Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
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Hopf algebras
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Sheet 4

In this sheet $\mathbb{K}$ is a field.

Problem 1. Let C a coalgebra and $I \subset C$ a vector subspace.

1. Show that the map

$$\begin{aligned}\overline{\Delta} : C/I &\to C/I \otimes C, \\ x + I &\mapsto \sum_{(x)} (x_{(1)} + I) \otimes x_{(2)}\end{aligned}$$

is a well-defined counital coaction of the coalgebra C on the quotient vector space C/I, iff I is a right coideal.

Solution. *Let $\pi : C \to C/I$ be the canonical projection. Consider the linear map $f := (\pi \otimes \mathrm{id}_C) \circ \Delta : C \to C/I \otimes C$. There is a unique map $F : C/I \to C/I \otimes C$ with $F \circ \pi = f$, iff $I \subset \ker f$. Now $I \subset \ker f$ is equivalent to $\Delta(I) \subset \ker(\pi \otimes \mathrm{id}_C) = I \otimes C$, i.e. I is a right coideal.*
Since $\overline{\Delta} \circ \pi = (\pi \otimes \mathrm{id}) \circ \Delta$ we have $F = \overline{\Delta}$.
Now we have

$$\begin{aligned}(\overline{\Delta} \otimes \mathrm{id}_C)\overline{\Delta}(x + I) &= \sum_{(x)} \overline{\Delta}(x_{(1)} + I) \otimes x_{(2)} \\ &= \sum_{(x)} \sum_{(x_{(1)})} ((x_{(1)})_{(1)} + I) \otimes (x_{(1)})_{(2)} \otimes x_{(2)} \\ &\overset{\text{coass.}}{=} \sum_{(x)} \sum_{(x_{(2)})} (x_{(1)} + I) \otimes (x_{(2)})_{(1)} \otimes (x_{(2)})_{(2)} \\ &= \sum_{(x)} (x_{(1)} + I) \otimes \Delta(x_{(2)}) = (\mathrm{id}_{C/I} \otimes \Delta)\overline{\Delta}(x + I) \quad ,\end{aligned}$$

hence $\overline{\Delta}$ is coassociative. Furthermore by

$$(\mathrm{id}_{C/I} \otimes \epsilon)\overline{\Delta}(x + I) = \sum_{(x)} (x_{(1)} + I) \otimes \epsilon(x_{(2)}) = x + I \quad ,$$

we see that $\overline{\Delta}$ is counital. □

2. Show that the comultiplication and counit of C define a coalgebra structure on the quotient vector space C/I by the induced maps, iff I is a two-sided coideal.

Solution. *Consider the linear map $(\pi \otimes \pi) \circ \Delta : C \to C/I \otimes C/I$. There exists a linear map*

$$\Delta_{C/I} : C/I \to C/I \otimes C/I$$

with $\Delta_{C/I} \circ \pi = (\pi \otimes \pi) \circ \Delta$, iff

$$I \subset \ker((\pi \otimes \pi)\Delta) \iff \Delta(I) \subset \ker(\pi \otimes \pi) = I \otimes C + C \otimes I \quad .$$

Now consider the linear map $\epsilon : C \to \mathbb{K}$. There exists a linear map $\epsilon_{C/I} : C/I \to \mathbb{K}$ with $\epsilon_{C/I} \circ \pi = \epsilon$, iff

$$I \subset \ker(\epsilon) \quad \iff \quad \epsilon(I) = 0 \quad .$$

If the induced maps $\Delta_{C/I} : C/I \to C/I \otimes C/I$ and $\epsilon_{C/I} : C/I \to \mathbb{K}$ exist, we see by similar calculations as in the first question that they give the structure of a coalgebra on the quotient vector space C/I. $\square$

Problem 2. Let (C, Δ, ϵ) be a coalgebra and x be an element of C.

1. Prove that for all $n \in \mathbb{N}$ and all i in $[1, n+1]$, we have:

$$\sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(n)} = \sum_{(x)} x_{(1)} \otimes \cdots \otimes x_{(i-1)} \otimes \epsilon(x_{(i)}) \otimes x_{(i+1)} \otimes \cdots \otimes x_{(n+1)}$$

Problem 3 (Frobenius[1] algebra)**.** Let A be a finite dimensional $\mathbb{K}$-algebra. Let $\eta : A \to \mathbb{K}$ be a $\mathbb{K}$-linear map, we suppose that the composition $\eta \circ \mu =: \langle \cdot, \cdot \rangle$ is a non-degenerate[2] bilinear form (A is then called a *Frobenius algebra*).

1. Prove that A is then naturally endowed with a co-algebra structure.
2. Prove $\mathrm{Mat}_{n \times n}(\mathbb{K})$ is a Frobenius algebra.
3. If G is a finite group, prove that $\mathbb{K}G$ is a Frobenius algebra.
4. (A little more difficult) Prove that $\mathbb{K}[X, Y]/(X^2, Y^2, XY)$ is not a Frobenius algebra.

Problem 4. Let $C := \mathbb{K}[X]$ be the vector space of polynomials in one variable and let us consider the following linear maps $\Delta(X^n) = \sum_{p+q=n} X^p \otimes X^q$ and $\epsilon(X^n) = \delta_{n,0}$.

1. Show that (C, Δ, ϵ) is a counital coalgebra.

Solution. We only have to check equalities on the basis $\{X^n\}_{n \in \mathbb{N}}$. Both $(\Delta \otimes \mathrm{id})\Delta(X^n)$ and $(\mathrm{id} \otimes \Delta)\Delta(X^n)$ are equal to

$$\sum_{\substack{p,q,r \in \mathbb{N} \\ p+q+r=n}} X^p \otimes X^q \otimes X^r \quad .$$

For the counitality $(\epsilon \otimes \mathrm{id})\Delta(X^n) = \sum_{k=0}^{n} \epsilon(X^k) \otimes X^{n-k} = 1 \otimes X^n = X^n$. In the same way we see $(\mathrm{id} \otimes \epsilon)\Delta(X^n) = X^n$. $\square$

2. We know that C, with the usual multiplication of polynomials, is an associative algebra. Is C with the comultiplication Δ a bialgebra?

Solution. We have $\epsilon(1) = 1$ and for $n + m > 0$: $\epsilon(X^n \cdot X^m) = 0 = \epsilon(X^n) \cdot \epsilon(X^m)$, so ϵ is a unital algebra homomorphism. Since $\Delta(1) = 1 \otimes 1$, Δ is unital, but since

$$\Delta(X^2) = X^2 \otimes 1 + X \otimes X + 1 \otimes X^2 \quad \textit{and}$$

$$\Delta(X)\Delta(X) = (X \otimes 1 + 1 \otimes X)(X \otimes 1 + 1 \otimes X) = X^2 \otimes 1 + 2(X \otimes X) + 1 \otimes X^2$$

the comultiplication is not a map of algebras, thus we do not have a bialgebra. $\square$

[1]Georg Frobenius (1849 – 1917), was a german Mathematician.

[2]I mean here that for every x, there exists y such that $\langle x, y \rangle \neq 0_{\mathbb{K}}$

3. Define $\mu(X^p \otimes X^q) := \binom{p+q}{p} X^{p+q}$. Show that this defines an associative multiplication on C. What is the unit?

4. Show that C is a bialgebra with the product μ and coproduct Δ.

Problem 5. Let C be a $\mathbb{K}$-coalgebra. And let us denote by $C^\star$ the dual of C.

1. (Re)-prove that $C^\star$ is naturally endowed with a structure of algebra.

Solution. In this direction there is no problem even if C is not finite dimensional: If Δ and ϵ where the co-product and the co-unity, we define $\mu_{C^\star}$ and $\eta_{C^\star}$ via:

$$\begin{array}{cccc} \mu_{C^\star}: & C^\star \otimes C^\star & \to & C^\star \\ & f \otimes g & \to & \left\{ \begin{array}{cccc} fg: & C & \to & \mathbb{K} \\ & x & \mapsto & \sum_{(x)} f(x_{(1)})g(x_{(2)}) \end{array} \right. \end{array} \qquad \begin{array}{cccc} \eta_{C^\star}: & \mathbb{K} & \to & C^\star \\ & \lambda & \mapsto & \lambda \cdot \epsilon \end{array}$$

One easily checks that $(C^\star, \mu_{C^\star}, \eta_{C^\star})$ is indeed an associative unital algebra. □

2. Let M be a comodule-C (I mean here a right C-comodule), (re)-prove that M is naturally endowed with a structure of $C^\star$-module.

Solution. By naturally, *I meant here that there is a formula. Let m be an element of M and f an element of $C^\star$, we define:*

$$f \rightharpoonup m = \sum_{(m)} m_{(0)} \cdot f(m_{(1)}) = \sum_{(m)} f(m_{(1)}) \cdot m_{(0)},$$

where $\cdot$ denote the product of m by element of the ground field. Let us show that $f \rightharpoonup (g \rightharpoonup m)$ is equal to $(fg) \rightharpoonup m$:

$$\begin{aligned} f \rightharpoonup (g \rightharpoonup m) &= f \rightharpoonup \left(\sum_{(m)} g(m_{(1)}) \cdot m_{(0)} \right) \\ &= \sum_{(m)} f \rightharpoonup \left(g(m_{(1)}) \cdot m_{(0)} \right) \\ &= \sum_{(m)} g(m_{(1)}) \cdot \left(f \rightharpoonup m_{(0)} \right) \\ &= \sum_{(m)} g(m_{(2)}) f(m_{(1)}) m_{(0)} \\ &= \sum_{(m)} m_{(0)} f(m_{(1)}) g(m_{(2)}) \end{aligned}$$

and

$$\begin{aligned} (fg) \rightharpoonup m &= \sum_{(m)} fg(m_{(1)}) \cdot m_{(0)} \\ &= \sum_{(m)} f(m_{(1)}) g(m_{(2)}) \cdot m_{(0)} \\ &= \sum_{(m)} m_{(0)} f(m_{(1)}) g(m_{(2)}) \end{aligned}$$

One easily checks that $1_{C^\star}$ acts trivially on the M:

$$\begin{aligned} 1 \rightharpoonup m &= \sum_{(m)} \epsilon(m_{(1)}) m_{(0)} \\ &= m. \end{aligned}$$

□

3. From now on M will be a $C^\star$-module. Prove that there exists a natural embedding ι of $M \otimes C$ in $\mathrm{Hom}(C^\star, M)$.

Solution. We define ι by the following formula:

$$\begin{array}{rccl} \iota: & M \otimes C & \to & \mathrm{Hom}(C^\star, M) \\ & m \otimes c & \mapsto & \left\{ \begin{array}{rccl} \iota(m \otimes c): & C^\star & \to & M \\ & f & \mapsto & f(c) \cdot m \end{array} \right. \end{array}.$$

This is clearly an embedding. □

4. Prove that from $C^\star$-module structure of M, one can naturally define a map ρ :. A module such that $\rho(M) \subseteq \iota M \otimes C$ is called a *rational* module.

Solution. The map ρ is defined by:

$$\begin{array}{rccl} \rho: & M & \to & \mathrm{Hom}(C^\star, M) \\ & m & \mapsto & \left\{ \begin{array}{rccl} \rho(m): & C^\star & \to & M \\ & f & \mapsto & f \cdot m \end{array} \right. \end{array}.$$

□

5. Prove that if the $C^\star$-module structure of M is obtained by the construction of question 2, then M is rational.

Solution. Let M be a comodule-C. It is endowed via $\rightharpoonup$ with a structure of $C^\star$-module. Let m be an element of m. Let us compute $\rho(m)$:

$$\rho(m)(f) = f \rightharpoonup m = \sum_{(m)} f(m_{(1)}) \cdot m_{(0)} = \iota(\sum_{(m)} m_{(0)} \otimes m_{(1)})(f).$$

This shows that $\rho(m)$ is in the image of ι. Since this is valid for all m, this shows that M is rational. □

6. Prove that if a $C^\star$-module M is rational, it can be naturally endowed with a comodule-C structure.

Solution. *[This was not so easy.] Of course ρ (almost) provides the structure of comodule-C: let $\pi\mathrm{Hom}(C^\star, M) \to M \otimes C$ be a linear left inverse to ι, then we define $\Delta_M : M \to M \otimes C$ by $\Delta = \pi \circ \rho$. The definition of Δ_M can be sum up by the following formula:*

$$\iota(\Delta_M(m))(f) = f \cdot m = \sum_{(m)} m_{(0)} f(m_{(1)}) \tag{1}$$

Let us first prove that defines a coaction on M. First the counity, we have to show that $(\mathrm{id}_M \otimes \epsilon) \circ \Delta_M = Id_M$. the counity ϵ is the 1 of the algebra $C^\star$, so that we have: $\epsilon \cdot m = m$ for all m in M. On the other hand, by (1), we have: $\epsilon \cdot m = \sum_{(m)} m_{(0)}\epsilon(m_{(1)})$ so that:

$$\sum_{(m)} m_{(0)}\epsilon(m_{(1)}) = m.$$

Let us now prove that $(\Delta_M \otimes \mathrm{id}_C) \circ \Delta_M = (\mathrm{id}_M \otimes \Delta_C) \circ \Delta_M$. Let m be an element of m and f and g be two elements of $C^\star$.

$$\begin{aligned}
(\mathrm{id}_M \otimes f \otimes g) \circ ((\Delta_M \otimes \mathrm{id}_C) \circ \Delta_M)(m) &= (\mathrm{id}_M \otimes f) \circ ((\Delta_M \otimes g) \circ \Delta_M)(m) \\
&= (\mathrm{id}_M \otimes f)(\sum_{(m)} (\Delta_M(m_{(0)}) g(m_{(1)}))) \\
&= (\mathrm{id}_M \otimes f)(\Delta_M(\sum_{(m)} (m_{(0)}) g(m_{(1)}))) \\
&= (\mathrm{id}_M \otimes f)((\sum_{(m)} (m_{(0)}) \otimes m_{(1)} g(m_{(2)}))) \\
&= ((\sum_{(m)} (m_{(0)}) \otimes f(m_{(1)}) g(m_{(2)}))) \\
&= (fg) \cdot m \\
&\textit{and} \\
(\mathrm{id}_M \otimes f \otimes g) \circ ((\mathrm{id}_M \otimes \Delta_C) \circ \Delta_M)(m) &= (\mathrm{id}_M \otimes fg) \circ \Delta_M(m) \\
&= (fg) \cdot m
\end{aligned}$$

I claim that if $x = (\Delta_M \otimes \mathrm{id}_C) \circ \Delta_M(m) - ((\mathrm{id}_M \otimes \Delta_C) \circ \Delta_M)(m) \neq 0$, then it would exist f and g such that $(\mathrm{id}_M \otimes f \otimes g)(x)$ would be non zero. This contradicts the computations we just did. To prove the claim, we suppose that x is non zero, so we might write $x = \sum_{i,j} m_{i,j} \otimes c_i \otimes d_i$ and suppose that the c_i are linearly independent and the d_j are linearly independent, and we might suppose that $m_{1,1}$ is not zero. Then choosing f and g such that $f(c_{i,1}) = \delta_{i,1}$ and $f(c_{j,1}) = \delta_{j,1}$. We find the f and g as we wished.

This proves that C is endowed with a structure of comodule-C.

One can check that the original $C^\star$-module structure and the $\rightharpoonup$-$C^\star$-module structure coincide. $\square$

7. If M is a rational module, prove that $N \subset M$ a sub-vector space is a submodule if and only if $\rho(N) \subseteq \iota(N \otimes C)$.

Solution. *Let us suppose first that that $\rho(N) \subseteq \iota(N \otimes C)$. Let n be an element of N, we write: $\pi \circ \rho(n) = \sum_i n_i \otimes c_i$. Let f be an element of $C^\star$. By definition of ρ, we have:$f \cdot n = \sum_i f(c_i)n_i$, and this shows that N is a sub-module.*

Let us now suppose that $\rho(n) \notin N \otimes C$ for some n in N. We might right $\rho(n) = \sum_i n_i \otimes c_i$ with all the c_i linearly independent and $n_1 \notin N$. We consider $f \in C^\star$ a linear for on C, such that $f(c_i) = \delta_{i1}$. Now we can compute $f \cdot n$:

$$f \cdot n = \sum_i \sum_i f(c_i)n_i = n_1 \notin N.$$

So that N is not a sub-module of M. $\square$