Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
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Hopf algebras
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Sheet 3

Problem 1. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras over a field $\mathbb{K}$. Recall that the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ was constructed in the lecture as the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal $I \subset T(\mathfrak{g})$ generated by the vectors $x \otimes y - y \otimes x - [x, y]$ with $x, y \in \mathfrak{g}$. The canonical embedding $\iota_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$ was given by the map $x \mapsto x + I$.

1. Show that for every Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ there is a unique morphism $U(\varphi) : U(\mathfrak{g}) \to U(\mathfrak{h})$ of associative algebras, such that $\iota_{\mathfrak{h}} \circ \varphi = U(\varphi) \circ \iota_{\mathfrak{g}}$.

2. Let $\varphi : \mathfrak{g} \to \mathfrak{g}'$ and $\psi : \mathfrak{g}' \to \mathfrak{g}''$ be Lie algebra homomorphisms. Show that the equalities $U(\mathrm{id}_{\mathfrak{g}}) = \mathrm{id}_{U(\mathfrak{g})}$ and $U(\psi \circ \varphi) = U(\psi) \circ U(\varphi)$ hold. (Hint: Use the universal property of the enveloping algebra)

3. Show the existence of an isomorphism $U(\mathfrak{g}^{\mathrm{opp}}) \to U(\mathfrak{g})^{\mathrm{opp}}$ of associative algebras. (Hint: Show that $U(\mathfrak{g})^{\mathrm{opp}}$ together with the linear map $\iota : \mathfrak{g}^{\mathrm{opp}} \to U(\mathfrak{g})^{\mathrm{opp}}, x \mapsto x + I$ fulfills the universal property of the enveloping algebra of $\mathfrak{g}^{\mathrm{opp}}$.)

Problem 2. Let G be a finite group, $\mathbb{C}[G]$ its associated $\mathbb{C}$-algebra. A $\mathbb{C}[G]$-module is also called a representation of G (:= Darstellung von G).

1. Let M be a finite dimensional $\mathbb{C}[G]$-module. Prove that the $\mathbb{C}[G]$-module structure of M induces a group homomorphism $\rho_M : G \to \mathrm{End}(M)$. Prove the reciprocal statement: if V is a vector space and $\rho : G \to \mathrm{End}(V)$ a group homomorphism, prove that we can endow V with a structure of $\mathbb{C}[G]$-module.

2. Let M be a finite dimensional $\mathbb{C}[G]$-module and N a sub-module of N. Let us consider N' a supplement of M as a vector space (in general N' is NOT a $\mathbb{C}[G]$-module), and denote $p : M \to N'$ the projection on N'. By using the map
$$\pi := \frac{1}{\#G} \sum_{g \in G} \rho_M(g) \circ p \circ \rho_M(g)^{-1},$$
prove[1] that we can find a submodule N'' of M such that $M = N \oplus N''$.

3. Let M_1 and M_2 be two simple $\mathbb{C}[G]$-module and $f : M_1 \to M_2$ a morphism of $\mathbb{C}[G]$-modules. Suppose that f is different from 0. Prove that M_1 and M_2 are isomorphic.

4. With the same notations and the same hypothesis as the previous question, and by considering the eigenvalues of f, prove that f is an homothety (that is a multiple of the identity)[2].

Problem 3 (Burau representations of the braid group)**.** We consider B_n the braid group on n strands and with its standard generators $(\sigma_i)_{1 \leq i \leq n-1}$. Let t be a non-zero complex number.

[1] If A is an algebra, we say that a A-module N is *simple* if N does not contain non-trivial sub-modules. And that an object is *indecomposable* if it cannot be expressed as a direct sum of two sub-modules. This question shows that in the case of group algebras for finite groups, these two notions coincide (why?), this is NOT true in general.

[2] This is Schur's lemma. Schur (1875 – 1945) was a German mathematician.

1. Prove that the following data yields a well-defined complex n-dimensional representation of B_n:

$$\sigma_i \mapsto \begin{pmatrix} I_{i-1} & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & I_{n-i-1} \end{pmatrix}$$

   It is called the *Burau*[3] *representation* of the braid group.

2. Prove that this representation is not irreducible (look for a common eigenvector).

3. Let us denote by $b_0, b_2, \dots b_{n-1}$ the standard basis of $\mathbb{C}^n$. Prove that the $(n-1)$-dimensional space spanned by $(t^i b_i - t^{i+1} b_{i+1})_{0 \le i \le n-2})$ is invariant by the action of B_n. This is a new representation of the braid group called *reduced Burau representation* of the braid group.

4. Compute the matrix associated to σ_i by the reduced Burau representation in the given base.

[3]Werner Burau (1906 – 1994) was a german mathematician and was professor in Hamburg.