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DER FORSCHUNG | DER LEHRE | DER BILDUNG
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Hopf algebras
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Sheet 5

In this sheet $\mathbb{K}$ is a field, and all algebras, co-algebras and bialgebras are over $\mathbb{K}$

Problem 1.

1. Let H be a bialgebra and C be a sub-coalgebra of H, prove that the sub-algebra of H generated by C is a sub-bialgebra.

2. Show that a coalgebra C is cocommutative if and only if Δ is a coalgebra map.

3. Let C, D and D' be coalgebras such that C is cocommutative, and $f : C \to D$ and $f' : C \to D'$ two coalgebra maps. Define the canonical coalgebra maps $\pi : D \otimes D' \to D$ and $\pi' : D \otimes D' \to D$, and prove that there exists a unique map $F : C \to D \otimes D'$ such that the two following diagrams are commutatives.

$$\begin{tikzcd} & D \otimes D' \arrow[dr, "\pi"] & \\ C \arrow[ur, "F"] \arrow[rr, "f"'] & & D \end{tikzcd} \qquad \begin{tikzcd} & D \otimes D' \arrow[dr, "\pi'"] & \\ C \arrow[ur, "F"] \arrow[rr, "f'"'] & & D \end{tikzcd}$$

Problem 2 (Rational modules, second part, see sheet 4)**.** Let be C a coalgebra, and M and M' be two rational $C^\star$-modules and L a $C^\star$-module.

1. Prove that if N is a cyclic[1] sub-module of M then it is finite dimensional.

   Solution. *Let n be a generator of N. We write $\rho(n) = \sum_i n_i \otimes c_i$. Now if f is an element of $C^\star$, $f \cdot n = \sum_i f(c_i)n_i$. Hence N is spanned as a vector space by the n_i and is therefor finite dimensional.* □

2. Prove that every finitely generated rational module is finite dimensional.

   Solution. *Same proof as before, with a finite set of generators instead of just one.* □

3. Prove that if a $C^\star$-module N is a quotient of M, then it is rational.

   Solution. *Let us write $N = M/P$ and denote $\pi : M \to N$ the canonical projection. The module P is a submodule of P and hence is rational and is endowed with a comodule-C structure. There is a unique comodule-C structure on M/P making π a comodule-C morphism. From this comodule structure one can built an alternative (and rational) $C^\star$-module structure. But then π is a $C^\star$-module map for both $C^\star$-module structure on N. This structure have to be equal since π is surjective. This prove that N is rational.* □

[1]This a module generated by one element

4. Prove that $L^{\mathrm{rat}} = \rho^{-1}(L \otimes C)$ is a rational sub-module. Prove that it contains all rational sub-modules of L.

Solution. *Let $x \in L^{\mathrm{rat}}$ and f in $C^\star$. We want to show that $f \cdot x \in L^{\mathrm{rat}}$, this means that $\rho(f \cdot x) \in L \otimes C$. Let us write $\rho(x) = \sum_i x_i \otimes c_i$. Let g be an element of $C^\star$. We have:*

$$\begin{aligned} g \cdot (f \cdot x) &= (gf) \cdot x \\ &= \sum_i x_i (gf)(c_i) \\ &= \sum_i x_i (g \otimes f)(\Delta(c_i)) \\ &= \sum_i \sum_{(c_i)} x_i g(c_{i(1)}) f(c_{i(2)}) \\ &= \sum_j x'_j g(c'_j). \end{aligned}$$

This means exactly that $\rho(f \cdot x) = \sum_j x'_j \otimes c'_j$. Hence, we have shown that L^{rat} is a $C^\star$-module. It is trivial to show that $\rho(L^{rat} \subset L^{rat} \otimes C$ which means that L^{rat} is rational and that it contains any rational submodules of L. □

5. Prove that $\phi : M \to M'$ is a $C^\star$-module map iff f is a comodule-C map.

Solution. *Suppose $\phi : M \to M'$ is a comodule-C morphism. Let f be an element of $C^\star$ and m an element of M. We write:*

$$f \cdot m = \sum_{(m)} m_{(0)} f(m_{(1)}) \quad \textit{and} \quad f \cdot \phi(m) = \sum_{(\phi(m))} \phi_{(0)} f(\phi(m)_{(1)}).$$

The map ϕ is suppose to be a map of comodule-C. This means:

$$\Delta(\phi(m)) \sum_{(\phi(m))} \phi(m)_{(0)} \otimes \phi(m)_{(1)} = \phi(m_{(0)}) \otimes m_{(1)} = (\phi \otimes \mathrm{id}_C)(\Delta(m)).$$

Hence we have:

$$\begin{aligned} f \cdot \phi(m) &= \sum_{(\phi(m))} \phi(m)_{(0)} f(\phi(m)_{(1)}) \\ &= \sum_{(m)} \phi(m_{(0)}) f(m_{(1)}) \\ &= \phi\left(\sum_{(m)} m_{(0)} f(m_{(1)}) \right) \\ &= \phi(f \cdot m). \end{aligned}$$

Which means that ϕ is a morphism of $C^\star$-modules.

Let us now suppose that ϕ is a $C^\star$-module map. This implies that for any $m \in M$ and any $f \in C^\star$, we have:

$$f \cdot \phi(m) = \sum_{(\phi(m))} \phi(m)_{(0)} f(\phi(m)_{(1)}) = \sum_{(m)} \phi(m_{(0)}) f(m_{(1)}) = \phi(f \cdot m).$$

This implies that $\Delta(\phi(m)) = \sum_{(m)} \phi(m_{(0)}) \otimes m_{(1)} = (\phi \otimes \mathrm{id})(\Delta(m))$ and thus that ϕ is a comodule map. □

Problem 3 (Fundamental theorem on Coalgebras)**.** In this exercise C is a coalgebra and c an element of C.

1. Let $(C_i)_{i\in I}$ a collection of subcoalgeras of C. Prove that $\bigcap_{i\in I} C_i$ is a coalgebra. Let S be a subset of C, define the notion of *coalgebra generated by S*.

   Solution. The first assertion is clear. For the second we simply consider the intersection of all sub-coalgebra of C containing S. □

2. Recall how C is naturally endowed with a structure of $C^\star$-module.

   Solution. This is clear, C is even a rational. □

3. Show that the sub-$C^\star$-module N generated by c is finite dimensional (one should use the exercise about rational modules).

   Solution. N is a submodule of a rational module, hence it is rational. Furthermore it is cyclic, so that it is finite dimensional. □

4. (A little more difficult) Let $J = \{a \in C^\star | a \cdot N = \{0\}\}$. Prove that $J^\perp$ is a sub-coalgebra of C and that it is finite dimensional.

   Solution. J is the kernel of $\pi : C^\star \to End_{\mathbb{K}}(N)$ which is an algebra map. Hence we can deduce that J is a ideal of $C^\star$ with finite codimension. We claim that with this single hypothesis, we can deduce $J^\perp$ is a finite dimensional sub-coalgebra of C: Indeed, one can check that if x is an element of $J^\perp$, then $\Delta(x)$ is in $J^\perp \otimes C \cap C \otimes J^\perp = J^\perp \otimes J^\perp$, so that $J^\perp$ is a co-algebra.
   Let us recall the definition of $J^\perp$ and of $(J^\perp)^\perp$ and prove that $J \subset (J^\perp)^\perp$[2]*:*

   $$J^\perp = \{b \in C | f(b) = 0 \text{ for all } f \in J\} \quad \text{and} \quad (J^\perp)^\perp = \left\{g \in C^\star | g(b) = 0 \text{ for all } b \in J^\perp\right\}.$$

   Let f be an element of J and let b be an element of $J^\perp$. By definition of $J^\perp$, $f(b) = 0$. This is valid for every b, hence f is in $(J^\perp)^\perp$. The converse does not hold (J might not be a vector space for example). Let us prove that $\dim J^\perp = \operatorname{codim}(J^\perp)^\perp$. Actually $V^\star \simeq C\star/V^\perp$ holds for every sub-space V of C. If V is finite dimensional, this gives what we wanted. In the end, we have $\dim(J^\perp) = \operatorname{codim}(J^\perp)^\perp \leq \operatorname{codim} J < +\infty$. □

5. Prove the following theorem:

   Theorem 1. *Every coalgebra is a sum*[3] *of finite dimensional coalgebras.*

   Solution. Every coalgebra is the sum of all the subcoalgebra generated by one element which are finite dimensional, as we have just seen it. □

[2] This was indeed completly obvious.
[3] Just a sum, not a direct sum

Problem 4. Let $(\mathcal{C}, \otimes, I, a, l, r)$ a (non-strict) monoidal category. In this problem we want to construct a strict monoidal category $\mathcal{D}$ such that $\mathcal{C}$ and $\mathcal{D}$ are tensor equivalent.

1. We start with a (useful) example. Let $\mathcal{V}$ be the monoidal category whose objects are non-negative integers and whose morphisms from m to n are matrices of size $n \times m$ with coefficient in a field $\mathbb{K}$, tensor products being given by the sum of integers. Prove that this category is tensor equivalent to $\mathbb{K}$-vect, the category of finite dimensional vector spaces over $\mathbb{K}$.

2. The objects of $\mathcal{D}$ are finite sequences (the empty sequence is allowed) of objects of $\mathcal{C}$. We construct at the same time a (tensor) functor $F : \mathcal{D} \to \mathcal{C}$ even if $\mathcal{D}$ is not completely defined. If $S = (V_1, V_2, \dots, V_l)$ is an object of $\mathcal{D}$, we set $F(S) = (\cdots((V_1 \otimes V_2) \otimes V_3) \otimes \cdots) \otimes V_l$ (what should be $F(\emptyset)$?). Define the hom-spaces of $\mathcal{D}$ and the tensor product on objects of $\mathcal{D}$, denoted by $\star$.

   Solution. The problem and its solution are derived from Quantum Groups from Christian Kassel. We define $F(\emptyset) = I$ and $\mathrm{Hom}_{\mathcal{D}}(S, S') = \mathrm{Hom}_{\mathcal{C}}(F(S), F(S'))$. The composition and the identity morphisms in $\mathcal{D}$ are given by the composition and the identity morphisms in $\mathcal{C}$. The (strict) tensor product on objects of $\mathcal{D}$ is given by the concatenation of sequences. The empty sequence being the unit. □

3. Finish the definition of F and prove that it is fully faithful and essentially surjective (see the script or sheet 1, for the definitions). This proves that F is an equivalence of category which admit $G : \mathcal{D} \to \mathcal{C}, G(V) = (V)$ as an inverse.

   Solution. The definition of F on the hom-spaces is completely trivial since it is really the identity map on each home-space. For this reason, the functor F is clearly fully faithful. It is as-well essentially surjective, since any object V of $\mathcal{C}$ is equal (and hence isomorphic) to the image by F of the sequence (V) of length one with V as the only element of this sequence. Thanks to a theorem we proved earlier this gives that F is an equivalence of category. The proof of this theorem shows that we can indeed take G as prescribed to be an inverse. □

4. For S and S' two objects of $\mathcal{D}$, let us define $\phi(S, S') : F(S) \otimes F(S') \to F(S \star S')$ inductively on the length of S' by:

$$\phi(\emptyset, S') = l_{S'}, \qquad \phi(S, \emptyset) = r_S, \qquad \phi(S, (V_1)) = \mathrm{id}_{F(S) \otimes V_1} \quad \text{and}$$
$$\phi(S, (V_1, \dots V_{l+1})) = (\phi(S, (V_1, \dots, V_l)) \otimes \mathrm{id}_{V_{l+1}}) \circ a^{-1}_{F(S), F((V_1, \dots, V_l)), V_{l+1}}.$$

   Prove that if S, S' and S'' are objects of $\mathcal{D}$, we have the following equality:

$$\phi(S, S' \star S'') \circ (\mathrm{id}_{F(S)} \otimes \phi(S', S'')) \circ a_{F(S), F(S'), F(S'')} = \phi(S, S' \star S'') \circ (\phi(S, S') \otimes \mathrm{id}_{F(S'')}).$$

   Solution. The maps ϕ should be thought as "re-parenthesisation".

   Note that this is exactly the "compatibility with the associativity" of definition 2.4.6 in the sckript because the associators in $\mathcal{D}$ are identity morphisms. This is done by induction on the length of S'': If $S'' = \emptyset$, we have:

$$\begin{aligned}
\phi(S, S') \circ (\mathrm{id}_S \otimes \phi(S', \emptyset)) \circ a_{F(S), F(S'), I} &= \phi(S, S') \circ (\mathrm{id}_S \otimes r_{F(S')}) \circ a_{F(S), F(S'), I} \\
&= \phi(S, S') \circ r_{F(S) \otimes F(S')} \\
&= r_{F(S) \otimes F(S')} \circ (\phi(S, S') \otimes \mathrm{id}_I) \\
&= \phi(S \star S', \emptyset) \circ (\phi(S, S') \otimes \mathrm{id}_I)
\end{aligned}$$

Let know V be an object of the category $\mathcal{C}$. Let us suppose that the equality holds for the sequences S, S' and S''.

$$\begin{aligned}
&\phi(S, S' \star S'' \star (V)) \circ \big(\mathrm{id}_{F(S)} \otimes \phi(S', S'' \star (V))\big) \circ a_{F(S),F(S'),F(S''\star(V))} \\
&\quad = (\phi(S, S' \star S'') \otimes \mathrm{id}_V) \circ a^{-1}_{F(S),F(S'\star S''),V} \circ \big(\mathrm{id}_{F(S)} \otimes (\phi(S', S'') \otimes \mathrm{id}_V)\big) \\
&\qquad\qquad \circ \left(\mathrm{id}_{F(S)} \otimes a^{-1}_{F(S'),F(S''),V}\right) \circ a_{F(S),F(S'),F(S'')\otimes V} \\
&\quad = (\phi(S, S' \star S'') \otimes \mathrm{id}_V) \circ \big((\mathrm{id}_{F(S)} \otimes \phi(S', S'')) \otimes \mathrm{id}_V\big) \circ a^{-1}_{F(S),F(S')\otimes F(S''),V} \\
&\qquad\qquad \circ \left(\mathrm{id}_{F(S)} \otimes a^{-1}_{F(S'),F(S''),V}\right) \circ a_{F(S),F(S'),F(S'')\otimes V} \\
&\quad = (\phi(S, S' \star S'') \otimes \mathrm{id}_V) \circ \big((\mathrm{id}_{F(S)} \otimes \phi(S', S'')) \otimes \mathrm{id}_V\big) \circ (a_{F(S),F(S'),F(S'')} \otimes \mathrm{id}_V) \circ a^{-1}_{F(S)\otimes F(S'),F(S''),V} \\
&\quad = (\phi(S \star S', S'') \otimes \mathrm{id}_V) \circ \big((\phi(S, S') \otimes \mathrm{id}_{F(S'')}) \otimes \mathrm{id}_V\big) \circ a^{-1}_{F(S)\otimes F(S'),F(S''),V} \\
&\quad = (\phi(S \star S', S'') \otimes \mathrm{id}_V) \circ a^{-1}_{F(S\star S'),F(S''),V} \circ \big(\phi(S, S') \otimes (\mathrm{id}_{F(S'')} \otimes \mathrm{id}_V)\big) \\
&\quad = \phi(S \star S', S'' \star (V)) \circ \big(\phi(S, S') \otimes \mathrm{id}_{F(S''\star(V))}\big)
\end{aligned}$$

□

5. Define the tensor product of two morphisms in $\mathcal{D}$ and prove that with this structure $C^\star$ is a strict monoidal category.

Solution. *In the end we want the following diagrams to commutes:*

$$\begin{tikzcd}
F(S) \otimes F(S') \arrow[r, "\phi(S,S')"] \arrow[d, "F(f)\otimes F(g)"'] & F(S \star S') \arrow[d, "F(f\star g)"] \\
F(T) \otimes F(T') \arrow[r, "\phi(T,T')"] & F(T \star T')
\end{tikzcd}$$

This defines the tensor product on $\mathcal{D}$ completely because F is trivial on the hom-space. One verify easily that $\star$ is a functor, and it is strictly associative by construction. □

6. Prove that F and G are tensor functors. Conclude.

Solution. *This is to be understood as "prove that F and G can be completed as tensor functors". The triple (F, id_I, ϕ) is a tensor functor since the question 4 tells us that the required equalities (of the definition 2.4.6 of the script) hold (right and left unit constraint follow from the definition of $\phi(S, \emptyset)$ and $\phi(\emptyset), S)$). The triple $(G, \mathrm{id}, \mathrm{id})$ is as well a tensor functor (the id's should be widely understood). Finally $FG = \mathrm{id}_{\mathcal{C}}$, and the natural isomorphism $\theta : GF \to \mathrm{id}_{\mathcal{D}}$ given by $\theta(S) = \mathrm{id}_{F(S)}$ is a tensor natural transformation so that $\mathcal{C}$ and $\mathcal{D}$ are tensor equivalent.* □