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DER FORSCHUNG | DER LEHRE | DER BILDUNG
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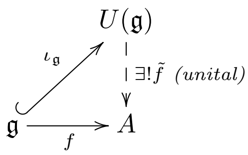
# Sheet 3

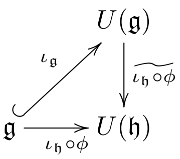
**Problem 1.** Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras over a field $\mathbb{K}$. Recall that the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ was constructed in the lecture as the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal $I \subset T(\mathfrak{g})$ generated by the vectors $x \otimes y - y \otimes x - [x, y]$ with $x, y \in \mathfrak{g}$. The canonical embedding $\iota_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$ was given by the map $x \mapsto x + I$.

1. Show that for every Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ there is a unique morphism $U(\varphi) : U(\mathfrak{g}) \to U(\mathfrak{h})$ of associative algebras, such that $\iota_{\mathfrak{h}} \circ \varphi = U(\varphi) \circ \iota_{\mathfrak{g}}$.

*Solution.* *Let us recall that the universal property of the universal enveloping algebra $(U(\mathfrak{g}), \iota_{\mathfrak{g}})$ of the Lie algebra $\mathfrak{g}$ reads like as follows. For every (unital associative)*[1] *$\mathbb{K}$-algebra, and every morphism (of Lie algebras) $f : \mathfrak{g} \to A$, there exists a unique unital*[2] *morphism (of algebras) $\tilde{f} : U(\mathfrak{g}) \to A$. This can be summarized by the following diagram:*



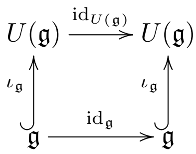

*The algebra $U(\mathfrak{h})$ is unital and associative. The map $\iota_{\mathfrak{h}} \circ \phi : \mathfrak{g} \to U(\mathfrak{h})$ is a Lie algebra map, hence, thanks to the universal property we know that there exist a unital map $U(\phi) := \widetilde{\iota_{\mathfrak{h}} \circ \phi}$ such that the following diagram commutes:*




*This is what we wanted. The uniqueness follows from the fact that, as an algebra, $U(h)$ is generated by $\iota_{\mathfrak{g}}(\mathfrak{g})$ and by 1 and the image hence the images of these element by $U(\phi)$ are determined by the required equality.* ☐

2. Let $\varphi : \mathfrak{g} \to \mathfrak{g}'$ and $\psi : \mathfrak{g}' \to \mathfrak{g}''$ be Lie algebra homomorphisms. Show that the equalities $U(\mathrm{id}_{\mathfrak{g}}) = \mathrm{id}_{U(\mathfrak{g})}$ and $U(\psi \circ \varphi) = U(\psi) \circ U(\varphi)$ hold. (Hint: Use the universal property of the enveloping algebra)

*Solution.* *This says that $U$ is a functor from the category of Lie $\mathbb{K}$-algebra to the category of $\mathbb{K}$-algebra. The diagram*




[1] When not mentioned this hypotheses are implicit.
[2] This means 1 is mapped to 1, and this is NOT an implicit hypothesis!

*commutes and the uniqueness of the previous question implies that* $\mathrm{id}_{U(\mathfrak{g})} = U(\mathrm{id}_{\mathfrak{g}})$*. In the following diagram the two squares commutes:*

$$\begin{array}{ccccc} U(\mathfrak{g}) & \xrightarrow{U(\varphi)} & U(\mathfrak{g}') & \xrightarrow{U(\psi)} & U(\mathfrak{g}'') \\ \uparrow{\scriptstyle \iota_{\mathfrak{g}}} & & \uparrow{\scriptstyle \iota_{\mathfrak{g}'}} & & \uparrow{\scriptstyle \iota_{\mathfrak{g}''}} \\ \mathfrak{g} & \xrightarrow{\varphi} & \mathfrak{g}' & \xrightarrow{\psi} & \mathfrak{g} \end{array}$$

*This implies that the following diagram commutes:*

$$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{U(\psi)\circ U(\varphi)} & U(\mathfrak{g}'') \\ \uparrow{\scriptstyle \iota_{\mathfrak{g}}} & & \uparrow{\scriptstyle \iota_{\mathfrak{g}''}} \\ \mathfrak{g} & \xrightarrow{\psi\circ\phi} & \mathfrak{g} \end{array}$$

*And this gives* $U(\psi) \circ U(\varphi) = U(\psi \circ \varphi)$*, once more by the uniqueness of the first question.* □

3. Show the existence of an isomorphism $U(\mathfrak{g}^{\mathrm{opp}}) \to U(\mathfrak{g})^{\mathrm{opp}}$ of associative algebras. (Hint: Show that $U(\mathfrak{g})^{\mathrm{opp}}$ together with the linear map $\iota : \mathfrak{g}^{\mathrm{opp}} \to U(\mathfrak{g})^{\mathrm{opp}}, x \mapsto x + I$ fulfills the universal property of the enveloping algebra of $\mathfrak{g}^{\mathrm{opp}}$.)

*Solution. As vector spaces* $U(\mathfrak{g})^{\mathrm{opp}}$ *and* $\mathfrak{g}^{\mathrm{opp}}$ *are nothing but identical (I really mean identical, not isomorphic) to* $U(\mathfrak{g})$ *and* $\mathfrak{g}$*. Hence the map* $\iota_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$ *can be regarded as a map from* $\mathfrak{g}^{\mathrm{opp}}$ *to* $U(\mathfrak{g}^{\mathrm{opp}})$*. We will show that the pair* $(U(\mathfrak{g})^{\mathrm{opp}}, \iota_{\mathfrak{g}})$ *satisfies the universal property of the universal enveloping algebra for* $\mathfrak{g}^{\mathrm{opp}}$*. This will implies that there exists a unique isomorphism* $\lambda : U(\mathfrak{g}^{\mathrm{opp}}) \to U(\mathfrak{g})^{\mathrm{opp}}$ *such that* $\lambda \circ \iota_{\mathfrak{g}^{\mathrm{opp}}} = \iota_{\mathfrak{g}}$*. Let* $A$ *be a* $\mathbb{K}$*-algebra and* $f : \mathfrak{g}^{\mathrm{opp}} \to A$ *a (Lie algebra) map. This is as well a map of Lie algebra from* $\mathfrak{g}$ *to* $A^{\mathrm{opp}}$*, hence there exists a unital map of algebra* $\tilde{f} : U(\mathfrak{g}) \to A^{\mathrm{opp}}$ *such that* $\tilde{f} \circ \iota_g = f$*. The map* $\tilde{f}$ *can be regarded as a map from* $U(\mathfrak{g})^{\mathrm{opp}} \to A$*. Hence we have the following commutative diagram:*

$$\begin{array}{ccc} & & U(\mathfrak{g})^{\mathrm{opp}} \\ & \nearrow^{\iota_{\mathfrak{g}}} & \Big\downarrow {\scriptstyle \exists! \tilde{f} \ (unital)} \\ \mathfrak{g}^{\mathrm{opp}} & \xrightarrow[f]{} & A \end{array}$$

*This proves that* $U(\mathfrak{g})^{\mathrm{opp}}$ *fulfills the universal property of the universal enveloping algebra* $\mathfrak{g}^{\mathrm{opp}}$*.* □

**Problem 2.** Let $G$ be a finite group, $\mathbb{C}[G]$ its associated $\mathbb{C}$-algebra. A $\mathbb{C}[G]$-module is also called a representation of $G$ (:= Darstellung von $G$).

1. Let $M$ be a finite dimensional $\mathbb{C}[G]$-module. Prove that the $\mathbb{C}[G]$-module structure of $M$ induces a group homomorphism $\rho_M : G \to \mathrm{End}(M)$. Prove the reciprocal statement: if $V$ is a vector space and $\rho : G \to \mathrm{End}(V)$ a group homomorphism, prove that we can endow $V$ with a structure of $\mathbb{C}[G]$-module.

*Solution. Easy.* □

2. (Sorry there were a few typos in this questions) Let $M$ be a finite dimensional $\mathbb{C}[G]$-module and $N$ a sub-module of $M$. Let us consider $N'$ a supplement of $M$ as a vector space (in general $N'$ is NOT a $\mathbb{C}[G]$-module), and denote $p$ the projector from $M$ to $N$ along $N'$. By using the map

$$\pi := \frac{1}{\#G} \sum_{g \in G} \rho_M(g) \circ p \circ \rho_M(g)^{-1},$$

prove[3] that we can find a submodule $N''$ of $M$ such that $M = N \oplus N''$.

*Solution. Let us first prove that $\pi$ is a projector on $N$: for all $x \in M$, we have:*

$$\begin{aligned}
\pi \circ \pi(x) &= \frac{1}{\#G^2} \sum_{g_1, g_2 \in G} \rho_M(g_1) \circ p \circ \rho_M(g_1)^{-1} \circ \rho_M(g_2) \circ p \circ \rho_M(g_2)^{-1}(x) \\
&= \frac{1}{\#G^2} \sum_{g_1, g_2 \in G} \rho_M(g_1) \circ p(\rho_M(g_1)^{-1} \circ \rho_M(g_2) \circ p \circ \rho_M(g_2)^{-1}(x)) \\
&= \frac{1}{\#G^2} \sum_{g_1, g_2 \in G} \rho_M(g_1) \circ (\rho_M(g_1)^{-1} \circ \rho_M(g_2) \circ p \circ \rho_M(g_2)^{-1}(x)) \\
&= \frac{1}{\#G^2} \sum_{g_1, g_2 \in G} \rho_M(g_2) \circ p \circ \rho_M(g_2)^{-1}(x)) \\
&= \frac{1}{\#G} \sum_{g_2 \in G} \rho_M(g_2) \circ p \circ \rho_M(g_2)^{-1}(x)) \\
&= \pi(x).
\end{aligned}$$

*So that $\pi$ is a projector. It's image is clearly contained in $N$ and as its trace is equal to the trace of $p$ it's image is exactly $N$. Let us now show that it is a $\mathbb{C}[G]$-module map. It is enough to show that $\pi$ commutes with $\rho_M(h)$ for every $h$ in $G$. We have indeed:*

$$\begin{aligned}
\pi \circ \rho_M(h) &= \frac{1}{\#G} \sum_{g \in G} \rho_M(g) \circ p \circ \rho_M(g)^{-1} \circ \rho(h) \\
&= \frac{1}{\#G} \sum_{g \in G} \rho_M(g) \circ p \circ \rho_M(g^{-1}h)^{-1} \\
&= \frac{1}{\#G} \sum_{g \in G} \rho_M(g) \circ p \circ \rho_M(g^{-1}h) \\
&= \frac{1}{\#G} \sum_{g \in G} \rho_M(g) \circ p \circ \rho_M(h^{-1}g)^{-1} \\
&= \frac{1}{\#G} \sum_{g' = h^{-1}g \in G} \rho_M(hg') \circ p \circ \rho_M(g')^{-1} \\
&= \frac{1}{\#G} \sum_{g' = h^{-1}g \in G} \rho_M(h) \circ \rho_M(g') \circ p \circ \rho_M(g')^{-1} \\
&= \rho_M(h) \circ \pi.
\end{aligned}$$

*The projector $\pi$ is a $\mathbb{C}[G]$-module map, hence $N'' := \ker \pi$ is a $\mathbb{C}[G]$-module (why ?), and we have $M = N \oplus N'$.* □

[3] If $A$ is an algebra, we say that a $A$-module $N$ is *simple* if $N$ does not contain non-trivial sub-modules. And that an object is *indecomposable* if it cannot be expressed as a direct sum of two sub-modules. This question shows that in the case of group algebras for finite groups, these two notions coincide (why?), this is NOT true in general.

3. Let $M_1$ and $M_2$ be two simple $\mathbb{C}[G]$-module and $f : M_1 \to M_2$ a morphism of $\mathbb{C}[G]$-modules. Suppose that $f$ is different from 0. Prove that $M_1$ and $M_2$ are isomorphic.

   *Solution. The kernel and the image of $f$ are submodules of $M_1$ and $M_2$, but this two modules are simple, hence* $\ker f = \{0\}$ *or* $\ker f = M_1$ *and Im $f = \{0\}$ or Im $f = M_2$. As $f$ is non zero we have:* $\ker f = \{0\}$ *and Im $f = M_2$, so that $f$ is an isomorphism.* □

4. With the same notations and the same hypothesis as the previous question, and by considering the eigenvalues of $f$, prove that $f$ is an homothety (that is a multiple of the identity)[4].

   *Solution. The question is not completely clear (sorry): since $M_1$ and $M_2$ are different (isomorphic bu different!), one cannot speak about the identity morphism. So we have to suppose that $f$ is an endomorphism of $M_1$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $f$, $f - \lambda \mathrm{id}_{M_1}$ is a $\mathbb{C}[G]$-module map. Hence its kernel has to be $\{0\}$ or $M_1$, since it is not $\{0\}$, it is $M_1$ and $f$ is an homothety.* □

**Problem 3** (Burau representations of the braid group)**.** We consider $B_n$ the braid group on $n$ strands and with its standard generators $(\sigma_i)_{1\leq i\leq n-1}$. Let $t$ be a non-zero complex number.

1. Prove that the following data yields a well-defined complex $n$-dimensional representation of $B_n$:

$$\sigma_i \mapsto \begin{pmatrix} I_{i-1} & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & I_{n-i-1} \end{pmatrix}$$

   It is called the *Burau*[5] *representation* of the braid group.

2. Prove that this representation is not irreducible (look for a common eigenvector).

3. Let us denote by $b_0, b_2, \dots b_{n-1}$ the standard basis of $\mathbb{C}^n$. Prove that the $(n-1)$-dimensional space spanned by $(t^i b_i - t^{i+1} b_{i+1})_{0\leq i\leq n-2})$ is invariant by the action of $B_n$. This is a new representation of the braid group called *reduced Burau representation* of the braid group.

4. Compute the matrix associated to $\sigma_i$ by the reduced Burau representation in the given base.

[4] This is Schur's lemma. Schur (1875 – 1945) was a German mathematician.
[5] Werner Burau (1906 – 1994 ) was a german mathematician and was professor in Hamburg.