrm{strat}}$ consists of:

- Objects: $(n-1)$-dimensional closed stratified manifolds.
- Morphisms: $\mathrm{Bord}_n^{\mathrm{strat}}(\Sigma_1, \Sigma_2)$ is formed by equivalence classes of bordisms $\Sigma_1 \to \Sigma_2$.

For now, we restrict to $n = 3$ and consider stratified bordisms without any 0-strata in the interior[5].

A set of 3-dimensional *defect data* is a tuple $\mathbb{D} = (D_3, D_2, D_1, s, t, j)$ where D_i is a set of labels for i-strata in 3-bordisms and:

- $s, t : D_2 \times \{\pm\} \to D_3$ (source/target map) such that for $\epsilon \in \{\pm\}$ $s(f, \epsilon) = t(f, -\epsilon)$.
- $j : D_1 \times \{\pm\} \to D_3 \amalg \coprod_m P_m/\mathbb{Z}_m$ (junction map)

are functions that describe adjacent strata, where

$$P_m = \{(d_1, \dots, d_m) \in (D_2 \times \{\pm\})^m | s(d_i) = t(d_{i+1}), i \in \{0, m-1\}, s(d_m) = t(d_1)\}$$

and $\mathbb{Z}_m$ acts by cyclic permutations (see [CMS16] for details).

Decorated closed surfaces are closed 2-stratified manifolds with j-strata labeled by elements in D_{j+1}. Morphisms of decorated closed surfaces are morphisms of stratified surfaces which also respect the decorations. *Decorated 3-bordisms* between decorated surfaces are stratified 3-bordisms (with j-strata labeled by elements in D_j) between the underlying stratified surfaces whose boundary parameterization is also compatible with decorations. Two such bordisms are called equivalent if they are equivalent as stratified bordisms and the corresponding isomorphism (of stratified manifolds) is also an isomorphism of decorated manifolds. These data form the decorated bordism category $\mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D})$.

Definition 3.3. A *3-dimensional TQFT with defects* given by defect data $\mathbb{D}$ is a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}.$$

Defects in Reshetikhin-Turaev TQFT

Let $\mathcal{C}$ be a modular tensor category, which is *anomaly free*, i.e. $p^+ = p^-$. Then, $\mathcal{C}$ gives rise to a 3-dimensional TQFT $\mathcal{Z}^{\mathrm{RT}} : \mathrm{Bord}_3^{\mathrm{rib}}(\mathcal{C}) \to \mathrm{Vect}$. The category $\mathrm{Bord}_3^{\mathrm{rib}}(\mathcal{C})$ consists of $\mathcal{C}$-marked surfaces and bordisms with embedded $\mathcal{C}$-coloured ribbon tangles[6]. For the construction of the Reshetikhin-Turaev TQFT see [Tur10b, Chap. 4] and [BK01, Sec. 4.4].

[5]This is not a significant restriction as there is a canonical way to add 0-strata labels in a given 3-dimensional defect TQFT. The details can be found in [CMS16],[CRS19].

[6]When $p^+ \neq p^-$ one replaces the source category by the extended category $\overline{\mathrm{Bord}_3^{\mathrm{rib}}}(\mathcal{C})$, whose objects are pairs (Σ, λ) where Σ is an object in $\mathrm{Bord}_3^{\mathrm{rib}}(\mathcal{C})$ and $\lambda \subset H_1(N, \mathbb{R})$ a Lagrangian subspace, and its morphisms are pairs (M, n), where M is a morphism in $\mathrm{Bord}_3^{\mathrm{rib}}(\mathcal{C})$ and $n \in \mathbb{Z}$ (The same extension is made for RT TQFT with defects).

The goal is to obtain a TQFT with defects using the theory of RT. Surface defects were already studied in [KS10, FSV13]. In [FSV13], they considered surface defects, which separate two RT theories described by modular tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$ and found that such defects exist if and only if $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}}$ is braided equivalent to $Z(\mathcal{W})$ for some fusion category $\mathcal{W}$. A TQFT with defects as in Definition 3.3 was given in [CRS17], where every region is governed by the same RT theory. Let $\mathcal{C}$ be a modular tensor category. There is defect data $\mathbb{D}^{\mathcal{C}} = (D_3^{\mathcal{C}}, D_2^{\mathcal{C}}, D_1^{\mathcal{C}}, s, t, j)$ with:

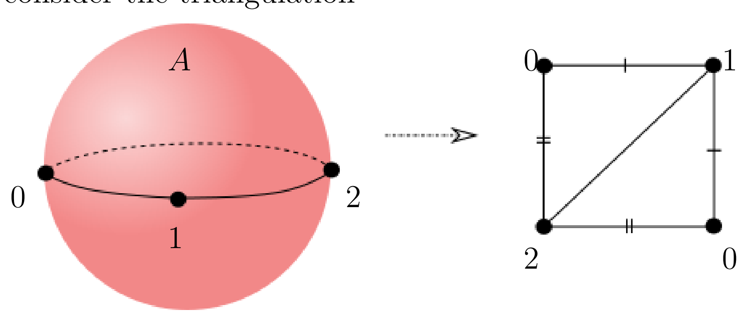
- $D_3^{\mathcal{C}} = \{\mathcal{C}\}$
- $D_2^{\mathcal{C}} = \{\Delta\text{-separable symmetric Frobenius algebras in } \mathcal{C}\}$
- $D_1^{\mathcal{C}} = \amalg_{n\in\mathbb{Z}_+} L_n$, where
$$L_0 = \{M \in \mathcal{C} | \theta_M = \mathrm{id}_M\}$$
and
$$\begin{aligned} L_n = \{((A_1, \epsilon_1), \dots, (A_n, \epsilon_n), M) | A_i \in D_2^{\mathcal{C}}, \epsilon_i \in \{\pm\}, \\ M \text{ cyclic multi-module for } ((A_1, \epsilon_1), \dots, (A_n, \epsilon_n))\}. \end{aligned}$$
- $s(A, \pm) = \mathcal{C} = t(A, \pm)\ \forall A \in D_2^{\mathcal{C}}$
- $j(((A_1, \epsilon_1), \dots, (A_n, \epsilon_n), M)) = [((A_1, \epsilon_1), \dots, (A_n, \epsilon_n))]$.

Theorem ([CRS17])**.** There is a 3-dimensional defect TQFT $\mathcal{Z} : \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}}) \to \mathrm{Vect}$.

The construction on a decorated bordism N goes roughly as follows (see [CRS17]):

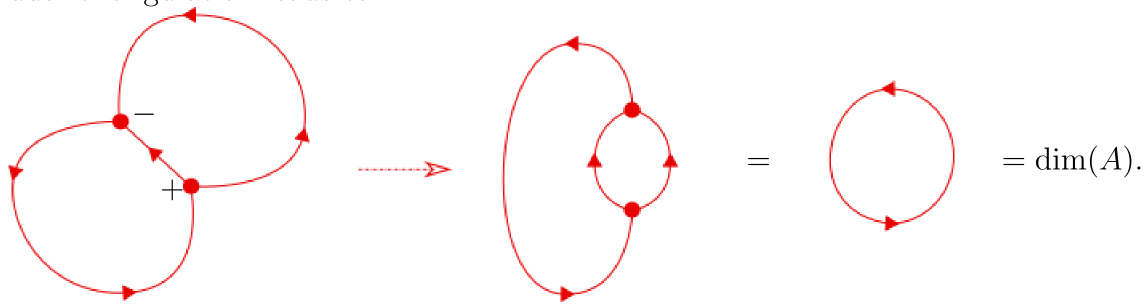
- Pick a triangulation[7] for every 2-stratum of N labeled by an algebra A. Its Poincare dual is thickened into a ribbon network (with coupons). Color each ribbon by A and each coupon by μ for a negative orientation and Δ for a positive orientation.
- Thicken every 1-stratum in N labelled by $((A_1, \epsilon_1), \dots, (A_n, \epsilon_n), M)$ into an M-colored ribbon and attach ribbons from the incident surface defects via the action maps ρ_i.
- Evaluate the RT TQFT on the resulting $\mathcal{C}$-marked bordism.
- This gives a construction, which is independent of the choice of triangulation in the interior. To get rid of the dependence on the boundary, one does a limit construction.

Example 3.1. Surface defects labeled by a Δ-separable symmetric Frobenius algebra A with the topology of a sphere $\mathbb{S}^2$ can be replaced by inserting in the theory the dimension of A. To see this, consider the triangulation



[7]By a *triangulation* we mean a Δ-complex with a total order on its vertices.

Its dual triangulation leads to



Orbifold Construction

From n-dimensional defect TQFTs and some orbifold data $\mathcal{A}$ subject to certain axioms one can construct a closed n-dimensional TQFT $\mathcal{Z}_{\mathcal{A}}$ called the associated *orbifold theory* [CRS19]. In [CRS18] such a construction is made for the previous mentioned defect RT TQFT.

A *(special) orbifold datum* $\mathcal{A}$ consists of labels $\mathcal{A}_3, \mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_0^+, \mathcal{A}_0^-$ in the corresponding defect label sets as well as point insertions ψ, ϕ for 2-strata resp. 3-strata. These data are subject to the orbifold axioms [CRS18, Sec. 2.2]. In Figure 3, the local neighborhoods of such defects are depicted.

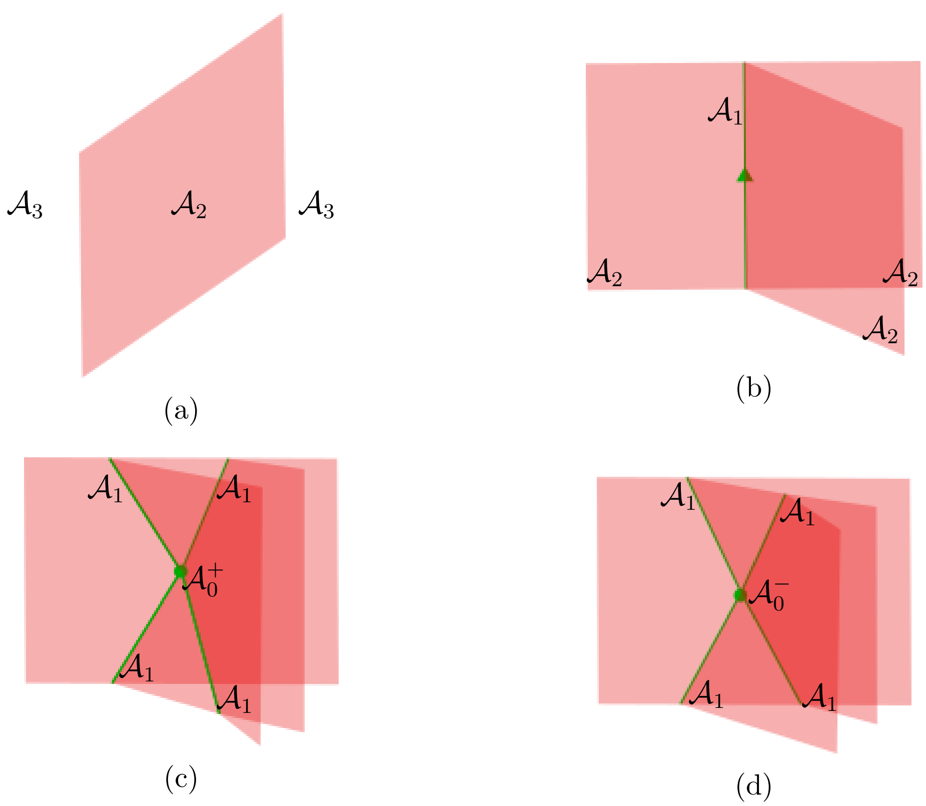


Figure 3: Local neighborhoods of orbifold defects.

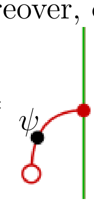
The construction of the orbifold theory $\mathcal{Z}_{\mathcal{A}}$ on bordisms goes roughly as follows:

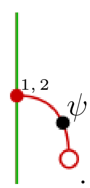
- For a bordism N pick a triangulation. Its Poincare dual gives a stratification of N.
- Label each j-stratum by $\mathcal{A}_j$ for $j > 0$ and each positively (negatively) oriented 0-stratum by $\mathcal{A}_0^+$ ($\mathcal{A}_0^-$). Each 3-stratum N_3^α comes with a $\phi^{\chi_{\mathrm{sym}}(N_3^\alpha)/2}$-insertion, while each 2-stratum N_2^α comes with a $\psi^{\chi_{\mathrm{sym}}(N_2^\alpha)}$-insertion, where χ_{sym} denotes the symmetric Euler characteristic[8] [CRS19].
- Evaluate the defect TQFT on this decorated bordism.
- Finally, one does a limit construction to get rid of the dependence of the triangulation.

An orbifold datum for a defect RT TQFT (with modular tensor category $\mathcal{C}$) consists of:

1. $\mathcal{A}_3 = \mathcal{C}$,
2. $\mathcal{A}_2 = A$ a Δ-separable symmetric Frobenius algebra in $\mathcal{C}$,
3. $\mathcal{A}_1 = T =_A T_{AA}$ an $A-(A\otimes A)$-bimodule (with left action ρ and right actions ρ_1, ρ_2),
4. $\mathcal{A}_0^+ = \alpha \in \mathrm{Hom}_{A_1,A_2A_5A_6}({}_{A_1}T_{A_2A_3} \otimes_{A_3} {}_{A_3}T_{A_5A_6}, {}_{A_1}T_{A_4A_6} \otimes_{A_4} {}_{A_4}T_{A_2A_5})$, [9]
5. $\mathcal{A}_0^- = \overline{\alpha} \in \mathrm{Hom}_{A_1,A_2A_5A_6}({}_{A_1}T_{A_4A_6} \otimes_{A_4} {}_{A_4}T_{A_2A_5}, {}_{A_1}T_{A_2A_3} \otimes_{A_3} {}_{A_3}T_{A_5A_6})$,
6. $\psi \in \mathrm{End}_{A,A}(A)^\times$,
7. $\phi \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})^\times$.

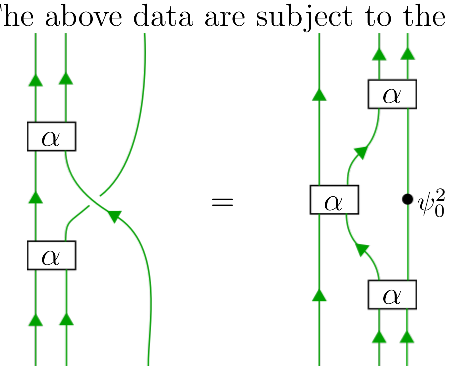
Moreover, define

$\psi_0 :=$ 

and $\psi_{1,2} :=$ 

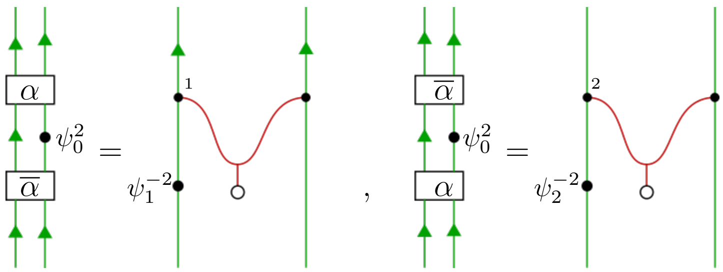
.

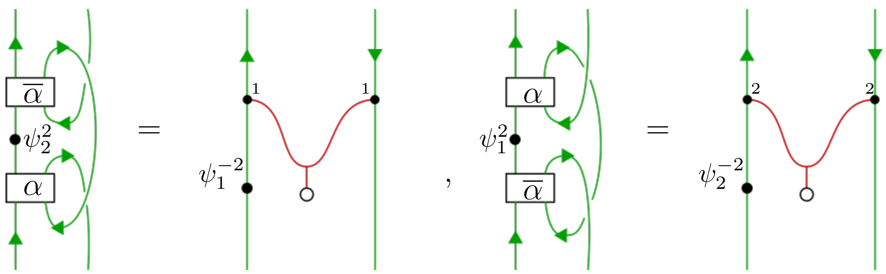
The above data are subject to the following string diagrammatic identities.

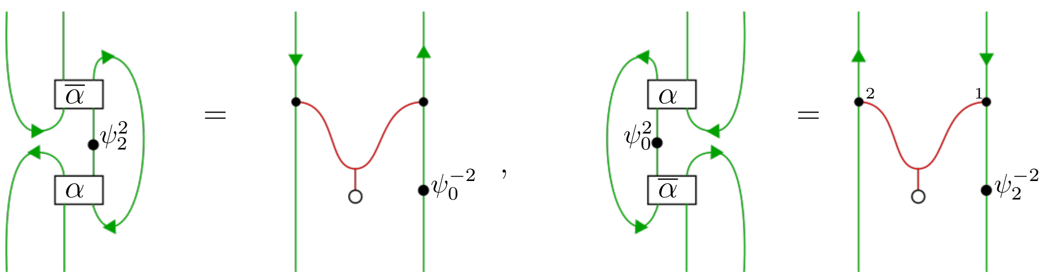


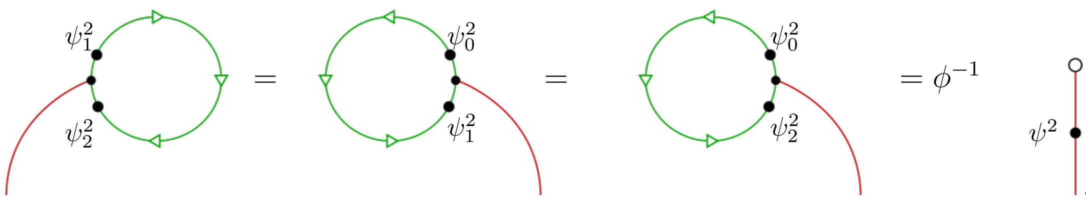
[8]The symmetric Euler characteristic of a stratum M_j^α in a manifold M is defined by $\chi_{\mathrm{sym}}(M_j^\alpha) = 2\chi(M_j^\alpha) - \chi(M_j^\alpha \cap \partial M)$.

[9]We keep the indices to keep track of the different actions of A on T.









Turaev-Viro Theory

Topological Field theories of Turaev-Viro type can be described via a certain orbifold theory. This is the result of [CRS18, Thm. 4.5].

Definition 3.4. Let $\mathcal{S}$ be the spherical fusion category with a set of representatives of simple objects I. Define

1. $A = \bigoplus_{i \in I} \mathbb{K}$
2. $T = \bigoplus_{i,j,k \in I} \mathcal{S}(i \otimes j, k)$
3. $\alpha : \lambda \otimes \mu \mapsto \sum_{d,\lambda',\mu'} d_d^{-1} F^{\lambda\lambda'}_{\mu\mu'} \lambda' \otimes \mu'$
4. $\overline{\alpha} : \lambda \otimes \mu \mapsto \sum_{d,\lambda',\mu'} d_d^{-1} (F^{\lambda\lambda'}_{\mu\mu'})^{-1} \lambda' \otimes \mu'$
5. $\psi^2 = \mathrm{diag}(d_1, \ldots, d_{|I|})$

6. $\phi = (\dim(\mathcal{S}))^{-1}$

For the details see [CRS18, Sec. 4].

Proposition 3.1 ([CRS18])**.** *The datum* $\mathcal{A}^{\mathcal{S}} := (\mathrm{Vect}, A, T, \alpha, \overline{\alpha}, \psi^2, \phi)$ *forms an orbifold datum.*

The orbifold theory assigned to $\mathcal{A}^{\mathcal{S}}$ describes the Turaev-Viro theory. This was proven in [CRS18, Theorem 4.5]

Theorem ([CRS18])**.** Let $\mathcal{S}$ be a spherical fusion category and $\mathcal{A}^{\mathcal{S}}$ the orbifold datum from Definition 3.4. Then, there is an isomorphism

$$\mathcal{Z}^{\mathrm{TV},\mathcal{S}} \cong \mathcal{Z}_{\mathcal{A}^{\mathcal{S}}}.$$

Orbifold Data from G-crossed ribbon extensions

Let $\mathcal{C}$ be a ribbon fusion category and let $\mathcal{D} = \bigoplus \mathcal{C}_g$ be some G-crossed extension. Then, there is a particularly interesting orbifold datum for $\mathcal{C}$, which is derived from this extension [CRS18, Sec. 5]. The algebra A is G-graded, i.e. $A = \bigoplus A_g$, where A_g as an object is the internal End ([Ost03]) of some simple object m_g in $\mathcal{C}_g$, i.e.

$$A_g = \underline{\mathrm{End}}(m_g) = m_g^* \otimes m_g.$$

The bimodule is $T = \bigoplus T_{g,h}$, where

$$T_{g,h} := m_{gh}^* \otimes m_g \otimes m_h$$

is an A_{gh}-$(A_g \otimes A_h)$ bimodule. The bimodule maps are $\alpha = \bigoplus_{g,h,k\in G} \alpha_{g,h,k}$ and $\overline{\alpha} = \bigoplus_{g,h,k\in G} \overline{\alpha}_{g,h,k}$, where

$$\alpha_{g,h,k} : T_{g,hk} \otimes T_{h,k} \to T_{gh,k} \otimes T_{g,h},$$
$$\overline{\alpha}_{g,h,k} : T_{g,hk} \otimes T_{h,k} \to T_{gh,k} \otimes T_{g,h}$$

and

$$\psi^2|_{A_g} = d_{m_g}^{-1} \cdot \mathrm{id}_{A_g},$$
$$\phi = |G|^{-1}.$$

The precise definition of the algebra structure, the bimodule structure as well as the bimodule maps will be given in section 3.2 for $G = \mathbb{Z}_2$. For arbitrary finite groups G see [CRS18].

Remark 3.1. In [CRS18] they conjecture that such orbifold theories, which are obtained via a G-crossed ribbon extension, are of Reshetikhin-Turaev type of the equivariantization $\mathcal{D}^G$, i.e.

$$\mathcal{Z}_{\mathcal{A}} \cong \mathcal{Z}^{\mathrm{RT},\mathcal{D}^G}.$$

This relates the construction mentioned here with *orbifoldizing* or *gauging* in other literature [CGPW16, BBCW14].

Example 3.2. Let $\mathcal{C} = \mathrm{Vect}$ be the trivial category and let G be a finite abelian group. Then, Vect_G is a G-crossed extension of Vect as discussed in Example 2.2. It is easy to check from the definition that the resulting orbifold datum $\mathcal{A}$ from this extension coincides with the orbifold datum $\mathcal{A}^{\mathcal{S}}$ from Definition 3.4 for the spherical fusion category $\mathcal{S} = \mathrm{Vect}_G$. Therefore, this is a Turaev-Viro theory

$$\mathcal{Z}_{\mathcal{A}} \cong \mathcal{Z}^{\mathrm{TV},\mathrm{Vect}_G}.$$

Recall by the results of [TV92] that the Turaev-Viro theory of a spherical fusion category $\mathcal{S}$ the RT TQFT of the Drinfeld center $Z(\mathcal{S})$. This implies that

$$\mathcal{Z}_{\mathcal{A}} \cong \mathcal{Z}^{\mathrm{RT},Z(\mathrm{Vect}_G)}. \tag{3.1}$$

This agrees with the above mentioned conjecture, since the equivariantization of Vect_G is equivalent to $\mathrm{Rep}(D(G)) \simeq Z(\mathrm{Vect}_G)$ (see Example 2.3).

3.2 $\mathbb{Z}_2$-crossed extension of $\mathcal{C} \boxtimes \mathcal{C}$ and orbifold data

3.2.1 Topological Bilayer Phase

The mathematical notions presented in Section 2.3 have relevance in physics in the so-called topological phases of matter, as described in [BBCW14]. If a topological phase carries some global symmetry described by a finite group G, it is natural to ask it is possible to gauge this into a local symmetry. Mathematically, this is provided by the theory of G-extensions [ENOM09]. As suggested in [BBCW14], simple objects in the twisted components $\mathcal{C}_g$ do not behave like intrinsic quasi-particles but rather than extrinsic quasi-particles (defects). Passing through the equivariantization of a G-crossed extension is referred to as *gauging* the symmetry, while deequivariantization is referred to as *condensing*.

A special topological phase is the so called bilayer phase. Mathematically, this is the Deligne product $\mathcal{C} \boxtimes \mathcal{C}$ of a modular tensor category $\mathcal{C}$ with itself. It has an obvious $\mathbb{Z}_2$-action, which permutes both layers. The physical realization of such systems is described in [BJQ13] and their study could bring several benefits to topological quantum computing. In [BFRS10], they gave a family of module category structures on $\mathcal{C}$ over $\mathcal{C} \boxtimes \mathcal{C}$. In [BS11], they gave explicitly the structure of a $\mathbb{Z}_2$-crossed extension on $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C} \oplus \mathcal{C}$ with all associated structure maps explicitly by applying the theory of G-*equivariant* modular functors [KP08]. This permutation action was later studied in the context of defects in [FS14].

In this section, we will combine the work of [BS11] and [CRS18] to determine the permutation orbifold data. Since the orbifold data in the latter are defined just by G-crossed graphical calculus, it is only a matter of replacing the corresponding structure maps of this extension. The structure maps are provided explicitly in the appendix B. For these string diagrammatic computations, we use the properties of fusion bases, which are given in Appendix A. For this part, let

$$D = \sum d_i^2$$

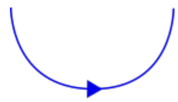
will be the dimension of $\mathcal{C}$. The category $\mathcal{C}$ is assumed to be anomaly free, i.e. $p^+ = p^-$. Furthermore, we write g for the non-trivial element in $\mathbb{Z}_2$.

3.2.2 Algebra

The algebra A_1 is the trivial Frobenius algebra $\mathbb{1}_{\mathcal{D}} = \mathbb{1} \boxtimes \mathbb{1}$. Now, let $m \equiv m_g$ be a simple object in the twisted component, i.e. a simple in $\mathcal{C}$. We determine the structure on $A_g = m^* \otimes m$ in the following way.

Unit:

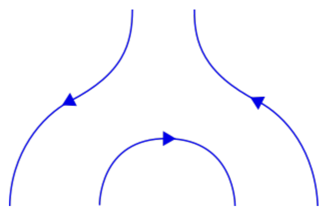
The unit $\eta_g : \mathbb{1} \boxtimes \mathbb{1} \to A_g$ is

$$\overrightarrow{coev}_m^{\mathcal{D}} =$$  (3.2)

Thus, we have $\eta_g = \overrightarrow{coev}_m \otimes_{\mathbb{K}} \mathrm{id}_{\mathbb{1}}$.

Multiplication:

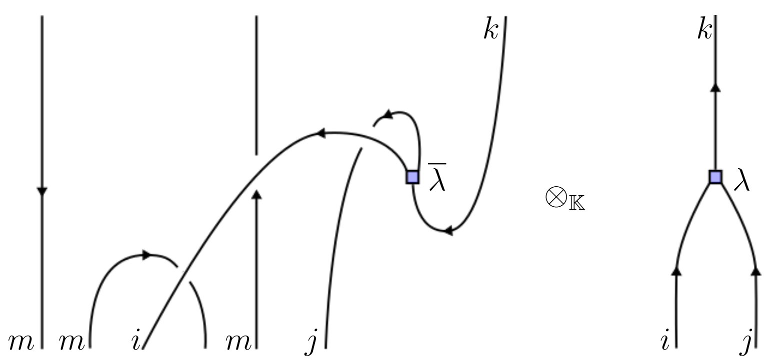
The multiplication $\mu_g : A_g \otimes A_g \to A_g$ is

$$\mu_g =$$  (3.3)

Therefore, we get

$$\mu_g = (\mathrm{id}_{m^*} \otimes \overrightarrow{ev}_m^{\mathcal{D}} \otimes \mathrm{id}_m) \circ (\mathrm{id}_{m^*} \otimes a^{-1}_{m,m^*,m}) \circ a_{m^*,m,m^* \otimes m}.$$

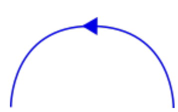
Graphically, this is

$$\mu_g = \bigoplus_{i,j,k} \sum_{\lambda=1}^{N_{ij}^k}$$ 

(3.4)

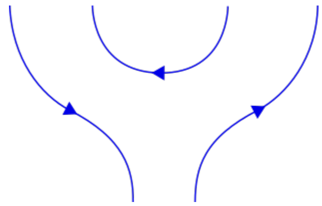
Counit:

The counit $\epsilon_g : A_g \to \mathbb{1} \boxtimes \mathbb{1}$ is defined by

$$d_m \cdot \overleftarrow{ev}_m^{\mathcal{D}} = d_m \cdot$$  (3.5)

Thus, we have $\epsilon_g = d_m D \cdot \overleftarrow{ev}_m \otimes_{\mathbb{K}} \mathrm{id}_{\mathbb{1}}$.

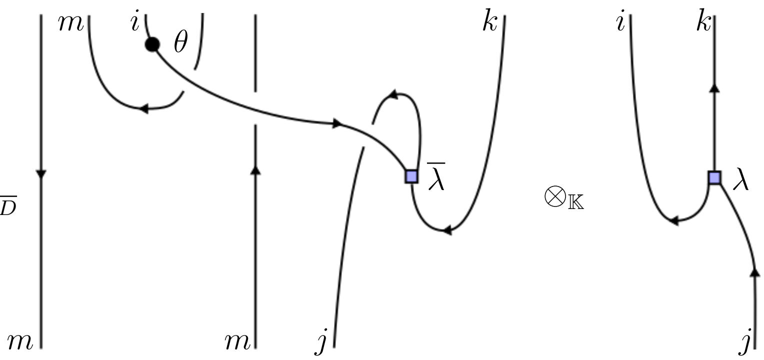
Comultiplication:
The comultiplication $\Delta_g : A_g \to A_g \otimes A_g$ is

$$\Delta_g = d_m^{-1} \cdot$$  (3.6)

i.e.

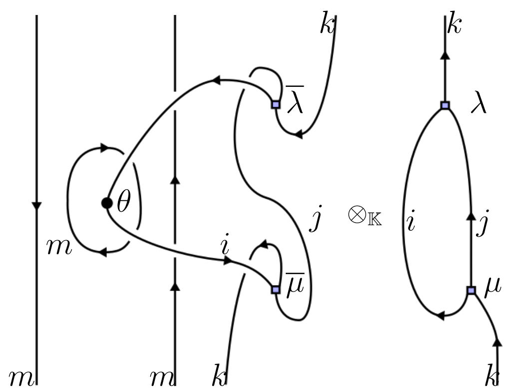
$$\Delta_g := d_m^{-1} \cdot a_{m^* \otimes m, m^*, m} \circ (a^{-1}_{m^*, m, m^*} \otimes \mathrm{id}_m) \circ (\mathrm{id}_{m^*} \otimes \overleftarrow{coev}^{\mathcal{D}}_m \otimes \mathrm{id}_m).$$

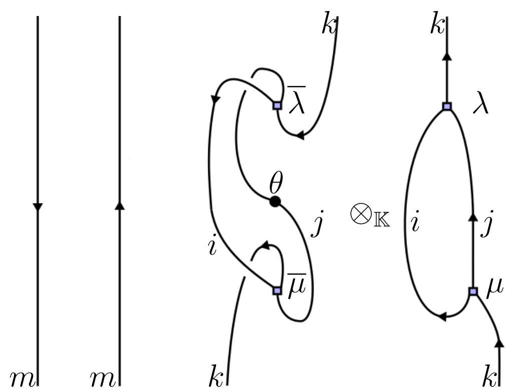
Graphically, this is

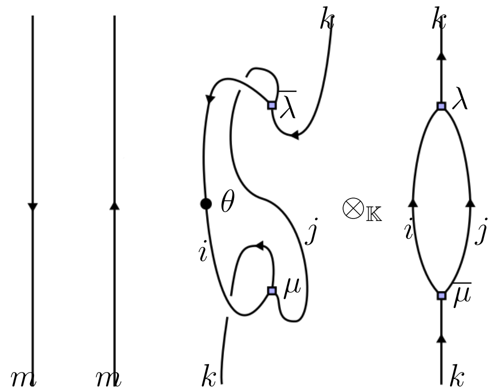
$$\Delta_g = \bigoplus_{i,j,k} \sum_{\lambda=1}^{N_{ij}^k} \frac{d_i}{d_m D}$$ 

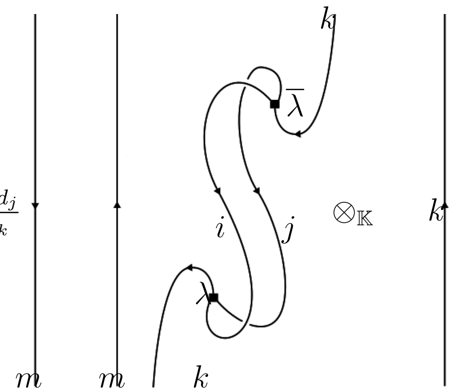
(3.7)

In fact, let us check for instance the Δ-separability of this algebra using the corresponding string diagrams in $\mathcal{C}$.

$$\mu_g \circ \Delta_g = \frac{1}{d_m D} \bigoplus_{i,j,k} \sum_{\lambda,\mu} d_i$$ 

$$= \frac{1}{D} \bigoplus_{i,j,k} \sum_{\lambda,\mu} d_i$$ 

$= \frac{1}{D} \bigoplus_{i,j,k} \sum_{\lambda,\mu} \frac{d_i d_j}{d_k}$ 

$= \frac{1}{D} \bigoplus_{i,j,k} \sum_{\lambda} \frac{d_i d_j}{d_k}$  $= \mathrm{id}_{m^* \otimes m}$

where in the last step we used that

$$\sum_{i,j} N_{ij}^{k} \frac{d_i d_j}{d_k} = \sum_{i,j} N_{i\bar{k}}^{\bar{j}} \frac{d_i d_j}{d_k} = \sum_i \frac{d_i^2 d_{k^*}}{d_k} = \sum_i d_i^2 = D.$$

We collect the above results into the algebra

$$A = A_1 \oplus A_g \tag{3.8}$$

which is a Δ-separable Frobenius algebra.

Remark 3.2. The Frobenius algebra A_g for $m = \mathbb{1}$ with multiplication (3.4) and comultiplication (3.7) is part of the family of Frobenius algebras in $\mathcal{C} \boxtimes \mathcal{C}$ determined in [BFRS10]. They also show that this Frobenius algebra is an Azumaya algebra.

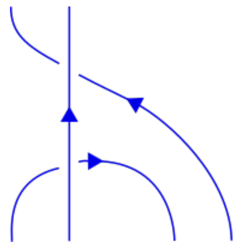
3.2.3 Bimodule

In the general case, the bimodule components were defined by $T_{g_1,g_2} = m^*_{g_1 g_2} \otimes m_{g_1} \otimes m_{g_2}$. Therefore, in our case we have four components to consider. As objects they are

- $T_{1,1} = \mathbb{1} \boxtimes \mathbb{1}$
- $T_{g,1} = m^* \otimes m$
- $T_{1,g} = m^* \otimes m$

- $T_{g,g} = m \otimes m$.

In all cases, we have trivial actions of $A_1 = \mathbb{1} \boxtimes \mathbb{1}$. The bimodules $T_{g,1}$ and $T_{1,g}$ are simply the algebra A_g as a $A_g - A_g$-bimodule. Finally, $T_{g,g}$ carries a right $(A_g \otimes A_g)$-module structure. The action $r_1 : T_{g,g} \otimes A_g \to A_g$ is defined by

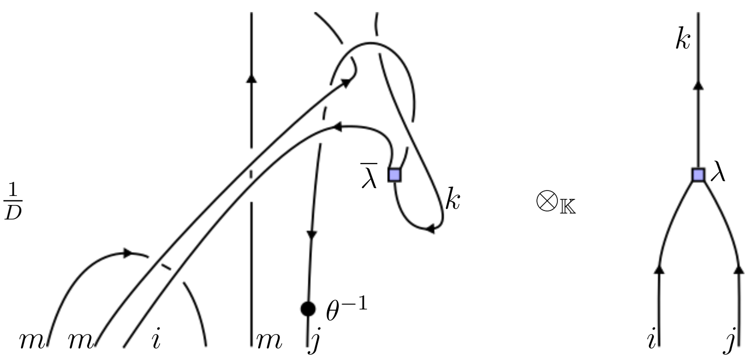
$r_1 =$ 

(3.9)

That is

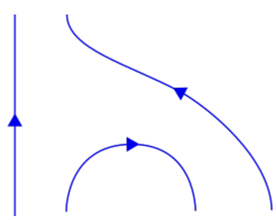
$$r_1 = c^R_{m,m} \circ (\mathrm{id}_m \otimes \overrightarrow{ev}^{\mathcal{D}}_m \otimes \mathrm{id}_m) \circ (\mathrm{id}_m \otimes a^{-1}_{m,m^*,m}) \circ a_{m,m,m^* \otimes m} \circ (\tilde{c}^R_{m,m} \otimes \mathrm{id}_{m^* \otimes m}).$$

Graphically, we have

$r_1 = \bigoplus_{i,j,k} \sum_{\lambda=1}^{N_{ij}^k} \frac{1}{D}$ 

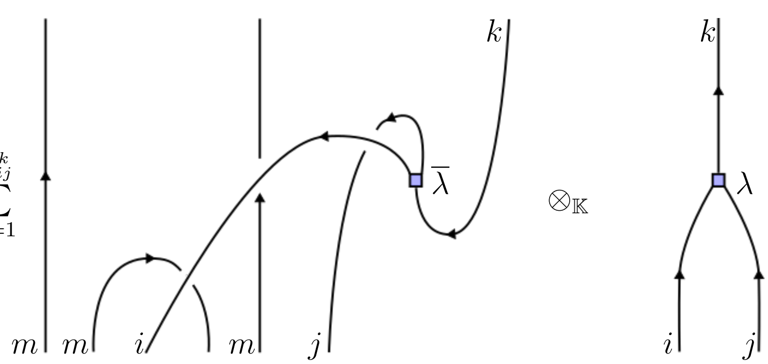
(3.10)

The action r_2 is given by

$r_2 =$ 

(3.11)

Thus, it is the same as the multiplication map up to the first strand, which has opposite orientation. Therefore, graphically

$r_2 = \bigoplus_{i,j,k} \sum_{\lambda=1}^{N_{ij}^k}$ 

(3.12)

The above results form the A-$A \otimes A$-bimodule

$$T = T_{1,1} \oplus T_{g,1} \oplus T_{1,g} \oplus T_{g,g}. \tag{3.13}$$

3.2.4 Bimodule Maps

The next step is to determine the bimodule maps

$$\alpha_{g_1,g_2,g_3} : T_{g_1,g_2g_3} \otimes T_{g_2,g_3} \to T_{g_1g_2,g_3} \otimes T_{g_1,g_2}$$

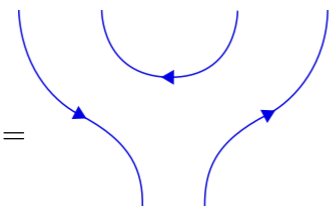
and

$$\overline{\alpha}_{g_1,g_2,g_3} : T_{g_1g_2,g_3} \otimes T_{g_1,g_2} \to T_{g_1,g_2g_3} \otimes T_{g_2,g_3}$$

for all $g_1, g_2, g_3 \in \mathbb{Z}_2$, i.e. eight components for each.

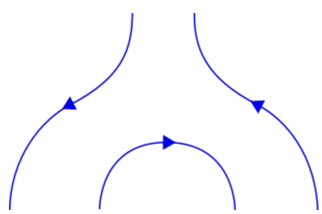
The map $\alpha_{1,1,1}$ is just the identity of $\mathbb{1} \boxtimes \mathbb{1}$. Next, we consider when there is exactly one non-trivial element. The map $\alpha_{g,1,1} : T_{g,1} \otimes T_{1,1} \to T_{g,1} \otimes T_{g,1}$ is given by

$\alpha_{g,1,1} =$



This diagram is the same as (3.6) up to the dimension factor. This implies $\alpha_{g,1,1} = d_m \cdot \Delta_g$, which we computed in (3.7). The bimodule map $\alpha_{1,1,g} : T_{1,g} \otimes T_{1,g} \to T_{1,g} \otimes T_{1,1}$ is given by

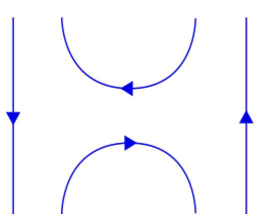
$\alpha_{1,1,g} =$



which is the multiplication from (3.3), i.e. $\alpha_{1,1,g} = \mu_g$ and the corresponding diagram in $\mathcal{C} \boxtimes \mathcal{C}$ in (3.4).

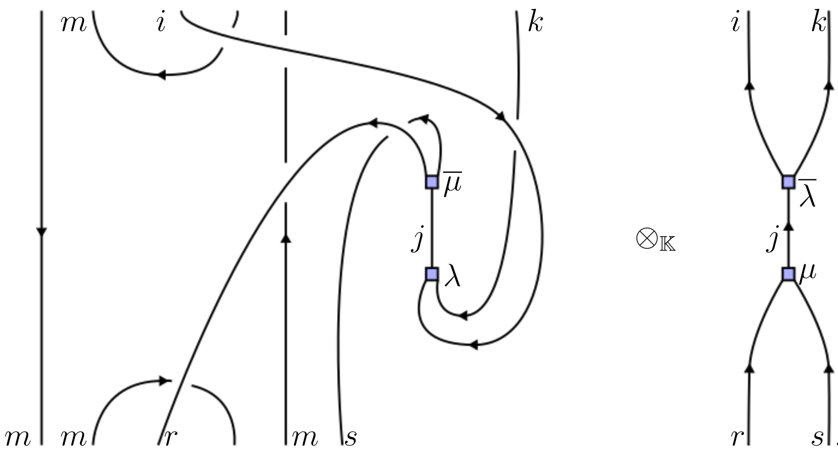
The bimodule map $\alpha_{1,g,1} : T_{1,g} \otimes T_{g,1} \to T_{g,1} \otimes T_{1,g}$ is defined by

$\alpha_{1,g,1} =$



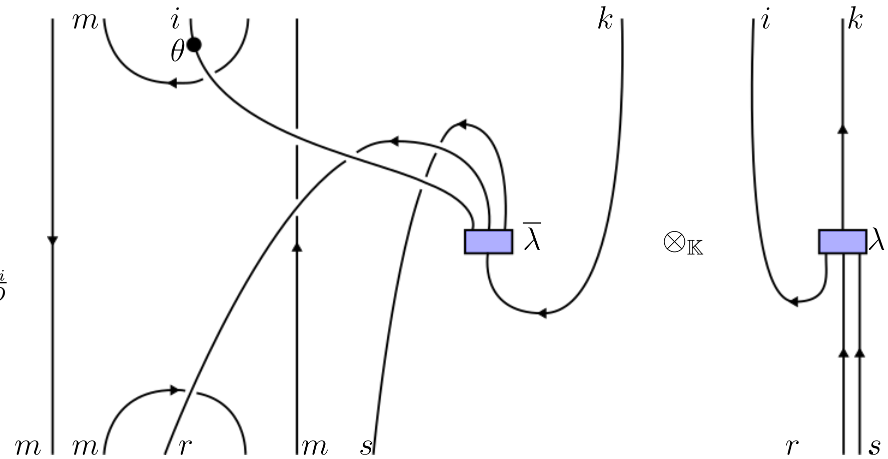
One can easily see, that $\alpha_{1,g,1} = \alpha_{g,1,1} \circ \alpha_{1,1,g} = d_m \Delta_g \circ \mu_g$. This results to

$$\alpha_{1,g,1} = \bigoplus_{i,k,r,s} \sum_{j} \sum_{\lambda,\mu} \frac{d_i d_k}{d_m D d_j}$$



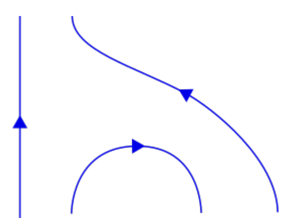
Using the associativity property of fusion (equation (A.4) in appendix A), this transforms to

$$\alpha_{1,g,1} = \bigoplus_{i,k,r,s} \sum_{\lambda} \frac{d_i}{D}$$



Moving on, we compute the components with two non-trivial group elements. The bimodule map $\alpha_{g,g,1} : T_{g,g} \otimes T_{g,1} \to T_{1,1} \otimes T_{g,g}$ is given by

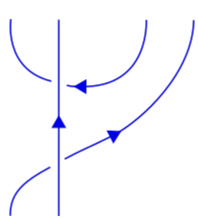
$$\alpha_{g,g,1} = \tag{3.14}$$



coincides with the definition of the action $\alpha_{g,g,1} = r_2$ (see equation (3.12)).

The map $\alpha_{1,g,g} : T_{1,1} \otimes T_{g,g} \to T_{g,g} \otimes T_{1,g}$ is given by

$$\alpha_{1,g,g} = \tag{3.15}$$

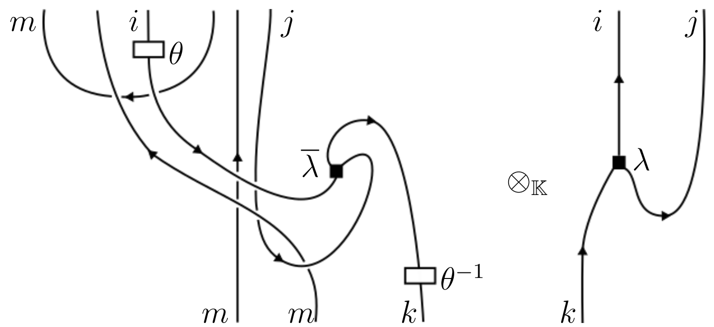


which translates to

$$\alpha_{1,g,g} = a^{-1}_{m,m,m^*\otimes m} \circ (\mathrm{id}_m \tilde{c}^R_{m^*\otimes m,m}) \circ (\mathrm{id}_m \otimes a^{-1}_{m^*,m,m}) \circ a_{m,m^*,m\otimes m} \circ (\overleftarrow{coev}_m \otimes \mathrm{id}_{m\otimes m}).$$

Therefore, we get

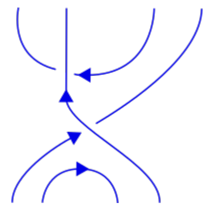
$$\alpha_{1,g,g} = \bigoplus_{i,j,k} \sum_{\lambda} \frac{d_j}{D}$$



(3.16)

The bimodule map $\alpha_{g,1,g} : T_{g,g} \otimes T_{1,g} \to T_{g,g} \otimes T_{g,1}$ is defined by

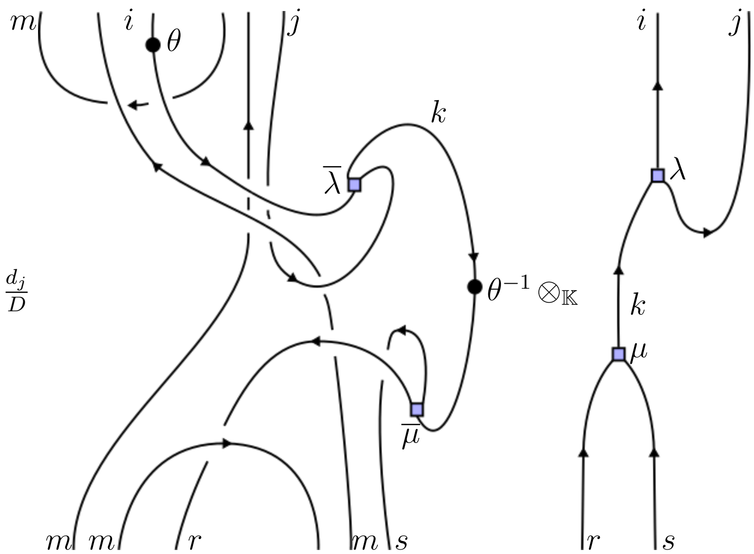
$$\alpha_{g,1,g} =$$



(3.17)

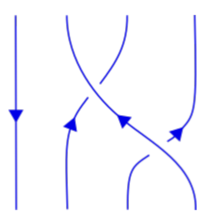
which is the concatenation of diagrams (3.16) and (3.14), i.e. $\alpha_{g,1,g} = \alpha_{1,g,g} \circ \alpha_{g,g,1}$. Graphically, this is

$$\alpha_{g,1,g} = \bigoplus_{i,j,r,s} \sum_{k} \sum_{\lambda,\mu} \frac{d_j}{D}$$



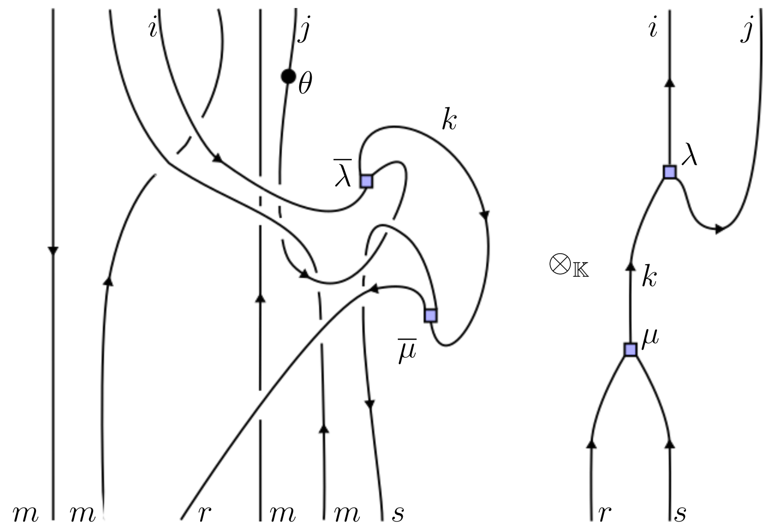
Finally, the map $\alpha_{g,g,g} : T_{g,1} \otimes T_{g,g} \to T_{1,g} \otimes T_{g,g}$ is

$$\alpha_{g,g,g} =$$



The result is

$$\alpha_{g,g,g} = \bigoplus_{i,j,r,s} \sum_{k} \sum_{\lambda,\mu} \frac{d_i d_j}{D d_k}$$



The above components collect to the bimodule map

$$\alpha : T \otimes T \to T \otimes T \tag{3.18}$$

The bimodule maps $\overline{\alpha}_{g_1,g_2,g_3}$ are computed similarly. Namely,

$$\overline{\alpha}_{1,1,1} = \mathrm{id}_{\mathbb{1}} \tag{3.19}$$

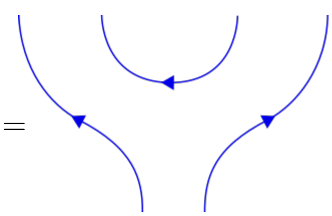
$$\overline{\alpha}_{1,1,g} = \alpha_{g,1,1} \tag{3.20}$$

$$\overline{\alpha}_{g,1,1} = \alpha_{1,1,g} \tag{3.21}$$

$$\overline{\alpha}_{1,g,1} = \alpha_{1,g,1} \tag{3.22}$$

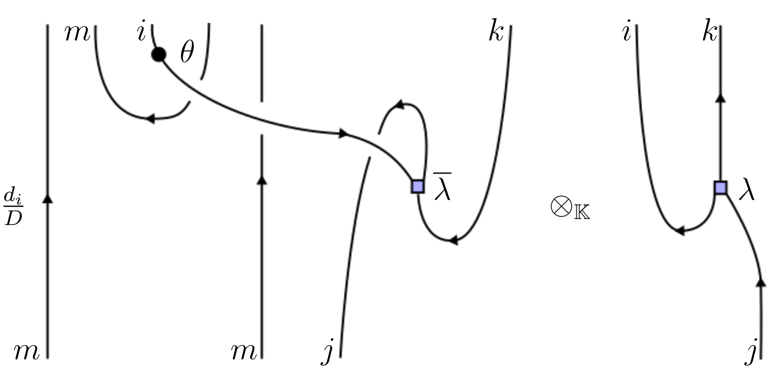
The bimodule map $\overline{\alpha}_{g,g,1} : T_{1,1} \otimes T_{g,g} \to T_{g,g} \to T_{g,1}$ is

$$\overline{\alpha}_{g,g,1} =$$



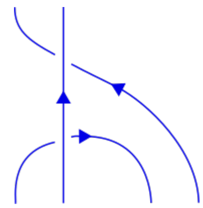
The result is

$$\overline{\alpha}_{g,g,1} = \bigoplus_{i,j,k} \sum_{\lambda=1}^{N_{ij}^{k}} \frac{d_i}{D}$$



The bimodule map $\overline{\alpha}_{1,g,g} : T_{g,g} \otimes T_{1,g} \to T_{1,1} \otimes T_{1,1} \to T_{g,g}$ is defined by

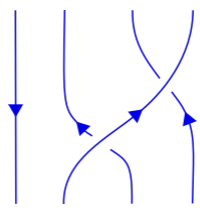
$$\overline{\alpha}_{1,g,g} =$$



This coincides with the diagram (3.9) which implies $\overline{\alpha}_{1,g,g} = r_1$.

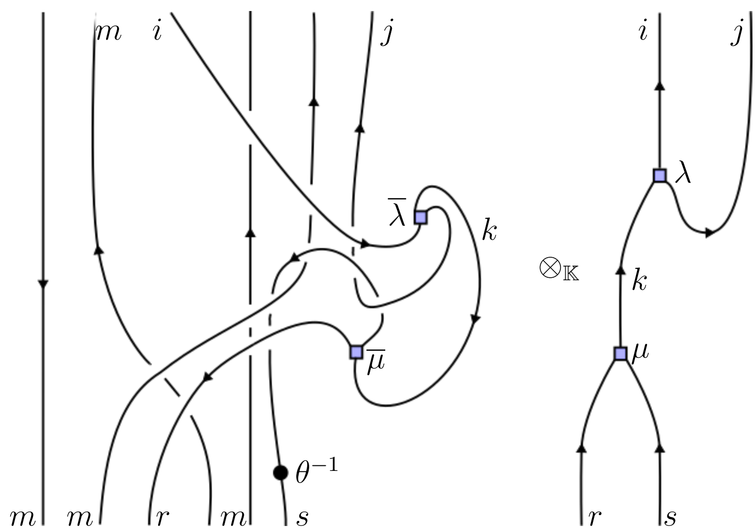
Finally, the bimodule map $\overline{\alpha}_{g,g,g} : T_{1,g} \otimes T_{g,g} \to T_{g,1} \otimes T_{g,g}$ is given by

$$\overline{\alpha}_{g,g,g} =$$



which implies

$$\overline{\alpha}_{g,g,g} = \bigoplus_{i,j,r,s} \sum_{k} \sum_{\lambda,\mu} \frac{d_j}{D}$$



The above data form the bimodule map

$$\overline{\alpha} : T \otimes T \to T \otimes T \tag{3.23}$$

The point insertions are

$$\psi = \mathrm{id}_{A_1} \oplus d_m^{-1/2} \mathrm{id}_{A_g} \tag{3.24}$$

and

$$\phi = \frac{1}{|\mathbb{Z}_2|} = \frac{1}{2}. \tag{3.25}$$

This concludes all the elements of the permutation orbifold datum, which is stated in the following theorem.

Theorem 1. *The datum* $\mathcal{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ *from* (3.8), (3.13), (3.18), (3.23), (3.24) *and* (3.25) *forms an orbifold datum for the bilayer* $\mathcal{C} \boxtimes \mathcal{C}$.

Example 3.3. Consider the trivial category $\mathcal{C} = \mathrm{Vect}$. The double copy $\mathrm{Vect} \boxtimes \mathrm{Vect}$ is equivalent to just Vect. Choosing the half-twist isomorphism $\sigma = \mathrm{id}$, the $\mathbb{Z}_2$-extension of Vect coincides with the category $\mathrm{Vect}_{\mathbb{Z}_2}$, see Example 2.1.

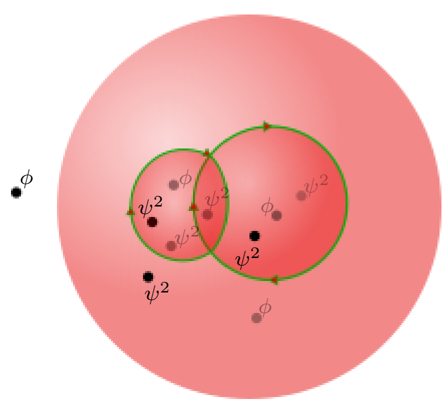
From Example 3.2, we know that the associated orbifold theory is isomorphic to the Turaev-Viro theory of $\mathrm{Vect}_{\mathbb{Z}_2}$ or equivalently the RT TQFT of $\mathrm{Rep}(D(\mathbb{Z}_2))$. In particular, this is an instance of the Kitaev's Toric Model [KK12].

3.3 Computations of Invariants of the Orbifold TQFT

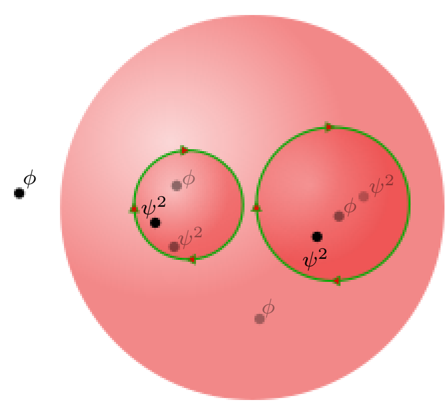
In this section, we will compute some invariants associated to the permutation orbifold theory. Since the construction of the orbifold theory involves oriented triangulations and their Poincare duals, it quickly becomes complicated and tedious to work with, even for cases like the state space of $\mathbb{S}^2$ or invariants of $\mathbb{S}^1 \times \mathbb{S}^2$ etc. Therefore, we will make the assumption, that every stratification with local neighborhoods as in Figure 3 and with contractible[10] strata in all dimensions is the result of a Poincare stratification by the use of the orbifold axioms. We refer to such stratifications as *orbifold stratifications*. Every surface defect appearing in this section, will inherit the paperplane orientation.

The 3-sphere

Consider the stratification of $\mathbb{S}^3$ in Figure 4a. One can think of the $\mathbb{S}^2$ as a surface defect, together with a bubble sitting on its surface and a smaller bubble on the side. In total, there are four separate regions, the interiors of the sphere and the bubbles as well as the exterior part. The surfaces give rise to six different strata and there are four line strata, all between the two 0-strata. Every stratum is contractible and in particular, the 3-strata and the 2-strata come with ϕ resp. ψ^2-insertions. Using the axioms of orbifold data, one can separate the bubbles in expense of two ψ^2-insertions and obtain the stratification in Figure 4b.



(a) Orbifold stratification of $\mathbb{S}^3$ with an $\mathbb{S}^2$-defect with two stacked bubbles.



(b) Stratification of $\mathbb{S}^3$ with an $\mathbb{S}^2$-defect and two separated bubbles.

Figure 4

[10] A j-stratum M_j^α in a bordism M is called *contractible*, if it is a contractible j-manifold and $\partial M \cap M_j^\alpha$ is contractible as a $(j-1)$-manifold.

Moreover, the stratification of Figure 4b can be simplified via the bubble move of orbifolds. The result is illustrated in Figure 5, where there is only the surface defect of $\mathbb{S}^2$ together with a ψ^4-insertion and two ϕ's.

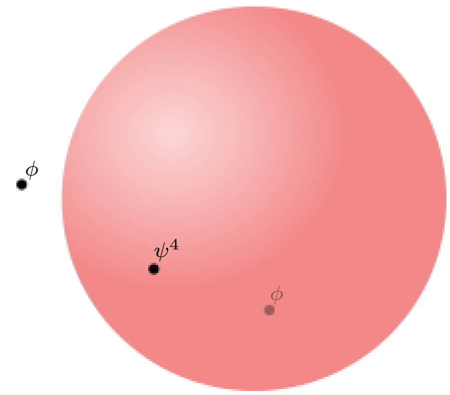


Figure 5: Stratification of $\mathbb{S}^3$ via an $\mathbb{S}^2$-defect.

In Example 3.1, we have seen that RT TQFT with defects replaces a sphere stratum by the dimension of the Frobenius algebra label. In the same way, a sphere stratum with point insertion ψ^4 gives a factor of $tr_A(\psi^4)$. Hence, the orbifold invariant is

$$\mathcal{Z}_{\mathcal{A}}(\mathbb{S}^3) = \phi^2 tr_A(\psi^4)\mathcal{Z}^{\mathrm{RT}}(\mathbb{S}^3) = \frac{\phi^2}{D} tr_A(\psi^4).$$

In the case, where the orbifold data is coming from a G-crossed extension, i.e.

$$\phi = \frac{1}{|G|}$$

and

$$\psi = \bigoplus_g d_m^{-1/2} \mathrm{id}_{A_g},$$

the orbifold theory on the sphere is

$$\mathcal{Z}_{\mathcal{A}}(\mathbb{S}^3) = \frac{1}{D_{\mathcal{C}}|G|^2} \sum_g \frac{1}{d_m^2} \dim(A_g) = \frac{1}{D_{\mathcal{C}}|G|^2} \sum_g 1 = \frac{1}{D_{\mathcal{C}}|G|}. \tag{3.26}$$

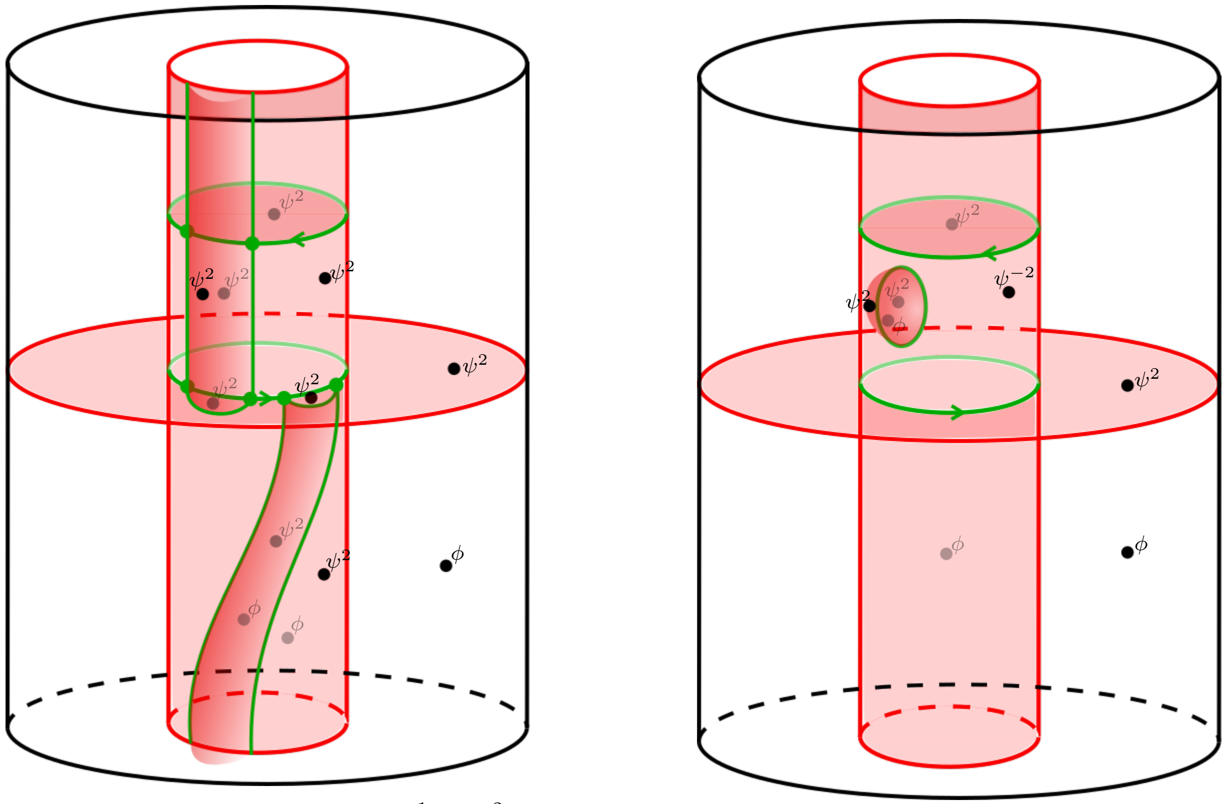
On the other hand, the gauge theory assigns

$$\mathcal{Z}^{\mathrm{RT},\mathcal{D}^G}(\mathbb{S}^3) = \frac{1}{D_{\mathcal{D}^G}} = \frac{1}{D_{\mathcal{C}}|G|},$$

where the last equality follows from equation (2.3), which agrees with the result of the orbifold theory as expected by Remark 3.1.

The space $\mathbb{S}^1 \times \mathbb{S}^2$

To compute the orbifold theory on $\mathbb{S}^1 \times \mathbb{S}^2$, we consider the stratification in Figure 6a. One can think of this stratifcation as the result of a torus defect with the meridian disks of



(a) Orbifold stratification of $\mathbb{S}^1 \times \mathbb{S}^2$ via a torus defect, two meridinan disks and a half-cylinder wrapping aroung the torus.

(b) Stratification of $\mathbb{S}^1 \times \mathbb{S}^2$ via a torus, two meridian disks and a bubble on the torus.

Figure 6

itself and its complement torus. Moreover, there is a *half-cylinder* that wraps around the torus as depicted in the figure. Obviously, this stratification is sufficient, but too complicated as it consists of many strata, including six 0-strata. However, we are able to use the orbifold axioms once again to obtain a simpler stratification. Using the axioms we can retract the outer half-cylinder into a bubble as depicted in Figure 6b. As before, the bubble move gives the stratification shown in Figure 7.

There are two 3-strata, which are open balls and therefore get a ϕ factor each. There are four 2-strata Σ_1, Σ_2, C_0 and C_2. The surface defects σ_1 and Σ_2 are disks and therefore carry ψ^2-insertions, while C_0 and C_2 are cylinders, hence there come without point insertions. The line defects separating these surfaces are the two green lines.

The next step is to evaluate this decorated manifold using the RT TQFT with defects as constructed in [CRS17]. For this, we need to obtain a ribbon network by stratifying each surface by its dual triangulation and then attaching each algebra ribbon to the corresponding module ribbons. If the surface is a disk carrying a ψ^2-insertion, then it is easy to see that by construction, this will only result to a ψ^2-insertion on the adjacent module via the corresponding action. This can be seen by the following identities, which make use of the

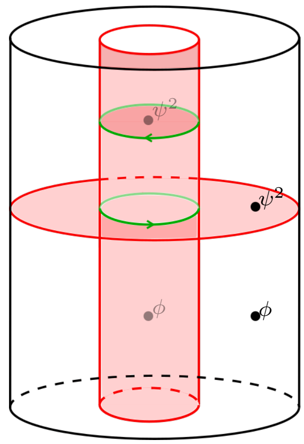
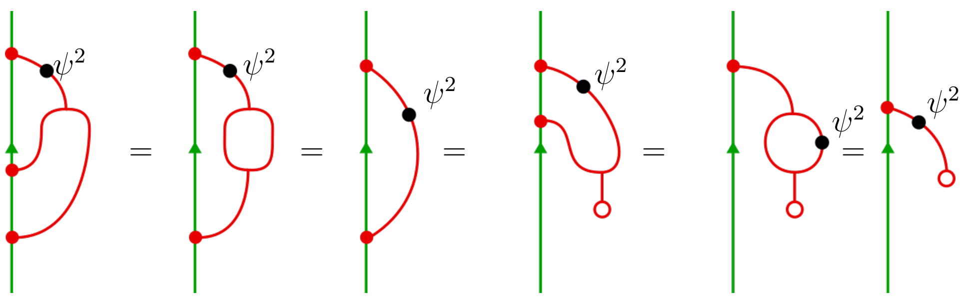


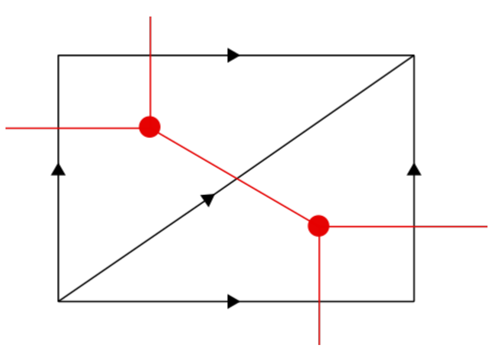
Figure 7: Stratification of $\mathbb{S}^1 \times \mathbb{S}^2$ via a torus surface and two meridian disks.

algebra properties and the fact that ψ is an A-A-bimodule map.

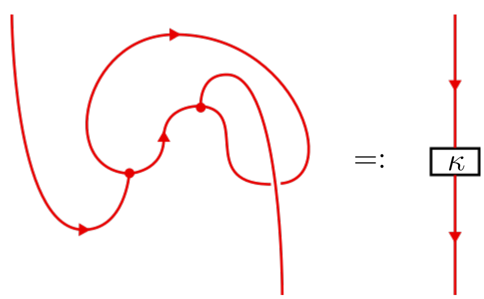


Hence, the disk defects Σ_1 and Σ_2 each insert ψ_1^2 resp. ψ_2^2 on the neighbouring line defects.

Consider now the cylinder defects. Choose the triangulation and its dual of a cylinder as shown in Figure 8a, where the left and right side get identified. The resulting ribbon graph gives rise to a morphism $\kappa : A^* \to A^*$. The two free ends of this ribbon graph will then attach via the action maps onto the adjacent line defects.

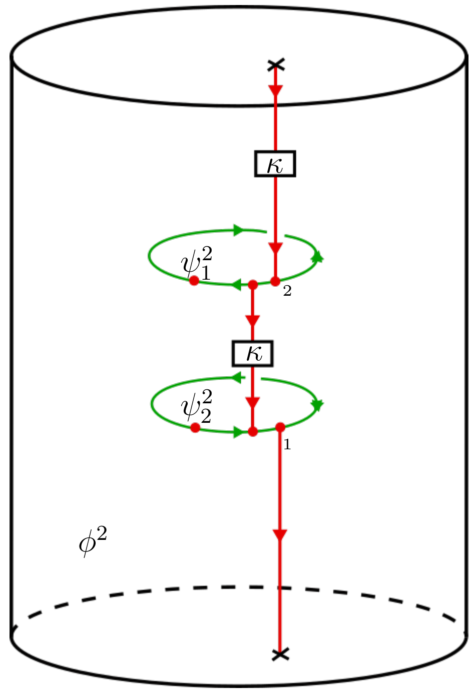


(a) Stratification



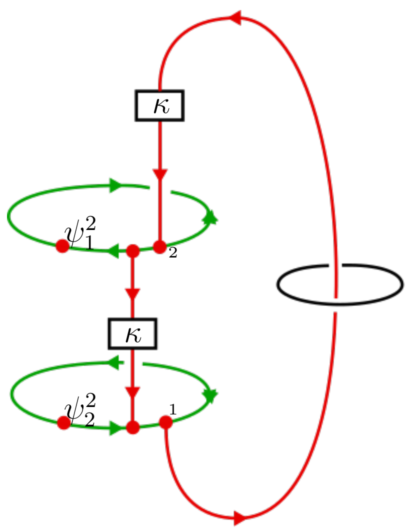
(b) Ribbon network

Hence, the resulting ribbon graph embedded in $\mathbb{S}^1 \times \mathbb{S}^2$ is

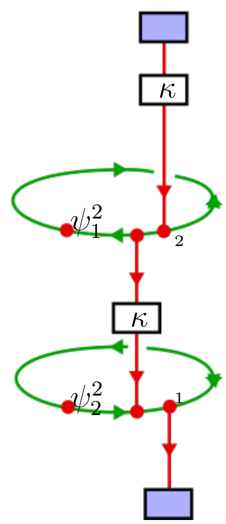


It only remains to evaluate the ordinary RT TQFT on this bordism. The space $\mathbb{S}^1 \times \mathbb{S}^2$ is obtained via surgery of $\mathbb{S}^3$ along the unknot (with no framing) (see [PS10] on surgery). Therefore, by the construction of the Reshetikhin-Turaev invariants we get the formula

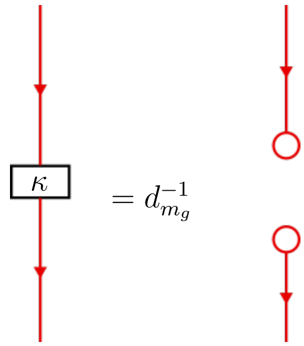
$\mathcal{Z}_{\mathcal{A}}(\mathbb{S}^1 \times \mathbb{S}^2) = D_{\mathcal{C}}^{-2}\phi^2$



$= \phi^2 \sum_\lambda$

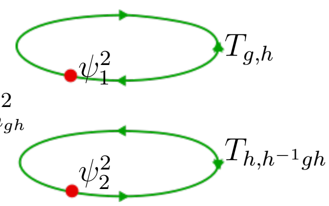


Let us now restrict to the G-graded case. One can show for the morphism κ restricted to the algebra A_g that



Then, the above formula becomes

$$\mathcal{Z}_{\mathcal{A}}(\mathbb{S}^1 \times \mathbb{S}^2) = |G|^{-2} \sum_{g,h \in G} d_{m_h}^{-2} d_{m_{gh}}^{-2}$$



$$\begin{aligned}
&= |G|^{-2} \sum_{g,h} d_{m_h}^{-2} d_{m_{gh}}^{-2} d_{m_g}^{-1} d_{m_{h^{-1}gh}}^{-1} \dim(T_{g,h}) \dim(T_{h,h^{-1}gh}) \\
&= |G|^{-2} \sum_{g,h} 1 = 1
\end{aligned}$$

since the dimension of $T_{g,h}$ is equal to $d_{m_{gh}} d_{m_g} d_{m_h}$. This result comes to no surprise as this invariant counts the dimension of the state space on $\mathbb{S}^2$.

Lens Space

We now consider the lens space $L(-2,1)$ and pick the stratification obtained via its Heegaard diagram [PS10] as shown in Figure 9. There are two regions corresponding to the interior of the solid tori. There are four 2-strata, which are all just disks. These are the two meridian disks as well as the two surface regions on the torus. The four line defects are the line segments on the meridians, which are separated by the two 0-strata placed in the intersection points of both meridians. One can check, that this is a valid stratification for our orbifold theory.

The next step is to replace each surface defect by ribbons colored by the algebra A and attach them to the ribbons of the bimodule T. The 0-strata will then be replaced by the bimodule maps $\overline{\alpha}$ (in both cases) because of their orientation. Since the surfaces are all disks, they only contribute via ψ^2-insertions on one adjacent line. The resulting ribbon graph embedded around the core of the inner torus is given in Figure 10, where we define

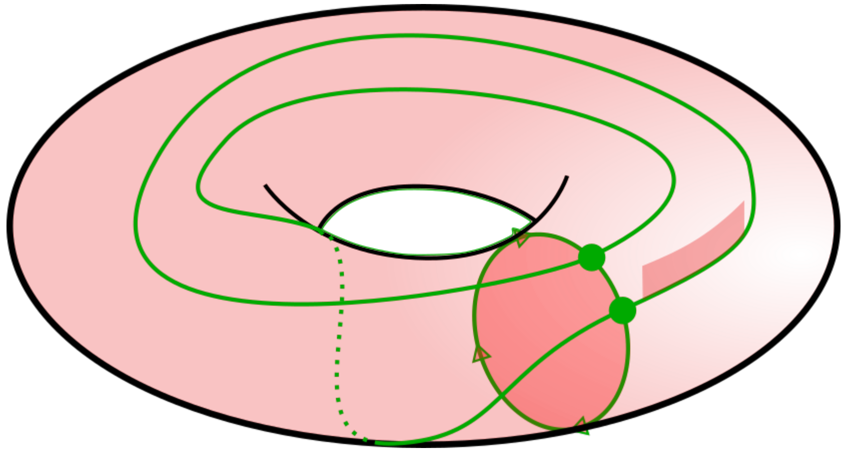
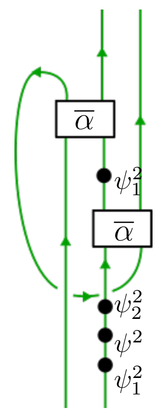


Figure 9: Heegaard diagram of a lens space

the morphism $\Lambda : T \otimes T \to T \otimes T$ by

$\Lambda :=$



(3.27)

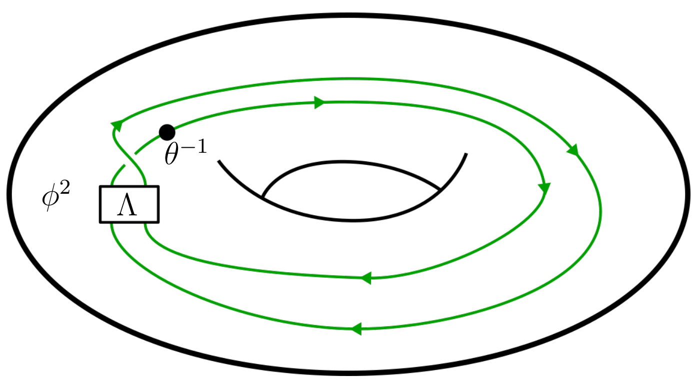
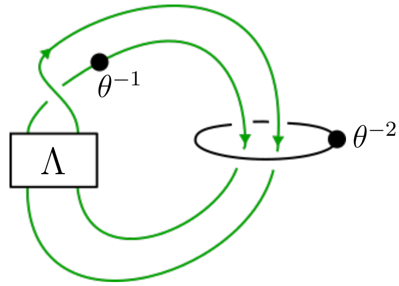


Figure 10: Lens Space with embedded graph.

Recall that a Lens space $L(n,1)$ for an integer n is obtained via surgery on $\mathbb{S}^3$ along the unknot with an n-framing, i.e. $L(n,1) = M_{\mathrm{O}^n}$ [BK01]. Hence, by definition of the Reshetikhin-Turaev invariants, we get

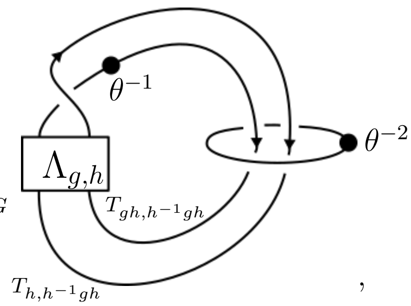
$$\mathcal{Z}_{\mathcal{A}}(L(-2,1)) = D_{\mathcal{C}}^{-2}\phi^2 \cdot$$



(3.28)

Consider now the G-graded case. Then, equation (3.28) becomes

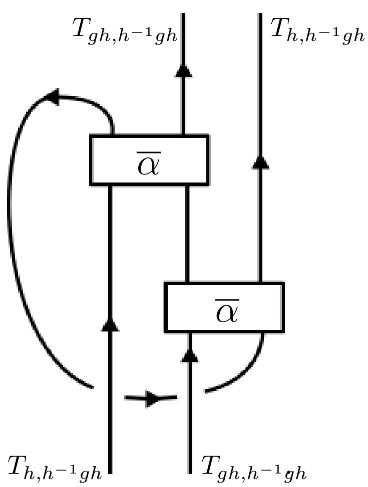
$$\mathcal{Z}_{\mathcal{A}}(L(-2,1)) = D_{\mathcal{C}}^{-2}|G|^{-2} \sum_{\substack{g,h\in G\\ g^2=1}}$$



, (3.29)

where $\Lambda_{g,h} : T_{h,h^{-1}gh} \otimes T_{gh,h^{-1}gh} \to T_{gh,h^{-1}gh} \otimes T_{h,h^{-1}gh}$ are the components of the morphism Λ, i.e.

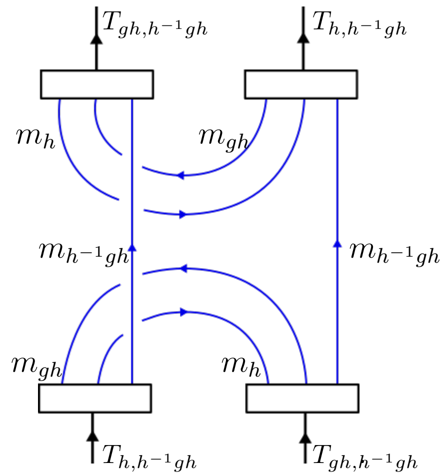
$$\Lambda_{g,h} = d_{m_g}^{-1} d_{m_h}^{-1} d_{m_{gh}}^{-1} d_{h^{-1}gh}^{-1} \cdot$$



(3.30)

After a short calculation, by inserting the definition of $\overline{\alpha}_{g,h,k}$, one gets

$$\Lambda_{g,h} = d_{m_h}^{-1} d_{m_{gh}}^{-1} d_{h^{-1}gh}^{-1} \cdot$$

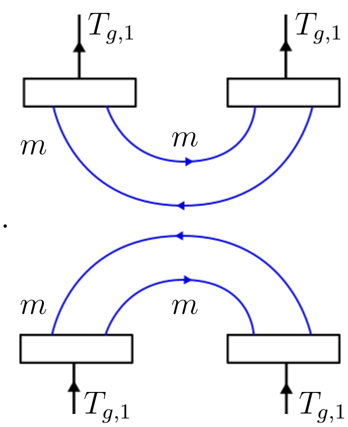


(3.31)

where the empty coupons are the identities and blue strands indicate the use of G-crossed graphical calculus.

For the group $G = \mathbb{Z}_2$, there are four summands in equation (3.29). One can easily see that the summand with $\Lambda_{1,1}$ gives just $\sum_i \theta_i^{-2} d_i^2$, which is the contribution of the uncolored link with framing -2. The map $\Lambda_{1,g} : T_{g,1} \otimes T_{g,1} \to T_{g,1} \otimes T_{g,1}$ is given by

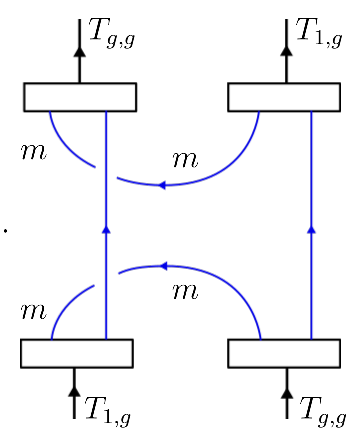
$$\Lambda_{1,g} = d_m^{-2} \cdot$$



(3.32)

It easily follows that the corresponding summand gives $\sum_i \theta_i^{-2} d_i^2$. Furthermore,

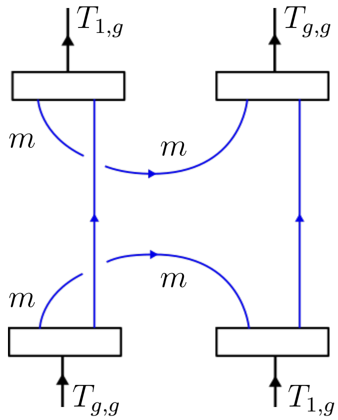
$$\Lambda_{g,1} = d_m^{-2} \cdot$$



(3.33)

and

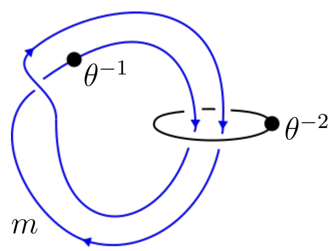
$$\Lambda_{g,g} = d_m^{-2} \cdot$$



(3.34)

After a short calculation, in both cases, inserting the above into the summands of equation (3.29), we find that the part remaining is

$$d_m^{-1} \cdot$$



(3.35)

Example 3.4. Let $\mathcal{C} = \mathrm{Vect}$ with the $\mathbb{Z}_2$-extension $\mathrm{Vect}_{\mathbb{Z}_2}$. From (3.29), we find

$$\mathcal{Z}_{\mathcal{A}}(L(-2,1)) = \frac{1}{4} \sum_{\substack{g,h \in \mathbb{Z}_2 \\ g^2=1}} 1 = 1.$$

Remark 3.3. To find an orbifold stratificaton for the Lens space $L(-2,1)$, we used its Heegaard diagram. One can do this for any lens space $L(p,q)$, where the number of strata depends on each space. For instance, the Heegaard diagram of $\mathbb{S}^3$ will contain exactly one 0-stratum.

4 A family of G-crossed categories and orbifold data

4.1 Parametrizing G-crossed categories

In this section, we provide modifications of the structure maps of G-crossed categories, which lead to a family of distinct G-crossed categories. This is inspired from passing from Vect_G to $\mathrm{Vect}_G^{\omega,\sigma}$ for an abelian group G with an abelian cocycle (ω, σ) [EGNO15].

Parametrized Associators

Let $\mathcal{D}$ be a (right) G-crossed category as described above. Then, we can modify the associators by defining for a 3-cochain $\omega : G \times G \times G \to \mathbb{K}^\times$ the associators

$$a^{\omega}_{X,Y,Z} = \omega(g_1, g_2, g_3)a_{X,Y,Z} \tag{4.1}$$

where g_1, g_2, g_3 are the degrees of the objects X, Y, Z. These are still natural isomorphisms as the category is G-graded. To ensure that the pentagon axioms are still satisfied, we impose the equation

$$\omega(g_1, g_2, g_3g_4)\omega(g_1g_2, g_3, g_4) = \omega(g_2, g_3, g_4)\omega(g_1, g_2g_3, g_4)\omega(g_1, g_2, g_3). \tag{4.2a}$$

Thus, ω is a 3-cocycle, i.e. $\omega \in Z^3(G)$. Moreover, the triangle axioms imply

$$\omega(g_1, 1, g_2) = 1, \tag{4.2b}$$

which by the cocycle equation (4.2a) implies

$$\omega(g_1, 1, g_2) = \omega(g_1, g_2, 1) = \omega(1, g_1, g_2) = 1,$$

i.e. ω is a normalised cocycle. So far, this modification ensures that $\mathcal{D}^\omega$ is a G-graded monoidal category. The compatibility condition with the action is satisfied

$$\omega(g^{-1}g_1g, g^{-1}g_2g, g^{-1}g_3g) = \omega(g_1, g_2, g_3), \tag{4.2c}$$

i.e. invariant under conjugation.

Modified Braidings Let $\mathcal{D}$ be now a G-crossed braided category with braidings $c_{X,Y}$. Let $\sigma : G \times G \to \mathbb{K}^\times$ be a 2-cochain. As before, we now modify the braiding by defining

$$c^{\sigma}_{X,Y} := \sigma(g_1, g_2)c_{X,Y}, \tag{4.3}$$

where g_1, g_2 are the degrees of X and Y. The unitality of the braiding is satisfied, if

$$\sigma(g, 1) = \sigma(1, g) = 1. \tag{4.4a}$$

The hexagon axioms are satisfied if

$$\begin{aligned}(g_1g_2, g_3)\omega(g_1, g_2, g_3)^{-1} =& \omega(g_3, g_3^{-1}g_1g_3, g_3^{-1}g_2g_3)\sigma(g_1, g_3)\\ & \omega(g_1, g_3, g_3^{-1}g_2g_3)^{-1}\sigma(g_2, g_3)\end{aligned} \tag{4.4b}$$

$$\begin{aligned}\sigma(g_1, g_2g_3)\omega(g_1, g_2, g_3) =& \omega(g_2, g_3, (g_2g_3)^{-1}g_1(g_2g_3))^{-1}\sigma(g_2^{-1}g_1g_2, g_3)\\ & \omega(g_2, g_2^{-1}g_1g_2, g_3)\sigma(g_1, g_2)\end{aligned} \tag{4.4c}$$

The action coherence condition is satisfied if

$$\sigma(g^{-1}g_1g, g^{-1}g_2g) = \sigma(g_1, g_2), \tag{4.4d}$$

i.e. if σ is invariant under conjugacy.

Remark 4.1. Notice that if G is an abelian group, this is the same as the notion of abelian cocycles $(\omega, \sigma) \in Z^3_{\mathrm{ab}}(G)$ (see [EGNO15]).

Example 4.1. Consider $G = \mathbb{Z}_2 = \{1, g\}$. It is well known, that the abelian cohomology of $\mathbb{Z}_2$ is given as $H^3_{ab}(\mathbb{Z}_2) \cong \mathbb{Z}_4$. Namely, after normalizing the four inequivalent abelian cocycles, they are given by

$$\omega(g, g, g) := e^{i\pi j}$$

and

$$\sigma(g, g) := e^{i\pi j/2}$$

for $j \in \{0, 1, 2, 3\}$.

If in addition $\mathcal{D}$ is rigid, then the dualities pick up the modification made to the associator. Namely, let $\mathcal{D}$ be rigid with evaluation and coevaluation maps $\overrightarrow{ev}, \overrightarrow{coev}, \overleftarrow{ev}, \overleftarrow{coev}$. The rigidity conditions are:

1. $(\overrightarrow{ev}_X \otimes \mathrm{id}_X) \circ a^{-1}_{X,X^*,X} \circ (\mathrm{id}_X \otimes \overrightarrow{coev}_X) = \mathrm{id}_X$
2. $(\mathrm{id}_{X^*} \otimes \overrightarrow{ev}_X) \circ a_{X^*,X,X^*} \circ (\overrightarrow{coev}_X \otimes \mathrm{id}_{X^*}) = \mathrm{id}_{X^*}$
3. $(\mathrm{id}_X \otimes \overleftarrow{ev}_X) \circ a_{X,X^*,X} \circ (\overleftarrow{coev}_X \otimes \mathrm{id}_X) = \mathrm{id}_X$
4. $(\overleftarrow{ev}_X \otimes \mathrm{id}_{X^*}) \circ a^{-1}_{X^*,X,X^*} \circ (\mathrm{id}_{X^*} \otimes \overleftarrow{coev}_X) = \mathrm{id}_{X^*}$

Hence, the evaluation and coevaluation maps change according to the associator modification. For an object $X \in \mathcal{C}_g$, we set:

$$\overrightarrow{ev}'_X = \overrightarrow{\kappa}(g) \cdot \overrightarrow{ev}_X \tag{4.5a}$$

$$\overrightarrow{coev}'_X = \overrightarrow{\kappa}(g) \cdot \overrightarrow{coev}_X \tag{4.5b}$$

and similarly with $\overleftarrow{\kappa}(g)$ for the left evaluation and coevaluation maps, where we fix $\overrightarrow{\kappa}(g)$ as a square root of $\omega(g, g^{-1}, g)$ and $\overleftarrow{\kappa}(g)$ as a square root of $\omega(g^{-1}, g, g^{-1})$.

In this way, $\mathcal{D}^\omega$ (and $\mathcal{D}^{\omega,\sigma}$ if braided) becomes G-crossed (braided) rigid category. Notice also, that by definition the dual functor on morphisms remains the same as for $f \in \mathcal{C}_g(X, Y)$,

$$f^* := (\mathrm{id}_{X^*} \otimes \overrightarrow{ev}_Y) \otimes (\mathrm{id}_{X^*} \otimes (f \otimes \mathrm{id}_{Y^*})) \circ a_{X^*,X,Y^*} \circ (\overrightarrow{coev}_X \otimes \mathrm{id}_{Y^*})$$

Modified Twists

Let $\mathcal{D}$ be now a G-crossed ribbon category with twist θ. The left dualities are induced using the twist, i.e. ${}^*X = X^*$ with maps

- $\overleftarrow{ev}_X := \overrightarrow{ev}_X \circ c_{X^*,X} \circ (\mathrm{id}_{X^*} \otimes \theta_X)$

- $\overleftarrow{coev}_X := (\theta_X^{-1} \otimes \mathrm{id}_{X^*}) \circ \tilde{c}_{X^*,X} \circ \overrightarrow{coev}_X$

just like in the non-crossed case.

Thus, it is pivotal with ${}^*f = f^*$. Let us now modify the twist by introducing a function $\nu : G \to \mathbb{K}^\times$ and define for $X \in \mathcal{C}_g$

$$\theta_X^\nu := \nu(g)\theta_X. \tag{4.6}$$

This defines a natural isomorphism. To ensure that the twist axioms are satisfied, we impose

$$\nu(1) = 1 \tag{4.7a}$$

$$\nu(h^{-1}gh) = \nu(g) \tag{4.7b}$$

$$\nu(gh) = \sigma(h, h^{-1}gh)\sigma(g,h)\nu(g)\nu(h) \tag{4.7c}$$

and for the ribbon property

$$\nu(g^{-1}) = \nu(g). \tag{4.7d}$$

The modification on the twist implies

$$\overleftarrow{\kappa}(g) = \overrightarrow{\kappa}(g)\sigma(g^{-1}, g)\nu(g)$$

and

$$\overleftarrow{\kappa}(g) = \nu(g)^{-1}\sigma(g, g^{-1})^{-1}\overrightarrow{\kappa}(g).$$

All in all, out of a given G-crossed ribbon extension of $\mathcal{C}$ one obtains a family of distinct G-crossed extensions by modifying the structure maps. This is formulated in the following proposition.

Proposition 4.1. *Let $\mathcal{D}$ be a G-crossed ribbon fusion extension of a ribbon fusion category $\mathcal{C}$ with associativity constraints $a_{X,Y,Z}$, bradings $c_{X,Y}$ and twist θ_X and let $\mathcal{D}^{\omega,\sigma,\nu}$ be the category equipped with the modified structure maps of equations* (4.1), (4.3), (4.6). *If ω, σ, ν satisfy equations* (4.2a)-(4.2c),(4.4a)-(4.4d) *and* (4.7a)-(4.7d), *then $\mathcal{D}^{\omega,\sigma,\nu}$ is a G-crossed ribbon extension of $\mathcal{C}$.*

Example 4.2. Continuing the example of $G = \mathbb{Z}_2$ we find that the modification of the twist should satisfy $\nu(1) = 1$ and $\sigma(g,g)^2\nu(g)^2 = 1$, i.e. $\nu(g) = \pm\sigma(g,g)$. Thus we have

$$\nu(g) = e^{i\pi(j/2+k)}$$

for j as before and $k \in \{0, 1\}$. We also fix for the dualities

$$\overrightarrow{\kappa}(g) = e^{-i\pi j/2}$$

as a square root of $\omega(g,g,g)$ and thus

$$\overleftarrow{\kappa}(g) = e^{i\pi(j/2+k)}.$$

Since $j \in \{0, 1, 2, 3\}$ and $k \in \{0, 1\}$, we have a family of eight distinct $\mathbb{Z}_2$-crossed ribbon categories $\mathcal{D}^{(j,k)}$. The dimension of an object in the twisted sector is then modified as $d_X^{(j,k)} = e^{i\pi k}d_X$.

4.2 Parametrized orbifold data

Let $\{\mathcal{D}^{j,k}\}$ be the family of $\mathbb{Z}_2$-crossed ribbon fusion categories from example 4.2. Let $\mathcal{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ be the orbifold datum of $\mathcal{C}$ derived by $\mathcal{D} \equiv \mathcal{D}^{0,0}$ as in [CRS18]. We now describe a family of orbifold data $\mathcal{A}^{j,k} = (A^{j,k}, T^{j,k}, \alpha^{j,k}, \overline{\alpha}^{j,k}, \psi^{j,k}, \phi^{j,k})$ which are obtained by the different $\mathbb{Z}_2$-crossed structures.

The underlying objects of $A^{j,k}$ and $T^{j,k}$ are the same as those of A and T respectively. In what follows, we only give the non-trivial changes. The component algebra $A_g^{j,k}$ has the following structure maps.

$$\begin{aligned} \mu_g^{(j,k)} &= e^{i\pi j/2}\mu_g \\ \eta_g^{(j,k)} &= e^{-i\pi j/2}\eta_g \\ \Delta_g^{(j,k)} &= e^{-i\pi j/2}\Delta_g \\ \epsilon_g^{(j,k)} &= e^{i\pi j/2}\epsilon_g \end{aligned} \tag{4.8}$$

The bimodule components $T_{g,1}^{j,k}$ and $T_{1,g}^{j,k}$ are just the algebra object $A_g^{j,k}$ and therefore, their left and right non-trivial actions are given by $\mu^{j,k}$. The bimodule $T_{g,g}^{j,k}$ is a right module over $A_g^{j,k} \otimes A_g^{j,k}$. The actions are given by

$$\begin{aligned} r_1^{(j,k)} &= e^{i\pi j/2} r_1 \\ r_2^{(j,k)} &= e^{i\pi j/2} r_2 \end{aligned} \tag{4.9}$$

The bimodule maps $\alpha^{j,k} : T \otimes T \to T \otimes T$ are given by

$$\begin{aligned} \alpha_{g,1,1}^{(j,k)} &= e^{-i\pi j/2 + i\pi k}\alpha_{g,1,1} \\ \alpha_{1,1,g}^{(j,k)} &= e^{i\pi j/2}\alpha_{1,1,g} \\ \alpha_{1,g,1}^{(j,k)} &= e^{i\pi k}\alpha_{1,g,1} \\ \alpha_{1,g,g}^{(j,k)} &= e^{-i\pi j/2 + i\pi k}\alpha_{1,g,g} \\ \alpha_{g,g,1}^{(j,k)} &= e^{i\pi j/2}\alpha_{g,g,1} \\ \alpha_{g,1,g}^{(j,k)} &= e^{i\pi k}\alpha_{g,1,g} \\ \alpha_{g,g,g}^{(j,k)} &= e^{i\pi j}\alpha_{g,g,g} \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \overline{\alpha}_{g,1,1}^{(j,k)} &= e^{i\pi j/2}\overline{\alpha}_{g,1,1} \\ \overline{\alpha}_{1,1,g}^{(j,k)} &= e^{-i\pi j/2 + i\pi k}\overline{\alpha}_{1,1,g} \\ \overline{\alpha}_{1,g,1}^{(j,k)} &= e^{i\pi k}\overline{\alpha}_{1,g,1} \\ \overline{\alpha}_{1,g,g}^{(j,k)} &= e^{i\pi j/2}\overline{\alpha}_{1,g,g} \\ \overline{\alpha}_{g,g,1}^{(j,k)} &= e^{-i\pi j/2 + i\pi k}\overline{\alpha}_{g,g,1} \\ \overline{\alpha}_{g,1,g}^{(j,k)} &= e^{i\pi k}\overline{\alpha}_{g,1,g} \\ \overline{\alpha}_{g,g,g}^{(j,k)} &= e^{i\pi j}\overline{\alpha}_{g,g,g} \end{aligned} \tag{4.11}$$

Moreover,

$$\psi^{(j,k)}|A_g = e^{i\pi k/2}\psi_{A_g}. \tag{4.12}$$

The scalar $\phi^{j,k} = \phi = 1/|G|$ remains as before.

Theorem 2. *The datum $\mathcal{A}^{j,k} = (A^{j,k}, T^{j,k}, \alpha^{j,k}, \overline{\alpha}^{j,k}, \psi^{j,k}, \phi)$ for any $j \in \{0,1,2,3\}$ and $k \in \{0,1\}$ forms an orbifold datum for $\mathcal{C}$. In particular, the orbifold datum $\mathcal{A} \equiv \mathcal{A}^{(0,0)}$ is part of a family of eight orbifold data.*

In particular, the orbifold datum of $\mathcal{C} \boxtimes \mathcal{C}$ in Theorem 1 is part of an eight element family of orbifold data.

Example 4.3. Consider $\mathcal{C} = \mathrm{Vect}$ with the extension $\mathrm{Vect}_{\mathbb{Z}_2}$. Then, the modification by the cocycle ω leads to the category $\mathrm{Vect}^{\omega}_{\mathbb{Z}_2}$. One can easily check, that the orbifold datum derived by the extension coincides, once again, with the orbifold datum for the Turaev-Viro theory of $\mathrm{Vect}^{\omega}_{\mathbb{Z}_2}$, which is isomorphic to the RT TQFT of $\mathrm{Rep}(D^{\omega}(\mathbb{Z}_2))$. Therefore, this datum describes Dijkgraaf-Witten theories.

If we restrict our attention to the algebras $A_g^{(j,k)}$, then we notice that they only depend on j, i.e. $A_g^{(j,0)} = A_g^{(j,1)}$. Moreover, they are isomorphic as Frobenius algebras with the isomorphism $f : A_g^{(j,k)} \xrightarrow{\sim} A_g = A_g^{(0,0)}$ given by $f = e^{i\pi j/2} \cdot \mathrm{id}$. Hence, the Frobenius algebras $A^{j,k}$, which are part of the orbifold data $\mathcal{A}^{j,k}$ are all pairwise isomorphic.

4.3 Comparison of the modified orbifold data

In [CRS18] they give the notion of a Morita transport of an orbifold datum. Let $\mathcal{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ be an orbifold datum for a MTC $\mathcal{C}$ and let B be a symmetric Δ-separable Frobenius algebra, such that A and B are Morita equivalent with Morita module $X \in A$-B-Mod, i.e. $X^* \otimes_A X \cong B$ as B-B-bimodules and $X \otimes X^* \cong A$ as A-A-bimodules. The *Morita transport* of $\mathcal{A}$ along X is the list

$$\mathcal{A}^X = (B, T^X, \alpha^X, \overline{\alpha}^X, \psi^X, \phi),$$

where $T^X = X^* \otimes_A T \otimes_{AA} (X \otimes X)$ is an B-$(B \otimes B)$-bimodule. For the precise definition of each object, we refer to [CRS18, Def. 3.7].

We consider now the case where B and A are even isomorphic as Frobenius algebras with $f : A \to B$ such an isomorphism. In particular, they are Mortia equivalent with Morita module $X = B$. We write $\tilde{T}$ for the object T with the induced B-$(B\otimes B)$-bimodule structure. Define $\psi_B := f \circ \psi \circ f^{-1}$.

Lemma 1. *The list $\mathcal{B} := (B, \tilde{T}, \alpha, \overline{\alpha}, \psi_B, \phi)$ forms an orbifold datum and its orbifold theory is isomorphic to the orbifold theory of the Morita transport $\mathcal{A}^X$, i.e.*

$$\mathcal{Z}_{\mathcal{B}} \cong \mathcal{Z}_{\mathcal{A}^X}.$$

Proof. To prove that $\mathcal{B}$ is an orbifold datum, one needs to check the orbifold equations. One can easily check these by using that f is an isomorphism of Frobenius algebras and the definition of the induced B-actions.

Next, we want to show that the associated orbifold theory is isomorphic to that of the Morita transport. To see this, we first notice that

$$\begin{aligned} T^X :=& B^* \otimes_A T \otimes_{AA} (B \otimes B) \\ \cong& B^* \otimes_B \tilde{T} \otimes_{BB} (B \otimes B) \\ \cong& B^* \otimes \tilde{T} \cong B \otimes_B \tilde{T} \cong \tilde{T} \end{aligned} \tag{4.13}$$

as B-$(B \otimes B)$-bimodules, where we used, that $B^* \cong B$ as Frobenius algebras (from equation (2.2)). Using this bimodule isomorphisms, we find that the diagrams

$$\begin{tikzcd} T^X \otimes T^X \arrow[r, "\alpha^X"] \arrow[d, "\cong"'] & T^X \otimes T^X \arrow[d, "\cong"] \\ T \otimes T \arrow[r, "\alpha"] & T \otimes T \end{tikzcd}$$

and

$$\begin{tikzcd} T^X \otimes T^X \arrow[r, "\overline{\alpha}^X"] \arrow[d, "\cong"'] & T^X \otimes T^X \arrow[d, "\cong"] \\ T \otimes T \arrow[r, "\overline{\alpha}"] & T \otimes T \end{tikzcd}$$

commute.

This is exactly the notion of a *T-compatible isomorphism* between the orbifold data $\mathcal{A}^X$ and $\mathcal{B} := (B, T, \alpha, \overline{\alpha}, \psi^X, \phi)$ [CRS18, Def. 3.12]. Therefore, by [CRS18, Lem. 3.13, Prop. 3.11] we have isomorphic orbifold theories

$$\mathcal{Z}_{\mathcal{A}} \cong \mathcal{Z}_{\mathcal{A}^X} \cong \mathcal{Z}_{\mathcal{B}}.$$

□

Let us now return to our case, where we have the algebra isomorphism $f : A^{(j,k)} \to A$. Then, we get by the above lemma the orbifold datum

$$\mathcal{B}^{(j,k)} = (A, T, \alpha^{(j,k)}, \overline{\alpha}^{(j,k)}, \psi^{(j,k)}, \phi),$$

which is T-isomorphic to the Morita transport of $\mathcal{A}^{j,k}$ along A and describes the same orbifold theory as $\mathcal{A}^{j,k}$. Thus, to compare the orbifold theories of the distinct orbifold data, it is sufficient to compare the ones associated to $\mathcal{B}^{(j,k)}$ and the original orbifold theory $\mathcal{A}$. These differ only in their bimodule maps and ψ insertions.

Comparison of Invariants

In section 3.3, we computed the invariants of the orbifold theory on $\mathbb{S}^3$, $\mathbb{S}^1 \times \mathbb{S}^2$ and $L(-2,1)$. The first two invariants do not change under the different orbifold data $\mathcal{A}^{j,k}$ as they are just equal to $\frac{1}{2D}$ respectively 1. However, there are certain non-trivial terms of the bimodule maps and the dimensions in equation (3.29) which depend on the parameters j,k. Namely, for the different orbifold data $\mathcal{B}^{j,k}$, Λ gets modified as

- $\Lambda^{j,k}_{1,1} = \Lambda_{1,1}$
- $\Lambda^{j,k}_{1,g} = \Lambda_{1,g}$
- $\Lambda^{j,k}_{g,1} = e^{i\pi j}\Lambda_{g,1}$
- $\Lambda^{j,k}_{g,g} = e^{i\pi j}\Lambda_{g,g}$

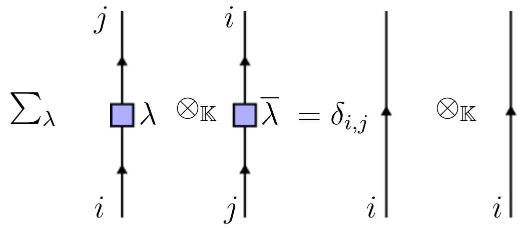
Example 4.4. Consider the orbifold data for Vect derived by the $\mathbb{Z}_2$-extension. Then, we get

$$\mathcal{Z}_{\mathcal{B}^{j,k}}(L(-2,1)) = \frac{1}{4}(2 + 2e^{i\pi j}) = \frac{1}{2}(1 + e^{i\pi j}).$$

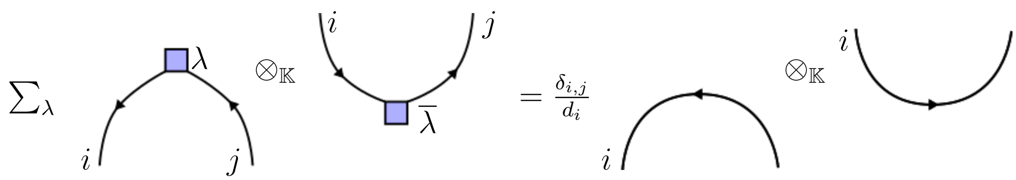
Therefore, the lens space $L(-2,1)$ detects the 3-cocycle ω.

A Partition Properties

Let $\mathcal{C}$ be a ribbon fusion category and I the set of representatives of isomorphism classes of simple objects. For an object X and $i \in I$, let $\{p_\lambda\}$ and $\{q_\lambda\}$ be an i-partition as introduced in Section 2.1. Then, such partitions satisfy the following equalities, which can be found in [TV17, Lemma 4.9].



(A.1)



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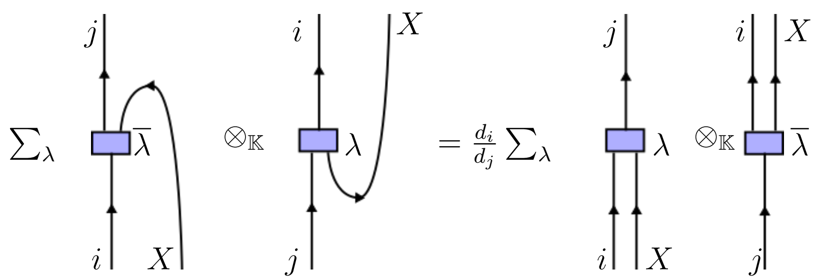

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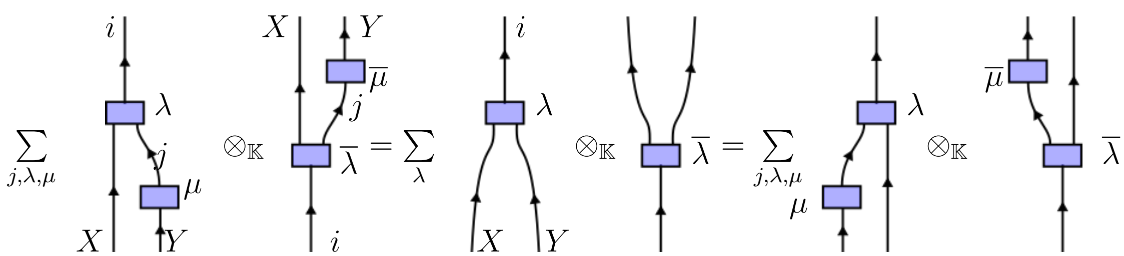

(A.4)

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B Right $\mathbb{Z}_2$- crossed structure on $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C} \oplus \mathcal{C}$

B.1 From left to right G-crossed categories

Let $\mathcal{D} = \bigoplus_{g \in G} \mathcal{C}_g$ denote a G-crossed category with left action functor $L : \underline{G} \to \mathrm{Aut}(\mathcal{D})$ [EGNO15]. We will write $\mathcal{D}^L \equiv \mathcal{D}$ for this left stucture. However, we also get a right G-crossed structure by defining the right action functor by the composite

$$\underline{G}^{\mathrm{mop}} \to \underline{G} \xrightarrow{L} \mathrm{End}(\mathcal{D}),$$

where we pass through group inverses. Thus, we have $R_g := L_{g^{-1}}$ and structure maps $\rho_{g,h} := \lambda_{h^{-1},g^{-1}}$. This functor is monoidal as a composition of monoidal functors. We will also write $\mathcal{D}^R$ for $\mathcal{D}$ seen as a right G-crossed category.

If in addition, $\mathcal{D}^L$ is left G-crossed monoidal, then $\mathcal{D}^R$ is right G-crossed monoidal with the same underlying monoidal structure, since $R_g(\mathcal{C}_h) := L_{g^{-1}}(\mathcal{C}_h) \subset \mathcal{C}_{g^{-1}hg}$. The coherence isomorphisms β^R_g are given by the (left) coherence isomorphisms $\beta^L_{g^{-1}}$. The same happens if we have a rigid structure. If $\mathcal{D}^L$ is G-crossed braided with braiding $c^L_{X,Y} : X \otimes Y \to g.Y \otimes X$, then we find braidings for its right crossed counterpart. The (right) crossed braiding $c^R_{X,Y}$: $X \otimes Y \to Y \otimes X.h := Y \otimes h^{-1}.X$ is defined via the (left) under-crossing

$$c^R_{X,Y} := \tilde{c}^L_{X,Y}.$$

We notice that by doing that, the neutral sector is given the opposite braiding. Thus, if $\mathcal{D}^L$ was a braided extension of the braided category $\mathcal{C}$, then its right counterpart $\mathcal{D}^R$ is a right crossed extension of $\mathcal{C}^{\mathrm{rev}}$.

If in addition, we have a (left) twist $\theta^L_X : X \to g.X$, then we get a (right) twist $\theta^R_X : X \to X.g := g^{-1}.X$ by

$$\theta^R_X := g^{-1}.(\theta^L_X)^{-1} \circ \lambda^{-1}_{g^{-1},g}$$

.

Again, restricted to the neutral sector, this corresponds to changing the twist by the opposite twist. Therefore, if $\mathcal{D}^L$ was the G-crossed ribbon extension of a ribbon fusion category $\mathcal{C}$, then $\mathcal{D}^R$ is a right G-crossed ribbon extension of $\mathcal{C}^{\mathrm{rev}}$.

B.2 $\mathbb{Z}_2$-crossed extension of $\mathcal{C} \boxtimes \mathcal{C}$

In[BS11], they input an anomaly free modular tensor category $\mathcal{C}$ and they obtain a (left) $\mathbb{Z}_2$-crossed extension of $\mathcal{C} \boxtimes \mathcal{C}$. Let $\mathcal{C}$ be the modular tensor category of interest. In this section, we input $\mathcal{C}^{\mathrm{rev}}$ in their results, to obtain a left crossed extension of $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}^{\mathrm{rev}}$. Passing to the convention of right crossed extensions as instructed in the previous section, we obtain, as wished, a right crossed extension of $\mathcal{C} \boxtimes \mathcal{C}$.

We adopt the same convention as in [BS11] and omit writing the tensor product of $\mathcal{C}$ on objects, i.e. $X \otimes_{\mathcal{C}} Y \equiv XY$ and by $\otimes$ we denote the tensor product of $\mathcal{D}$. The tensor product of two objects in the neutral sector is just the tensor product in the Deligne product, i.e.

$$(A_1 \boxtimes A_2) \otimes (B_1 \boxtimes B_2) = A_1 B_1 \boxtimes A_2 B_2.$$

For the tensor product of a neutral and a twisted object, we have

$$(A_1 \boxtimes A_2) \otimes X = A_1 A_2 X$$

and

$$X \otimes (A_1 \boxtimes A_2) = X A_1 A_2.$$

Finally, the tensor product of two twisted objects is given by

$$X \otimes Y = \bigoplus_{i \in I} X Y i^* \otimes i.$$

The unit $\mathbb{1}_{\mathcal{D}}$ is the unit in $\mathcal{C} \boxtimes \mathcal{C}$, i.e. $\mathbb{1} \boxtimes \mathbb{1}$ and the left and right unitality constraints are identities.

Associators:

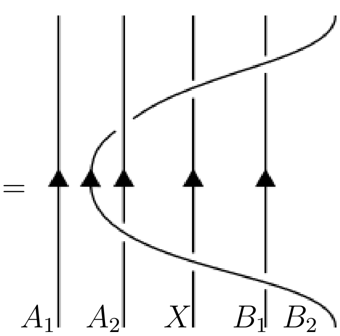
The associativity constraint on three neutral objects is

$$a_{A_1 \boxtimes A_2, B_1 \boxtimes B_2, C_1 \boxtimes C_2} = \mathrm{id}_{A_1 B_1 C_1 \boxtimes A_2 B_2 C_2}.$$

For the associativity constraints of two neutral objects and one twisted object, we get:

$$a_{A_1 \boxtimes A_2, B_1 \boxtimes B_2, X} = \mathrm{id}_{A_1} \otimes_{\mathcal{C}} c^{-1}_{A_2, B_1} \otimes_{\mathcal{C}} \mathrm{id}_{B_2 X}$$
$$a_{X, A_1 \boxtimes A_2, B_1 \boxtimes B_2} = \mathrm{id}_{X A_1} \otimes_{\mathcal{C}} c_{A_2, B_1} \otimes_{\mathcal{C}} \mathrm{id}_{B_2}$$

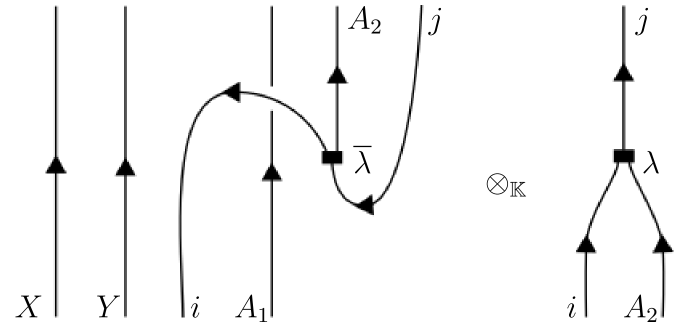
$$a_{A_1 \boxtimes A_2, X, B_1 \boxtimes B_2} =$$



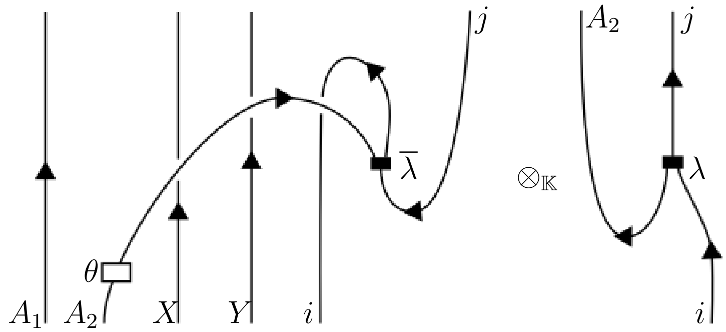
Remark B.1. The above associativity constraints endow $\mathcal{C}$ with a bimodule structure over $\mathcal{C} \boxtimes \mathcal{C}$. They are part of a family of such bimodule structures, found in [BFRS10].

The associativity constraints between one neutral object and two twisted objects are

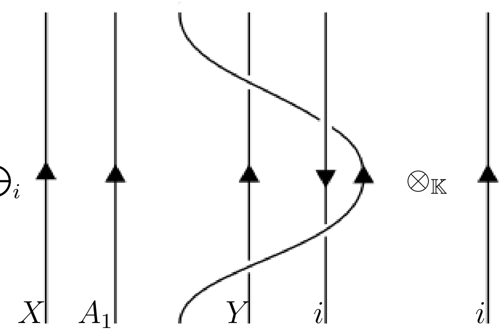
$$a_{X,Y,A_1\boxtimes A_2} = \bigoplus_{i,j} \sum_\lambda$$



$$a_{A_1\boxtimes A_2,X,Y} = \bigoplus_{i,j} \sum_\lambda$$

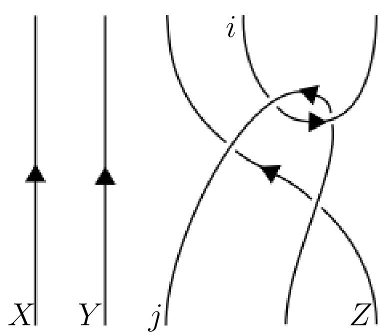


$$a_{X,A_1\boxtimes A_2,Y} = \bigoplus_i$$



Finally, the associativity constraint of three twisted objects is

$$a_{X,Y,Z} = \bigoplus_i \frac{d_i}{D}$$

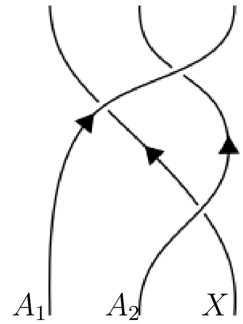


Braiding:
The braiding of two neutral objects is just the braiding in $\mathcal{C} \boxtimes \mathcal{C}$, i.e.

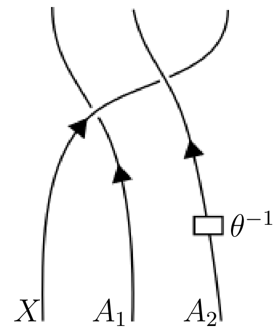
$$c^R_{A_1\boxtimes A_2, B_1\boxtimes B_2} = c_{A_1,B_1} \otimes_{\mathbb{K}} c_{A_2,B_2}.$$

The braiding between a neutral and a twisted object is
and

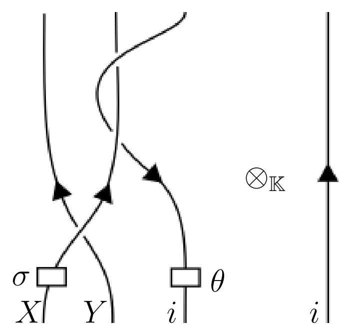
$c^R_{A_1 \boxtimes A_2, X} =$



$c^R_{X, A_1 \boxtimes A_2} =$



$c^R_{X,Y} = \bigoplus_i$



The braiding of two twisted objects is

where σ is the morphism described in [BS11] coming from the half-twist, which squares to θ.

The twist of a neutral object coincides with the twist in $\mathcal{C} \boxtimes \mathcal{C}$, i.e.

$$\theta_{A_1 \boxtimes A_2} = \theta_{A_1} \otimes_{\mathbb{K}} \theta_{A_2},$$

while the twist of a twisted object is given by the half-twist morphism $\theta_X = \sigma_X$.

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