Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
Mathematics department

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Hopf algebras
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Sheet 6

Problem 1. 1. Prove that the left (or right) dual of an object is essentially unique. (It is unique up to a unique isomorphism).

2. Suppose $\mathcal{C}$ is autonomous and V is an object of $\mathcal{C}$. Show that there are canonical isomorphisms: ${}^{\vee}(V^{\vee}) \simeq V \simeq ({}^{\vee}V)^{\vee}$.

3. Let us consider the category $\mathbb{K}$-Vect of $\mathbb{K}$-vector spaces, with its usual tensor structure. Prove that an object of $\mathbb{K}$-Vect is left (or right) dualizable if and only if it is finite dimensional.

4. Let us consider the category $\mathsf{Cob}(k+1)$ whose objects are oriented k-dimensional manifold and whose morphisms are $(k+1)$-dimensional cobordism. Prove that the disjoint union turns this category into a tensor category.

5. Prove that every object in $\mathsf{Cob}(k+1)$ is left (or right) dualizable.

6. A tensor catogory $(\mathcal{C}, I, a, r, l)$ is *symmetric* if there exists a natural isomorphism between s between $\bullet \otimes \bullet$ and $\bullet \overset{\tau}{\otimes} \bullet$ (wehre $A \overset{\tau}{\otimes} B := B \otimes A$, and similarly for morphisms) such that for any triple of objects (A, B, C) of $\mathcal{C}$ such that the following diagrams commute:

$$\begin{tikzcd} A \otimes I \arrow[rr, "s_{A,I}"] \arrow[dr, "r_A"'] & & I \otimes A \arrow[dl, "l_A"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "s_{A,B} \otimes \mathrm{id}_C"] \arrow[d, "a_{A,B,C}"'] & (B \otimes A) \otimes C \arrow[d, "a_{B,A,C}"] \\ A \otimes (B \otimes C) \arrow[d, "s_{A,B \otimes C}"'] & B \otimes (A \otimes C) \arrow[d, "\mathrm{id}_B \otimes s_{A,C}"] \\ (B \otimes C) \otimes A \arrow[r, "a_{B,C,A}"'] & B \otimes (C \otimes A) \end{tikzcd} \qquad \begin{tikzcd} A \otimes B \arrow[rr, "\mathrm{id}_{A \otimes B}"] \arrow[dr, "s_{A,B}"'] & & A \otimes B \\ & B \otimes A \arrow[ur, "s_{B,A}"'] & \end{tikzcd}$$

Prove that in a symmetric tensor category every the notion of left dual and of right dual coincide.

7. Prove that in a symmetric tensor category there is a good notion of trace.

8. What is the trace in $\mathsf{Cob}(n)$?

Problem 2. Let H be a Hopf algebra over a field $\mathbb{K}$. Let $a \in H$ and define

$$\begin{aligned} \mathrm{ad}_a &: H \to H, \\ \mathrm{ad}_a(x) &:= \sum_{(a)} a_{(1)} \cdot x \cdot S(a_{(2)}). \end{aligned}$$

1. Show that $\mathrm{ad} : H \otimes H \to H, a \otimes x \mapsto \mathrm{ad}_a(x)$ defines the structure of a left H-module $H_{\mathrm{ad}} = (H, \mathrm{ad})$ on H. H_{ad} is called the *adjoint module of* H.

2. Show that the multiplication $\mu : H_{\mathrm{ad}} \otimes H_{\mathrm{ad}} \to H_{\mathrm{ad}}$ is a homomorphism of H-modules.

3. Show that if $\epsilon(a) = 1$, ad_a preserves the counit and the unit.

4. Suppose a is group-like. Show that ad_a preserves the comultiplication, i.e.

$$(\mathrm{ad}_a \otimes \mathrm{ad}_a) \circ \Delta = \Delta \circ \mathrm{ad}_a.$$

Problem 3. Let H be a bialgebra and V a sub-space of H. Let us denote by I_l, I_r and I_2 respectively the left, right and bi-sided ideal generated by V.

1. Prove that if $\Delta(V) \subset I_\bullet \otimes H$ then $\Delta(I_\bullet) \subset I_\bullet \otimes H$ for $\bullet = l$, or 2.
2. Prove that if $\Delta(V) \subset H \otimes I_\bullet$ then $\Delta(I_\bullet) \subset H \otimes I_\bullet$ for $\bullet = l$, or 2.
3. Prove that if $\Delta(V) \subset H \otimes I_\bullet + I_\bullet \otimes H$ then $\Delta(I_\bullet) \subset H \otimes I_\bullet + I_\bullet \otimes H$ for $\bullet = l$, or 2.
4. Prove that if $\epsilon(V) = \{0\}$, then $\epsilon(I_\bullet) = \{0\}$.
5. From now on we suppose that H is a Hopf algebra with antipode S. Prove that if $S(V) \subset I_l$ then $S(I_r) \subset I_l$.
6. Prove that if $S(V) \subset I_r$ then $S(I_l) \subset I_r$.
7. Prove that if $S(V) \subset I_2$ then $S(I_2) \subset I_2$.

Problem 4. In this problem we will construct Hopf algebra with antipode of any even order. Let F be the free non-commutative algebra with on three variable X, Y and Z.

1. Prove that the following data yields a well defined bi-alegra:

$$\begin{aligned} \Delta(X) &= X \otimes X, & \epsilon(X) &= 1, \\ \Delta(Y) &= Y \otimes Y, & \epsilon(Y) &= 1, \\ \Delta(Z) &= 1 \otimes Z + Z \otimes X, & \epsilon(Z) &= 0. \end{aligned}$$

2. Prove that the two sided ideal I generated by $XY - 1$ and $YX - 1$ is a bi-ideal. We write $H = F/I$.
3. Prove that H is a Hopf algebra (find the antipode S).
4. Prove that S has infinite order.
5. Let, n be a natural number. Starting from H construct a Hopf algebra with antipode of order $2n$.