Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
Mathematics department

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Hopf algebras
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Sheet 12

Problem 1. Let H be a finite dimensional Hopf algebra with an invertible antipode. We Set $E = \mathrm{End}(H)$.

1. Explain how to identify $H \otimes H^*$ and E (as vector spaces). Explain how to identify $E \otimes E$ with $\mathrm{End}(H \otimes H)$.
2. Prove that E endowed with the Δ_E and the convolution (that we denote μ_E or $*$) is a bialgebra.
3. Prove that the following formula define an antipode on E:
$$S_E(f)(x) = \sum_{(x)} S(x_{(1)})(S \circ f \circ S^{-1}(x_{(2)}))x_{(3)}.$$
4. Prove that the maps $p_H : E \to H$ and $p_{H^*} : E \to H^{*\mathrm{cop}}$ defined by:
$$p_H(f) = f(1) \qquad \text{and} \qquad p_{H^*}(f) = \epsilon \circ f$$
are morphisms of Hopf algebras.
5. Prove that the map $p_H \otimes p_{H^*} \circ \Delta_E$ is the identification of the first question.
6. Prove that the following equality holds: $r * \mu_E = \mu_E^{\mathrm{op}} * r$ where $*$ denote the convolution on $E \otimes E$.
7. Prove that the dual of E is naturally isomorphic to $D(H)$.

Problem 2. Let A and B be Hopf algebras over a field $\mathbb{K}$. A Hopf-pairing is a linear map $\sigma : A \otimes B \to \mathbb{K}$ such that for all $a, a' \in A$ and $b, b' \in B$

$$\sigma(aa' \otimes b) = \sigma(a \otimes b_{(2)}) \cdot \sigma(a' \otimes b_{(1)}) \qquad \sigma(1 \otimes b) = \epsilon(b)$$
$$\sigma(a \otimes bb') = \sigma(a_{(1)} \otimes b) \cdot \sigma(a_{(2)} \otimes b') \qquad \sigma(a \otimes 1) = \epsilon(a)$$

1. Prove that $A \otimes B$ becomes an associative, unital algebra with unit $1_A \otimes 1_B$, if we set
$$(a \otimes b)(a' \otimes b') := \sigma(a'_{(1)}\, S(b_{(1)})) \cdot \sigma(a'_{(3)} \otimes b_{(3)}) aa'_{(2)} \otimes b_{(2)}b'.$$
2. Show that $A \otimes B$ becomes a Hopf algebra with
$$\Delta(a \otimes b) := a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)} \qquad \epsilon(a \otimes b) := \epsilon(a) \cdot \epsilon(b)$$
$$S(a \otimes b) := \sigma(a_{(1)} \otimes b_{(1)}) \cdot \sigma(a_{(3)} \otimes S(b_{(3)}))S(a_{(2)}) \otimes S(b_{(2)}).$$
3. Let H be a finite dimensional Hopf algebra over $\mathbb{K}$. Show that the evaluation $V^* \otimes V \to \mathbb{K}$ defines a Hopf pairing $\sigma : (H^{\mathrm{cop}})^* \otimes H \to \mathbb{K}$.

Problem 3. Let G be a finite group and $D(G)$ the Drinfel'd double of the group (Hopf) algebra $\mathbb{K}[G]$ over a field $\mathbb{K}$. Assume also $|G| \nmid \mathrm{char}\, \mathbb{K}$. Due to this assumption, the category $\mathcal{C}$ of finite dimensional left $D(G)$-modules over $\mathbb{K}$ can be shown to be semisimple.[1]

1. Determine the isomorphism classes of simple objects in $\mathcal{C}$ for an abelian group G.
2. Determine the isomorphism classes of simple objects in $\mathcal{C}$ for $G = S_3$, the symmetric group on three letters.
3. Determine the isomorphism classes of simple objects in $\mathcal{C}$ for general G.

[1]Equivalently one can consider the category ${}_H\mathcal{YD}^H$ of Yetter Drinfel'd module (left module and right comodule), where the compatibility condition reads: $h_{(1)}v_0 \otimes h_{(2)}v_1 = (h_{(2)}v)_{(0)} \otimes (h_{(2)}v)_{(1)}h_{(1)}$.