Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
Mathematics department

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Hopf algebras
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Sheet 11

Problem 1. We consider the $\mathbb{C}$-algebra H generated by C and X with the relations:

$$C^2 = 1, \quad X^2 = 0 \quad \text{and} \quad CX + XC = 0.$$

1. Show that setting $\Delta(C) = C \otimes C$ and $\Delta(X) = 1 \otimes X + X \otimes C$ yields a well-defined Hopf-algebra.
2. What is the order of S?
3. Show that
$$R = \frac{1}{2}(1 \otimes 1 + 1 \otimes C + C \otimes 1 - C \otimes C) + \frac{1}{2}(X \otimes X + X \otimes CX + CX \otimes CX - CX \otimes X)$$
is a universal R-matrix.
4. Deform R_1 into R_q with q in $\mathbb{C}$ to obtain a one parameter family of universal R-matrices.
5. Relate R_q^{-1} and R_q.

Problem 2. Let H be a quasi-triangular Hopf algebra with R-matrix $R = \sum_{(R)} R_{(1)} \otimes R_{(2)}$. Let X be a right H-module and define $\delta : X \to X \otimes H$ by

$$v \mapsto \sum_R vR_{(1)} \otimes R_{(2)}.$$

1. Show that (X, δ) is a right H-comodule.
2. Show that the right action and the right coaction on X fulfill the (right-right) Yetter-Drinfeld condition:
$$\begin{aligned}&(\mathrm{id}_X \otimes \mu)(\tau_{H,X} \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes (\delta\rho))(\tau_{X,H} \otimes \mathrm{id}_H)(\mathrm{id}_X \otimes \Delta)\\ =&(\rho \otimes \mu)(\mathrm{id}_X \otimes \tau_{H,H} \otimes \mathrm{id}_H)(\delta \otimes \Delta).\end{aligned}$$

Problem 3. Let H be a bialgebra in a strict braided category $\mathcal{C}$ with braiding c, i.e. H is equipped with an algebra and a coalgebra structure which are compatible in the following way

$$\Delta\mu = (\mu \otimes \mu)(\mathrm{id} \otimes c_{H,H} \otimes \mathrm{id})(\Delta \otimes \Delta), \quad \Delta \circ \eta = \eta \otimes \eta, \quad \epsilon\mu = \epsilon \otimes \epsilon, \quad \epsilon\eta = \mathrm{id}_1.$$

A right-right Yetter-Drinfeld module over H is an object X in $\mathcal{C}$ together with an (associative, unital) action $\rho : X \otimes H \to X$ and a (coassociative, counital) coaction $\delta : X \to X \otimes H$ such that

$$\begin{aligned}&(\mathrm{id}_X \otimes \mu)(c_{H,X} \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes (\delta\rho))(c_{X,H} \otimes \mathrm{id}_H)(\mathrm{id}_X \otimes \Delta)\\ =&(\rho \otimes \mu)(\mathrm{id}_X \otimes c_{H,H} \otimes \mathrm{id}_H)(\delta \otimes \Delta).\end{aligned}$$

1. Assume that H is a Hopf algebra, i.e. there is a morphism $S : H \to H$ such that
$$\mu(S \otimes \mathrm{id})\Delta = \eta\epsilon = \mu(\mathrm{id} \otimes S)\Delta.$$
Show that X is a Yetter-Drinfeld module, if and only if
$$\begin{aligned}\delta\rho =&(\mathrm{id}_X \otimes \mu)(c_{H,X} \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes \rho \otimes \mu)(S \otimes \mathrm{id}_X \otimes c_{H,H} \otimes \mathrm{id}_H)\\ &(\mathrm{id}_H \otimes \delta \otimes \Delta)(c_{X,H} \otimes \mathrm{id}_H)(\mathrm{id}_X \otimes \Delta)\end{aligned}$$

2. Let H be a Hopf-algebra. Show that H is a Yetter-Drinfeld module with $\delta := \Delta$ and $\rho := \mu(S \otimes \mu)(c_{H,H} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)$.
   Hint: The following equality holds $(S \otimes S) \circ \Delta = c_{H,H}^{-1} \circ \Delta \circ S$.

Problem 4. Let $\mathbb{K}$ be a field and let H, L be two bi-algebras over $\mathbb{K}$ and $\phi : H \to L$ a morphism of bialgebras. Denote by H-Mod resp. L-Mod the category of left modules over H resp. L and by Comod-H resp. Comod-L the category of right H resp. L comodules.

1. Show that ϕ induces a functor $\Phi : L\text{-mod} \to H\text{-mod}$.
2. Show that the functor Φ is strict monoidal.
3. Show that ϕ induces a functor $\Psi : \text{comod-}H \to \text{comod-}L$. Is this functor monoidal?
4. Let H, L be quasi-triangular with R-matrices R, R'. Show that in this case the functor Φ is braided, if and only if $(\phi \otimes \phi)(R) = R'$.

Problem 5. Let H be a quasi-triangular Hopf algebra, with antipode S, R-matrix $R = R_{12}$ and Drinfeld element $u = \sum_R S(R_{(2)})R_{(1)}$. We denote $\Delta' = \tau \circ \Delta$.

1. Show that the following formula endow $H \otimes H$ with a structure of module-$H^{\otimes 4}$:
$$(x \otimes y) \bullet (a \otimes b \otimes c \otimes d) = S(b)xa \otimes S(d)yc.$$
2. Compute $R_{21} \bullet R_{23}$ and $R_{21} \bullet (R_{23}R_{13}R_{12}R_{14})$.
3. Prove the following equality in $H^{\otimes 4}$: $R_{12}(\Delta \otimes \Delta')(R) = R_{23}R_{13}R_{12}R_{14}R_{24}$.
4. Prove that:
$$\Delta(u) = (R_{21}R)^{-1}(u \otimes u) = (u \otimes u)(R_{21}R)^{-1}$$
5. Prove that $g = u(S(u))^{-1}$ is group like, and that S^4 is an inner automorphism.