Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
Mathematics department

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Hopf algebras
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Sheet 9

Problem 1. Let A and B be Hopf algebras. Consider the tensor categories $A-\mathsf{mod}$ and $B-\mathsf{mod}$ of finite dimensional left modules over A and B. A functor $F : A-\mathsf{mod} \to B-\mathsf{mod}$ is called exact, if for any short exact sequence

$$0 \to X \to Y \to Z \to 0$$

in $A-\mathsf{mod}$ the sequence

$$0 \to FX \to FY \to FZ \to 0$$

is exact in $B-\mathsf{mod}$.

Recall that an A-module P is called projective, if $\mathrm{Hom}_A(P, \bullet) : \mathcal{C} \to \mathsf{Vect}_{\mathbb{K}} = \mathbb{K}-\mathsf{mod}$ is an exact functor.

1. If P is projective, then $\bullet \otimes P$ is exact.
2. If P is projective, then $P^\vee$ is projective.

Problem 2. Let A be an algebra over $\mathbb{K}$. An A-module M is called indecomposable, if $M = N \oplus N'$ implies that either N or N' is the zero module. An A-module M is called simple, if M and 0 are its only submodules.

Show that for a semi-simple algebra A every indecomposable module is simple.

Problem 3. We consider the following Hopf algebra H (called Sweedler's Hopf algebra): as an algebra it is given by the following quotient:

$$\mathbb{C}\langle C, X\rangle/(C^2 - 1, X^2, CX + XC)$$

where $\mathbb{C}\langle C, X\rangle$ is the algebra of non-commutative polynomials. The comultiplication is given by:

$$\Delta(C) = C \otimes C \quad \text{and} \quad \Delta(X) = C \otimes X + X \otimes 1.$$

1. Find a counity and an antipode and prove that H is indeed a Hopf algebra. Remark that H is neither commutative nor cocommutative.
2. Find all (up to isomorphism) simple H-modules.
3. Prove that the tensor product of two simple modules is simple.
4. Find all (up to isomorphism) projective indecomposable H-modules.
5. Prove that the tensor product of any two projective indecomposable H-modules is a direct sum of 2 projective indecomposable H-modules.

Problem 4. Let H be a finite dimensional Hopf algebra. We suppose that S as an odd order (ie the smallest positiv n such that $S^n = \mathrm{id}_H$ is odd).

1. Prove that H is commutative.
2. Prove that H is cocommutative.
3. Prove that $S = \mathrm{id}$
4. Give an example of such a Hopf algebra.