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Hopf algebras
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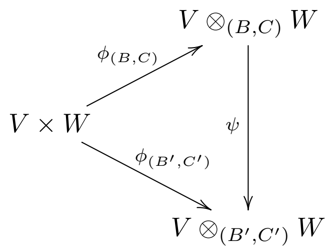
Sheet 1

Problem 1 (Tensor product)**.** Let $\mathbb{K}$ be a field, V and W be two $\mathbb{K}$-vector spaces. We consider $B = (b_i)_{i\in I}$ and $C = (c_j)_{j\in J}$ bases of V and W. Let us denote by $V \otimes_{(B,C)} W$ the $\mathbb{K}$-vector space spanned by the set $(b_i, c_j)_{i\in I, j\in J}$ and by $\phi_{(B,C)} : V \times W \to V \otimes_{(B,C)} W$ the bilinear map defined by $\phi_{(B,C)}(b_i, c_j) = (b_i, c_j)$.

1. Show that if B' and C' are other bases for V and W, there exists a unique isomorphism of vector spaces $\psi : V \otimes_{(B,C)} W \to V \otimes_{(B',C')} W$ such that the following diagram commutes:



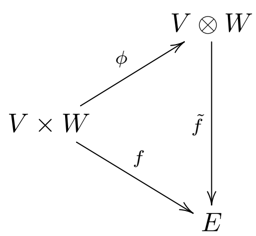
From now on, the symbol $V \otimes W$ denotes the vector space $V \otimes_{(B,C)} W$ for some arbitrary but fixed bases B and C. If (x, y) is an element of $V \times W$, the symbol $x \otimes y$ denotes the image of (x, y) by $\phi_{(B,C)}$ and is called an *elementary tensor*. In the following we write ϕ instead of $\phi_{(B,C)}$,. If we want to emphasize the ground field, we might write $V \otimes_{\mathbb{K}} W$ and $x \otimes_{\mathbb{K}} y$.

By definition $(b_i, c_j)_{i\in I, j\in J}$ is a base of $V \otimes W$, hence to define a morphism from $V \otimes_{(B,C)} W$ to $V \otimes_{(B',C')} W$, we only need to set the image of (b_i, c_j) for every $(i, j) \in I \times J$. As we want the diagram to commute we have to (this gives uniqueness) set $\psi(b_i, c_j) = \phi_{(B',C')}(b_i, c_j)$. It is routine to check that the diagram commutes. Note that the uniqueness of ψ allows us to speak about $x \otimes y$ without specifying in which space we consider it.

2. If V and W are finite dimensional, what is the dimension of $V \otimes W$?

It is of course the product of the dimension of V and W.

3. Prove that, for every $\mathbb{K}$-vector space E and for every bilinear map f from $V \times W$, there exists a unique linear map $\tilde{f}$ such that the following diagram commutes:



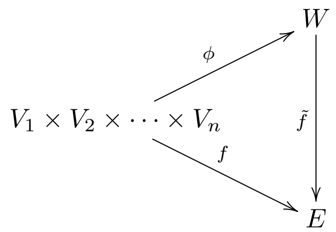
The bilinear map f is completely determined by the image of a of $B \times C$. This suggests to define $\tilde{f}$ by setting the images of $(b \otimes c)$ to be equal to $f((b, c))$ for b in B and c in C. It is easy to check that the diagram indeed commutes. The uniqueness is clear.

4. Prove that the property given in the previous question determines the pair $(V \otimes W, \phi)$ up to a unique isomorphism (meaning that if a pair (U, ρ) satisfies the property, then there exists a unique isomorphism π from $V \otimes W$ to U such that $\phi = \pi \circ \rho$.

   Let (U, ρ) be a pair such that for every vector space E and every bilinear map $f : V \times W \to E$ there exists a linear map $\hat{f} : U \to E$ such that $f = \hat{f} \circ \rho$. We can apply this property to the vector space $V \otimes W$ and the bilinear map ϕ. We find a (unique) map $\hat{\phi}$ such that $\phi = \hat{\phi} \circ \rho$. The map $\hat{\phi}$ is an isomorphism: indeed, if we apply the property of $V \otimes W$ to the vector space U and the bilinear map ρ we obtain a linear map $\tilde{\rho} : V \otimes W \to U$ such that: $\rho = \tilde{\rho} \circ \phi$. Using the uniqueness of the property twice, we obtain that $\tilde{\rho} \circ \hat{\phi} = \mathrm{id}_{V \otimes W}$ and $\hat{\phi} \circ \tilde{\rho} = \mathrm{id}_U$

5. Generalizing the previous questions, define the tensor product of a finite collection of vector spaces.

   Let $V_1, V_2, \ldots, V_n$ be a collection of $\mathbb{K}$-vector spaces, and suppose that there exist a vector space, W and a n-linear map ϕ, such that for every $\mathbb{K}$-vector space E and every n-linear map $f : V_1 \times V_2 \times \times \ldots V_n : E$, there exists a linear map $\tilde{f} : W \to E$ such that the following diagram commutes:



   The argument of last question shows that such a pair (W, ϕ) is unique. For the existence, this is clear: if $B_1, B_2, \ldots B_n$ are bases of $V_1, V_2, \ldots, V_n$ we consider the vector space spanned by the element of $B_1 \times B_2 \times \ldots B_n$ and the map ϕ is defined exactly as in the first question.

6. Suppose that V and W are finite dimensional, prove that $W^\star \otimes V$ is "canonically" isomorphic to $\mathrm{Hom}(W, V)$. This means that every linear map from W to V can be expressed as a finite linear combination of elementary tensors.

   Actually we only need W to be finite dimensional. Let us define χ the isomorphism between $W^\star \otimes V$ and $\mathrm{Hom}(W, V)$:

$$\begin{array}{rccc} \chi : & W^\star \otimes V & \to & \mathrm{Hom}(W, V) \\ & f \otimes v & \mapsto & (W \ni x \mapsto f(x)v \in V). \end{array}$$

   To prove that χ is an isomorphism, we exhibit its inverse: let $B = (b_1, \ldots b_k)$ be a base of W and $B^\star = (b_1^\star, \ldots, b_k^\star)$ its dual base, then we define:

$$\begin{array}{rccc} \chi^{-1} : & \mathrm{Hom}(W, V) & \to & W^\star \otimes V \\ & (g : W \to V) & \mapsto & \sum_{i=1}^{k} b_i^\star \otimes g(b_i). \end{array}$$

   One easily checks that these two morphisms are inverse from each others.

7. If V is finite dimensional and if g is an endomorphism of V, write a formula for the trace of g tanks to the identification of $\mathrm{End}(V)$ with $V^\star \otimes V$.

   Let g be an endomorphism of V. We want to express the trace of g thanks to $\chi^{-1}(g)$. If $\chi^{-1}(g) = \sum_l f_l \otimes v_l$ we claim that $\mathrm{tr}(g) = \sum_l f_l(v_l)$. Indeed the right-hand side of this formula defines a linear form from $W^\star \otimes V$ to $\mathbb{K}$, and it agrees with the trace on the base of $W^\star \otimes V$ given by $b_i^\star \otimes b_j$.

8. Let V_1, V_2, W_1 and W_2 be four $\mathbb{K}$-vector spaces, let $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$ two linear maps. Use the question 3 to define a "natural" linear map $f_1 \otimes f_2 : V_1 \otimes W_1 \to V_2 \otimes W_2$. If M_1 and M_2 are matrices of f_1 and f_2 in some bases, describe a matrix representing $f_1 \otimes f_2$ in some appropriate bases.

   In order to use the question 3, we should find a "natural" bilinear map from $V_1 \times V_2$ to $W_1 \otimes W_2$. The composition of the map $\widetilde{f_1 \times f_2} : V_1 \otimes V_2 \to W_1 \times W_2$ with the map $\phi : W_1 \times W_2 \to W_1 \otimes W_2$ is exactly what we want, and this defines a linear map from $V_1 \otimes V_2$ to $W_1 \otimes W_2$. Furthermore one easily checks that:

   $$f_1 \otimes f_2(x \otimes y) = f_1(x) \otimes f_2(y).$$

   Let us fix some bases B_1, B_2, C_1 and C_2 for V_1, V_2, W_1 and W_2. A base of $V_1 \otimes V_2$ is given by $B_1 \times B_2$ with the lexicographical order and a base of $W_1 \otimes W_2$ is given by $C_1 \times C_2$ with the lexicographical order. If V_1 and V_2 have dimension m_1 and m_2 and W_1 and W_2 have dimension n_1 and n_2, the matrix M_1 of f_1 has size $m_1 \times n_1$, the matrix M_2 of f_2 has size $m_2 \times n_1$ and the martix $M_{1 \otimes 2}$ of $f_1 \otimes f_2$ has dimension $m_1 m_2 \times n_1 n_2$ and is obtained by replacing every entry λ of the matrix M_1 by the matrix $\lambda \cdot M_2$.

Problem 2 (Group algebra)**.** Let $\mathbb{K}$ be a field and G a group. Let $\mathbb{K}[G]$ denote the $\mathbb{K}$-vector space with basis G.

1. Show that the multiplication of the group G induces a multiplication on $\mathbb{K}[G]$ making this vector space an (associative) $\mathbb{K}$-algebra. It is called the group algebra of G. Is $\mathbb{K}[G]$ unital? For which groups G is the algebra $\mathbb{K}[G]$ comutative?

   The multiplication of $\mathbb{K}[G]$ is commutative if and only if the group G is abelian.

2. Let n be a positive integer and let us denote by A_n the set of matrices with shape

   $$\begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_2 & \dots & \dots & a_n & a_1 \end{pmatrix}$$

   for $a_1, \dots a_n$ elements of $\mathbb{K}$. Prove that A_n is an algebra isomorphic to a group algebra.

   It is isomorphic to the algebra $\mathbb{K}[\mathbb{Z}/n\mathbb{Z}]$: an isomorphism is completely determined by the data:

   $$1 \cdot \bar{1} \mapsto \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

3. We denote by $\mathbb{K}[X^{\pm 1}]$ the set of Laurent polynomials over $\mathbb{K}$. It is defined by the following formula:

$$\mathbb{K}[X^{\pm 1}] = \{f(X) \in \mathbb{K}(X) | \exists l \in \mathbb{N} \text{ such that } X^l f(X) \in \mathbb{K}[X]\}$$

Prove that $\mathbb{K}[X^{\pm}]$ is isomorphic to a group algebra.

It is isomorphic to $\mathbb{K}[\mathbb{Z}]$ via sending $\lambda \cdot k$ to λX^k.

4. Suppose that G is finite of order n and $\mathbb{K}$ is of characteristic 0. Show that $\mathbb{K}[G]$ decomposes as a direct sum of an ideal of dimension $n-1$ and an ideal of dimension 1.

We consider the map

$$\begin{array}{rccc} \phi : & \mathbb{K}[G] & \to & \mathbb{K} \\ & \sum_{g \in G} \lambda_g g & \mapsto & \sum_g \lambda_g. \end{array}$$

This is a morphism of algebras, hence its kernel is an ideal. It is clear that it has dimension $n-1$. The other ideal we are looking for is generated by $\sum_{g \in G} g$ and consists of the element of $\mathbb{K}[G]$ of the form $\lambda \sum_{g \in G} g$ with λ element of $\mathbb{K}$. They are in direct sum since every element of $\mathbb{K}[G]$ can be decomposed (and the dimensions fit):

$$\sum_g \lambda_g g = \left(\sum_g \lambda_g g - \frac{1}{n} \left(\sum_g \lambda_g \right) \left(\sum_g g \right) \right) + \frac{1}{n} \left(\sum_g \lambda_g \right) \left(\sum_g g \right).$$

Problem 3. We consider the category $\mathcal{Z}$ given by the following data whose objects are compact oriented 0-manifold (ie collection of points with signs) and whose hom-sets are given by the following formula:

$$\mathrm{Hom}(N_1, N_2) = \begin{cases} \{f_{M_1, M_2}\} & \text{if there exists a compact oriented 1-manifold } W \text{ such that } \partial W \text{ is diffeomorphic to } -M_1 \sqcup M_2, \\ \emptyset & \text{else.} \end{cases}$$

1. Prove that every morphism is an isomorphism.

Done during the exercise session.

2. Give the isomorphism classes of the category $\mathcal{Z}$.

If N is a compact oriented 0-dimensional manifold, let us denote by $n(N) \in \mathbb{Z}$ the number of point of N counted with orientation (ie the number of positive points minus the number of negative points). We claim that N_1 and N_2 are isomorphic if and only if $n(N_1) = n(N_2)$. First, let us consider two objects N_1 and N_2, such that $n(N_1) = n(N_2)$. The manifold $-N_1 \sqcup N_2$ has as many negative points as positive points because $n(-N_1 \sqcup N_2) = -n(N_1) + n(N_2) = 0$, we label them m_i and p_i for $1 \leq i \leq k$. The manifold $\bigsqcup_{i=1}^{k} [0,1]$ and the obvious identification of the boundary with $-N_1 \sqcup N_2$ proves that N_1 and N_2 are equivalent. The other direction is similar: let N_1 and N_2 be two isomorphic objects. It means that there exists W a compact oriented 1-manifold such that ∂W is diffeomorphic to $-N_1 \sqcup N_2$ but W is a collection of circles and intervals, in particular its boundary has as many positive points as negatives points. This gives $n(N_1) = n(N_2)$.

3. What happens if we consider the same category but without the orientability / orientation conditions ?

   This leads to $\mathbb{Z}/2\mathbb{Z}$.

Problem 4. Let n be an integer greater than or equal to 2. In this problem, we will prove that the symmetric group S_n has the following presentation[1]:

$$\left\langle \tau_1, \ldots, \tau_{n-1} \;\middle|\; \begin{array}{ll} \tau_i^2 = 1 & \text{for } 1 \le i \le n-1, \\ \tau_i \tau_j = \tau_j \tau_i & \text{for } 1 \le i, j \le n-1 \text{ and } |i-j| \ge 2, \\ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} & \text{for } 1 \le i \le n-2. \end{array} \right\rangle$$

For the moment let us denote by G_n the group given by the presentation.

1. Prove that there exists a surjective homomorphism φ from G_n to S_n.

   Done in the Exercise session.

2. We want to prove that φ is an isomorphism. Why is it enough to show that the order or $G_n \le n!$?

   Done in the Exercise session.

3. Prove that every element of G_n has an expression as a product of τ_i with at most one τ_{n-1}.

   This is done by induction on n. If $n = 2$ this is clear since in this case $G_n \simeq \mathbb{Z}/2\mathbb{Z}$. Suppose now that the statement holds for n. Let σ be an element of G_{n+1} and let w be a word in the τ_i such that $\bar{w} = \sigma$ and such that the number of τ_n is minimal. Suppose that this number is greater than or equal to 2. Then we can write: $w = w_1 \tau_n w_2 \tau_n$ with w_2 a word in the letter $\tau_1, \ldots, \tau_{n-1}$. The word w_2 represent an element of G_n and this element has an expression w_2' as a product of τ_i with at most one τ_{n-1}. Hence we can write: $w = w_1 \tau_n w_2' \tau_n$. Now using the relations of the presentation of G_n, we found a contradiction (done in the exercise session).

4. Consider the canonical injection $\iota : G_{n-1} \hookrightarrow G_n$ and prove that the sets

   $$\iota(G_{n-1}),\ \tau_{n-1}\iota(G_{n-1}),\ \tau_{n-2}\tau_{n-1}\iota(G_{n-1}),\ \ldots, \tau_{n-i} \cdots \tau_{n-1}\iota(G_{n-1}),\ \ldots,\ \tau_1 \cdots \tau_{n-1}\iota(G_{n-1})$$

   cover G_n.

5. Conclude.

[1] It is the same presentation as the one of the braid group B_n with the additional relations $\tau_i^2 = 1$. One says that B_n is the Artin group and S_n the Coxeter group of the same Coxeter system.