Special representations and the two - dimensional McKay correspondence

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Introduction

The theory of KLEIN *singularities* establishes a 1-1–correspondence between the *conjugacy classes of finite subgroups* of SL (2, $\mathbb{C}$) (also called *binary polyhedral groups* by abuse of language) and the COXETER-DYNKIN-WITT *diagrams* (or DYNKIN–*diagrams*, as they are usually called) of type ADE via the following scheme:

$$\begin{array}{ccc} \{\,\text{finite subgroups}\ \Gamma \subset \mathrm{SL}\,(2,\,\mathbb{C})\,\}/\ \text{conjugacy} & & \{\,\text{CDW-diagrams of type ADE}\,\} \\ \updownarrow & & \updownarrow \\ \{\,\text{Klein singularities}\, X_\Gamma = \mathbb{C}^2/\Gamma\,\}/\sim & \longleftrightarrow & \{\,\text{minimal resolutions}\, \widetilde{X}_\Gamma\,\}/\sim \end{array}$$

Here, the symbol $\sim$ in the last row denotes *complex–analytic equivalence*. The arrow in the second column is given in the upward direction by associating to a minimal resolution $\widetilde{X}_\Gamma$ of X_Γ the *dual graph* of its *exceptional set* $E \subset \widetilde{X}_\Gamma$.

In 1979, MCKAY [16] constructed directly via representation theory the resulting bijection in the first row of this diagram (see Section 1). In particular, according to this so called MCKAY *correspondence*, each *(nontrivial) irreducible complex representation* of Γ corresponds uniquely to an *irreducible component* of the exceptional set E.

Of course, geometers wanted to understand this phenomenon *geometrically*, and the first who succeeded in this attempt were GONZALES-SPRINBERG and VERDIER [8] in 1983. They associated to each nontrivial irreducible representation of Γ a vector bundle $\mathcal{F}$ on $\widetilde{X}_\Gamma$ whose first Chern class $c_1(\mathcal{F})$ hits precisely one component of E transversally. Their proof was not completely satisfying since they had to check the details case by case. But in 1985, ARTIN and VERDIER [1] gave a conceptual proof using only standard facts on rational singularities, and in combination with the so called *multiplication formula* contained in the paper [6] of HÉLÈNE ESNAULT and KNÖRRER from the same year it became clear how to understand the *full strength* of the correspondence, i. e. how to

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reconstruct the dual graph of $E \subset \widetilde{X}_\Gamma$ from the representations of Γ, completely in geometrical terms.

Research then followed different directions. One trend (from 1992 on) proposed to treat the *higher dimensional* case $\Gamma \subset \mathrm{SL}(n, \mathbb{C})$ under the slogan: if X_Γ has a *crepant* resolution $\widetilde{X}_\Gamma$, i. e. if the canonical sheaf of $\widetilde{X}_\Gamma$ is trivial, then there should be a bijection

$$\{\text{nontrivial irreducible representations of } \Gamma\} \longleftrightarrow \text{basis of } H^*(\widetilde{X}_\Gamma, \mathbb{Z})$$

which - in case $n = 2$ - is just a rephrasing of the result for finite subgroups in $\mathrm{SL}(2, \mathbb{C})$. (C. f. the notes [18] of Reid).

Another path started earlier: to understand the more general 2–dimensional case of finite subgroups $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$. Here, one has to name Esnault with the paper [5] in 1985, a doctoral thesis of Wunram written under my supervision in 1986 (published 1988 in Mathematische Annalen [23]) and some unpublished notes of myself [19] from 1987.

In June 1997 I learned from a lecture given by Yukari Ito at RIMS in Kyoto that she and I. Nakamura constructed in joint work the minimal resolution $\widetilde{X}_\Gamma$ in the case of finite subgroups of the *special* linear group $\mathrm{SL}(2, \mathbb{C})$ by invariant theory of Γ acting on a certain Hilbert scheme. They were able, again by checking case by case, to produce the correct representations from the irreducible components of $E \subset \widetilde{X}_\Gamma$ (and even more). You may find this in the beautiful article [12] (see also [11] and Section 2 of the present notes). In 1999 Crawley-Boevey released a note (c.f. [4]) in which he proved the result of Ito–Nakamura without case by case checking; he uses the theory of *preprojective algebras* associated to McKay *quivers*. However, these methods are not available for general quotients.

A year later, Nakamura lectured in 1998 on this topic in Hamburg; I soon became aware of how one should generalize the statement to (small) subgroups of the *general* linear group $\mathrm{GL}(2, \mathbb{C})$ and developed some vague ideas how to prove this without too many calculations. This conjecture could be checked in the case of *cyclic* quotients by a simple computation which depended on the concrete results in the doctoral thesis of Rie Kidoh, written under the supervision of Nakamura. I sent her my results and asked her for inserting them into the version she was preparing for publication in this journal [14]. However, she decided not to follow my suggestion.

I gave some lectures on this topic in Japan during September 1999 and was very happy to learn from Akira Ishii in August 2000 that he succeeded in proving the conjecture via rephrasing the *multiplication formula* of Wunram in terms of a functor between certain derived categories (cf. [9] and Section 5).

It is the purpose of these notes to give a more detailed introduction to these results and to present the very easy proof in the case of cyclic quotients as an appendix to [14]. Besides the general proof of A. Ishii [9] which uses much heavier machinery there exists now another independend proof in the cyclic case via toric geometry by Y. Ito [10]; she doesn't use Kidoh's explicit construction but a characterization of special representations (see Theorem 5) whose (easy) proof, however, was never published in a journal. I therefore include it here in Section 6 as a benefit to the reader.

I would like to add the remark that it has been and still is one of my wishes to describe the *deformation theory* of X_Γ by using only the *representation theory* of Γ. This is possible for the *infinitesimal* deformations according to a result of PINKHAM [17] which was used by BEHNKE, KAHN and myself [2] for calculating the vector space T^1 of first order deformations yielding a nice *dimension formula*, but no clear image how to construct the *versal* deformation as - to some extent - has been achieved by KRONHEIMER [15] in the case of binary polyhedral groups. At least in the cyclic case, I found a rather strange way to produce the deformation space over the ARTIN *component* combining a quiver construction and some invariant theoretic ideas [20].

1. McKay's observation

Let me first recall the construction of the MCKAY *quiver* associated to a finite small subgroup $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$. Let $\mathrm{Irr}\,\Gamma := \{\rho_0, \rho_1, \ldots, \rho_r\}$ denote the set of irreducible complex representations of Γ, ρ_0 the trivial one, and c the natural representation on $\mathbb{C}^2$ given by the inclusion $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$. Then,

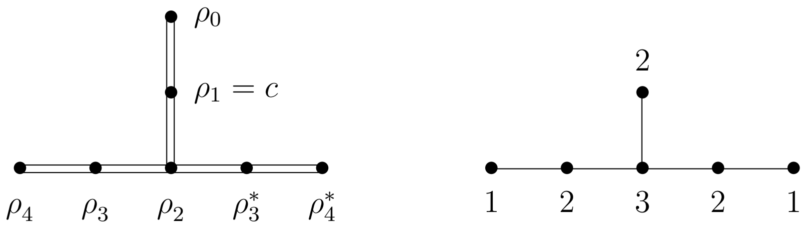
$$\rho_i \otimes c^* = \sum_j a_{ij}\, \rho_j$$

where c^* denotes the *dual* representation of c. The MCKAY quiver is formed in the following way: Associate to each representation a vertex and join the ith vertex with the jth vertex by a_{ij} arrows.

For the *binary tetrahedral* group $\mathbb{T}$, that is the preimage of the symmetry group $T \subset \mathrm{SO}(3, \mathbb{R})$ of a *regular tetrahedron* under the canonical group epimorphism

$$(0 \longrightarrow \{\pm 1\} \longrightarrow)\ \mathrm{SU}(2, \mathbb{C}) \longrightarrow \mathrm{SO}(3, \mathbb{R}) \longrightarrow 1\,,$$

the resulting quiver looks as on the left side of the following diagram where a subgraph stands for a *double arrow*, i. e. two arrows in opposite direction. Replacing such subgraphs by a simple line , forgetting ρ_0 and inserting the ranks of the corresponding representations, yields the other diagram on the right side which, in fact, is not only the CDW diagram of type E_6 but also represents the *fundamental cycle* Z of the singularity $\mathbb{C}^2/\,\mathbb{T}$.



MCKAY's observation may be formulated in the following way:

For every finite subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$, *one has* $a_{ji} = a_{ij} \in \{0, 1\}$. *Replacing each* double arrow *by a line, one finds exactly the (extended) CDW diagrams of correct type.*

2. The Ito - Nakamura construction

Let $\mathrm{Hilb}^n(\mathbb{C}^2)$ be the Hilbert scheme of all 0–dimensional subschemes on $\mathbb{C}^2$ of colength n. It is well known that the canonical Hilbert-Chow–*morphism*

$$\mathrm{Hilb}^n(\mathbb{C}^2) \longrightarrow \mathrm{Sym}^n(\mathbb{C}^2) = (\mathbb{C}^2)^n/\mathfrak{S}_n$$

is a resolution of singularities (Fogarty), and $\mathrm{Hilb}^n(\mathbb{C}^2)$ carries a holomorphic symplectic structure (Beauville). Let $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$ be a finite small subgroup of order $n = \mathrm{ord}\,\Gamma$, and take the invariant part of the natural action of Γ on $\mathrm{Hilb}^n(\mathbb{C}^2)$. The resulting space $\mathrm{Hilb}^n(\mathbb{C}^2)^\Gamma$ is smooth and maps under the canonical mapping

$$\mathrm{Hilb}^n(\mathbb{C}^2)^\Gamma \longmapsto \mathrm{Sym}^n(\mathbb{C}^2)^\Gamma \cong \mathbb{C}^2/\Gamma$$

to X_Γ. It may a priori have several components, but there is exactly one which maps *onto* X_Γ and thus constitutes a *resolution* of X_Γ which will be denoted by

$$\widetilde{X}_\Gamma = \mathrm{Hilb}^\Gamma(\mathbb{C}^2)\ .$$

In fact, $\mathrm{Hilb}^\Gamma(\mathbb{C}^2)$ is equal to the open subset of so–called Γ–invariant *n–clusters* in $\mathbb{C}^2$, and the resolution is *minimal* (c.f. [9]). The last fact has been known before in the case $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ by Ito-Nakamura and for cyclic subgroups of $\mathrm{GL}(2, \mathbb{C})$ by Kidoh; it has been conjectured for general finite small subgroups $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$ by Ginzburg-Kapranov [7].

Hence, a point on the exceptional set E of $\widetilde{X}_\Gamma$ may be regarded as a Γ–invariant ideal $I \subset \mathcal{O}_{\mathbb{C}^2}$ with support in 0. Now, let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{\mathbb{C}^2,0}$, $\mathfrak{m}_X$ that of $\mathcal{O}_{X,0} = \mathcal{O}^\Gamma_{\mathbb{C}^2,0}$ and $\mathfrak{n} = \mathfrak{m}_X\mathcal{O}_{\mathbb{C}^2,0}$. Put

$$V(I) := I/(\mathfrak{m}I + \mathfrak{n})\ .$$

This is a (finite–dimensional) Γ–module. For a (nontrivial) irreducible representation $\rho \in \mathrm{Irr}^0\,\Gamma := \mathrm{Irr}\,\Gamma \setminus \{\rho_0\}$ with representation space V_ρ put

$$E_\rho = \{ I : V(I) \quad \text{contains} \quad V_\rho \}\ .$$

In the case of Klein singularities, i. e. for finite subgroups $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$, one has the following beautiful result of Ito and Nakamura which opened up a new way to understand the McKay correspondence completely in terms of the binary polyhedral group Γ.

Theorem 1 (Ito - Nakamura [11]) *For $\rho \in \mathrm{Irr}^0\,\Gamma$, $E_\rho \cong \mathbb{P}_1$. Moreover, $E_\rho \cap E_{\rho'}$ is empty or consists of exactly one point for $\rho \neq \rho'$, and*

$$E = \bigcup_{\rho \in \mathrm{Irr}^0\Gamma} E_\rho\ .$$

More precisely, $V(I) = V_\rho$ for the ideals $I \in E_\rho$ corresponding to smooth points of E, and

$$E_\rho \cap E_{\rho'} \ni I \iff V(I) = V_\rho \oplus V_{\rho'}\ .$$

It is natural to ask how this result can be generalized to *finite small subgroups* $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$. The purpose of these notes is to make the following theorem plausible, in particular by proving it for *cyclic* quotients.

Theorem 2 (A. Ishii [9]) *The Ito–Nakamura construction yields the same result as above also for finite small subgroups* $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$ *if the set* $\mathrm{Irr}^0\Gamma$ *of* all *nontrivial irreducible representations is replaced by the subset of so–called* special *ones.*

Remark. For the definition of *special* representations, see Section 4. Theorem 2 has been conjectured by the author in [21]. Some ingredients of the proof by Akira Ishii will be reviewed in Section 5.

3. Kidoh's calculation in the cyclic case

We are going to investigate in this section the case of *cyclic* groups Γ. This is merely a simple observation and follows easily by a calculation using Kidoh's determination of the invariant ideals in this case in terms of *continued fraction expansions*.

Recall that cyclic quotient surface singularities of $\mathbb{C}^2$ are determined by two natural numbers n, q with $1 \le q < n$ and $\gcd(n, q) = 1$. The cyclic group $C_{n,q}$ acting is generated by the linear map with matrix

$$\begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^q \end{pmatrix} , \quad \zeta_n = \exp(2\pi i/n) \text{ an } n^{\mathrm{th}} \textit{ primitive root of unity}$$

that operates on the polynomial ring $\mathbb{C}[u, v]$ by $(u, v) \longmapsto (\zeta_n u, \zeta_n^q v)$. A monomial $u^\alpha v^\beta$ is invariant under this action if and only if

$$\alpha + q\beta \equiv 0 \mod n ,$$

e. g. for $(\alpha, \beta) = (n, 0), \ (n - q, 1), \ (0, n)$. The Hirzebruch - Jung *continued fraction*

$$\frac{n}{n-q} = a_1 - \frac{1}{a_2 - 1/\cdots} = a_1 - \underline{1 \rfloor a_2} - \cdots - \underline{1 \rfloor a_m} \ , \quad a_\mu \ge 2$$

gives a strictly decreasing sequence

$$\alpha_0 = n > \alpha_1 = n - q > \alpha_2 = a_1 \alpha_1 - \alpha_0 > \cdots$$

stopping with $\alpha_{m+1} = 0$, and a strictly increasing sequence

$$\beta_0 = 0 < \beta_1 = 1 < \beta_2 = a_1\beta_1 - \beta_0 < \cdots < \beta_{m+1} = n .$$

It is well known that the monomials

$$u^{\alpha_\mu} v^{\beta_\mu} , \quad \mu = 0, \ldots, m+1$$

generate the invariant algebra

$$A_{n,q} := \mathbb{C}\langle u, v\rangle^{C_{n,q}} = \mathcal{O}_{\mathbb{C}^2/C_{n,q},0}$$

minimally. In particular, $\operatorname{embdim} A_{n,q} = m+2$, hence, $\operatorname{mult} A_{n,q} = m+1$. The numbers a_μ are exponents in canonical equations for $A_{n,q}$. On the other hand, the continued fraction expansion

$$\frac{n}{q} = b_1 - \underline{1 \rceil b_2} - \cdots - \underline{1 \rceil b_r}\ , \quad b_k \geq 2$$

gives invariants for the minimal resolution of $\mathbb{C}^2 / C_{n,q}$ whose exceptional divisor consists of a string of rational curves with *selfintersection numbers* $-b_k$.

Define correspondingly the decreasing sequence

$$i_0 = n > i_1 = q > i_2 = b_1 i_1 - i_0 > \cdots > i_r = 1 > i_{r+1} = 0$$

and

$$j_0 = 0 < j_1 = 1 < j_2 = b_1 j_1 - j_0 < \cdots < j_{r+1} = n\ .$$

Theorem 3 (R. Kidoh [14]) *Let* (n, q) *be given. Then,* $\operatorname{Hilb}^{C_{n,q}}(\mathbb{C}^2)$ *consists of the following* $C_{n,q}$*–invariant ideals of colength* $n = \operatorname{ord} C_{n,q}$*:*

$$I_k(s_k, t_k) = (u^{i_{k-1}} - s_k v^{j_{k-1}}, v^{j_k} - t_k u^{i_k}, u^{i_{k-1}-i_k} v^{j_k - j_{k-1}} - s_k t_k),\ 1 \leq k \leq r+1,\ (s_k, t_k) \in \mathbb{C}^2.$$

Remarks. 1. These are in fact $C_{n,q}$–invariant ideals, since $i_k \equiv q\, j_k \bmod n$ and the functions $u^{i_{k-1}-i_k} v^{j_k - j_{k-1}}$ are invariant.

2. The $(r+1)$ copies of $\mathbb{C}^2$ patch together to form the *minimal resolution* of $\mathbb{C}^2 / C_{n,q}$, i. e. $I_k(s_k, t_k) = I_{k+1}(s_{k+1}, t_{k+1}) \Longleftrightarrow s_{k+1} t_k = 1$ and $t_{k+1} = t_k^{b_k} s_k$.

3. The exceptional divisor E equals

$$I_1(0, t_1) \cup \bigcup_{k=2}^{r} \{ I_k(s_k, t_k) : s_k t_k = 0 \} \cup I_{r+1}(s_{r+1}, 0)\ .$$

4. It is not difficult to deduce Kidoh's result by induction using the well known partial resolution of cyclic quotient singularities constructed by Fujiki.

What about the representations of $C_{n,q}$ on the $V(I_k)$? For $I_1(0, t_1)$ the first generator $u^{i_0} = u^n$ is an invariant. The third is such in all cases anyway. So, $C_{n,q}$ acts on $V(I_1(0, t_1)) \cong \mathbb{C}$ as the *character* χ_{i_1} where

$$\chi_i : z \longmapsto \zeta_n^i z$$

(recall that $q j_k \equiv i_k \bmod n$). This remains automatically true for $I_2(s_2, t_2)$ with $t_2 = 0$, $s_2 \neq 0$. The first normal crossing point of the exceptional set is the ideal $I_2(0, 0)$ which is generated by u^{i_1}, v^{j_2} and an invariant. Therefore, the corresponding representation is the sum

$$\chi_{i_1} \oplus \chi_{q j_2} = \chi_{i_1} \oplus \chi_{i_2}\ .$$

The ideal $I_2(0, t_2)$, $t_2 \neq 0$, is generated by u^{i_1}, $v^{j_2} - t_2 u^{i_2}$ and the invariant $u^{i_1 - i_2} v^{j_2 - j_1}$. Now,

$$t_2 u^{i_1} = v^{j_1} (u^{i_1 - i_2} v^{j_2 - j_1}) - u^{i_1 - i_2} (v^{j_2} - t_2 u^{i_2}) \in \mathfrak{m}\, I_2(0, t_2)\ .$$

Therefore, the representation is just the one–dimensional

$$\chi_{i_2} = \chi_{qj_2} .$$

It should be clear how this game goes on: We get precisely the r representations χ_{i_k}, $k = 1, \ldots, r$, resp. the correct sum of two of them at the intersection points.

What is so *special* about these representations under all representations χ_i ? The answer is known to me since a long time, although it is still rather mysterious.

4. Special full sheaves, special reflexive sheaves and special representations

Let Γ be a finite small subgroup of $\mathrm{GL}(2, \mathbb{C})$ and ρ a representation of Γ on the vector space $V = V_\rho$. Γ operates on $\mathbb{C}^2 \times V$ via the natural representation c and ρ, and the quotient is a vector bundle on $(\mathbb{C}^2 \setminus \{0\})/\Gamma$ whose (locally free) sheaf of holomorphic sections extends to a *reflexive sheaf* M_ρ on $\mathbb{C}^2/\Gamma = X_\Gamma$:

$$M_\rho := \mu_*(\mathcal{O}_{\mathbb{C}^2} \otimes V_{\rho^*})^\Gamma ,$$

where μ denotes the canonical projection $\mathbb{C}^2 \to X_\Gamma$ and ρ^* is the *dual* representation. In fact, one gets all reflexive modules M on X_Γ in this manner (for more details, see Theorem 12). M is indecomposable if and only if ρ is irreducible.

We can study more generally any *rational* surface singularity X and an arbitrary reflexive module M on it. Let $\pi : \widetilde{X} \longrightarrow X$ be a minimal resolution, and put

$$\widetilde{M} := \pi^* M / \,\mathrm{torsion} .$$

Such sheaves on $\widetilde{X}$ were baptized *full sheaves* by ESNAULT. By *local duality*, one has the following

Theorem 4 (Esnault [5]) *A sheaf $\mathcal{F}$ on $\widetilde{X}$ is full if and only if the following conditions are satisfied*:

1. $\mathcal{F}$ *is* locally free, *i. e. (the sheaf of holomorphic sections in) a* vector bundle,
2. $\mathcal{F}$ *is* generated by global sections, *in particular* $H^1(\widetilde{X}, \mathcal{F}) = 0$,
3. $H^1(\widetilde{X}, \mathcal{F}^* \otimes \omega_{\widetilde{X}}) = 0$, *where* $\omega_{\widetilde{X}}$ *denotes the* canonical sheaf *on* $\widetilde{X}$.

Under these assumptions, $M = \pi_* \mathcal{F}$ *is reflexive and* $\mathcal{F} = \widetilde{M}$. Moreover, $M^* = \pi_*(\mathcal{F}^*)$ (but, $\mathcal{F}^*$ is, in general, <u>not</u> a *full* sheaf).

Let $\mathcal{F} = \widetilde{M}$ be of rank r and full. Then one can construct an exact sequence

$$0 \longrightarrow \mathcal{O}^r_{\widetilde{X}} \longrightarrow \mathcal{F} \longrightarrow N \longrightarrow 0$$

with $D := \mathrm{supp}\, N$ a divisor in a neighborhood of the *exceptional* set E which cuts E transversally at regular points only. We call D the CHERN *divisor* $c_1(\mathcal{F})$.

Definition. A full sheaf $\widetilde{M}$ / reflexive module M / representation ρ is called *special* (perhaps better *exceptional*), if and only if $H^1(\widetilde{X}, (\widetilde{M})^*) = 0$ (where $M := M_\rho$ in case of a representation ρ).

Special full sheaves have been characterized in [23], special reflexive modules and representations in [19]. Notice that in [19], [22] and [23], we associated the module $\mu_*(\mathcal{O}_{\mathbb{C}^2} \otimes V_\rho)^\Gamma$ to a representation ρ instead of $\mu_*(\mathcal{O}_{\mathbb{C}^2} \otimes V_{\rho^*})^\Gamma$. Hence, we are dealing in the present paper with the *dual* representations which fit better into the framework of the Ito–Nakamura construction.

Theorem 5

1) $\widetilde{M}$ special $\Longleftrightarrow$ *the canonical map* $\widetilde{M} \otimes \omega_{\widetilde{X}} \to [(M \otimes \omega_X)^{**}]^{\widetilde{\ }}$ *is an isomorphism.*

2) M special $\Longleftrightarrow$ $M \otimes \omega_X/$ torsion *is reflexive.*

3) ρ special $\Longleftrightarrow$ *the canonical map* $(\Omega^2_{\mathbb{C}^2,0})^\Gamma \otimes (\mathcal{O}_{\mathbb{C}^2,0} \otimes V_{\rho^*})^\Gamma \to (\Omega^2_{\mathbb{C}^2,0} \otimes V_{\rho^*})^\Gamma$ *is surjective.*

Here, of course, two stars denote the double dual *(or* reflexive hull *) of a coherent analytic sheaf*, Ω^m_X *is the sheaf of* Kähler m–forms *and* $\omega_X := (\Omega^2_X)^{**}$ *the* dualizing sheaf *on a complex analytic surface* X .

Since the paper [19] is not easily available, we include a proof of Theorem 5 in section 6.

Theorem 6 (Wunram [23]) *There is a bijection*

$$\{ \textit{ special nontrivial indec. reflexive modules } \} \longleftrightarrow \{ \textit{ irreducible components } E_j \textit{ of } E \}$$

via

$$M \longmapsto c_1(\widetilde{M})\, E_k = \delta_{jk} \ .$$

The rank of M_j *equals the multiplicity* r_j *of the curve* E_j *in the* fundamental cycle $Z = \sum r_j\, E_j$.

As a *Corollary*, one gets once more the McKay *correspondence* since for the Klein singularities one has $\omega_X \cong \mathcal{O}_X$ (Gorenstein property) and $\omega_{\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}$.

Wunram has, in particular, computed the irreducible special representations in the case of cyclic quotients [22]. As it turns out, they coincide with the representations we have found in Section 3. In fact, one has to dualize twice to get the correct result. In our notation for M_ρ which is dual to Wunram's, the special representations are exactly the dual ones to those computed in section 3 with respect to the given action on the polynomial ring $\mathbb{C}\,[u,\, v]$. But this action, in turn, is dual to the action on $\mathbb{C}^2$ we should start with. Thus, Theorem 2 is proven (more or less by inspection) in the cyclic case.

5. The manuscript of A. Ishii

McKay correspondence may be understood as an equivalence of *derived categories*. This has been worked out by Kapranov and Vasserot for $\mathrm{SL}\,(2,\, \mathbb{C})$ [13] and by Bridgeland, King, Reid [3] in dimension 3. The last paper led A. Ishii to study more closely the canonical functor

$$\Psi :\, D^\Gamma_c(\mathbb{C}^2) \longrightarrow D_c(Y)$$

where $D_c^\Gamma(\mathbb{C}^2)$ denotes the derived category of Γ–equivariant coherent analytic sheaves with compact support on $\mathbb{C}^2$, Γ a finite small subgroup of $\mathrm{GL}\,(2,\,\mathbb{C})$, and $D_c(Y)$ the derived category of coherent analytic sheaves on $Y = \mathrm{Hilb}^\Gamma\,(\mathbb{C}^2)$ with compact support.

The main ingredient of his proof is Wunram's *multiplication formula* which generalizes the one of Esnault and Knörrer. We denote by M a reflexive module on $X = \mathbb{C}^2/\Gamma$, its Auslander-Reiten *translate*, i. e. the module $(M \otimes \omega_X)^{**}$, by $\tau\,(M)$, and finally, we write $N_M = (M \otimes \Omega^1_X)^{**}$. Then we have:

Theorem 7 (Wunram [23])

$$c_1(\widetilde{N}_M) \,-\, c_1(\widetilde{M}) \,-\, c_1\left(\widetilde{\tau(M)}\right) \;=\; \begin{cases} E_j\,, & M \,=\, M_j \ \text{special}\,,\ j \neq 0\,, \\ Z\,, & M \,=\, M_0 \,:=\, \mathcal{O}_X\,, \\ 0\,, & M \ \text{nonspecial}\,. \end{cases}$$

Here, Z *denotes the* fundamental cycle *of the minimal resolution of* X.

A. Ishii first restates and proves once more Wunram's *multiplication formula* in the following form.

Theorem 8 (A. Ishii [9]) *Let* ρ *be an irreducible representation of* $\Gamma \subset \mathrm{GL}\,(2,\,\mathbb{C})$ *and put* $\mathcal{O}_0 = \mathcal{O}_{\mathbb{C}^2,0}/\,\mathfrak{m}$, *where* $\mathfrak{m}$ *denotes the maximal ideal of* $\mathbb{C}^2$ *at the origin. Then*

$$\Psi\,(\mathcal{O}_0 \otimes V_{\rho^*}) \;=\; \begin{cases} \mathcal{O}_{E_j}(-1)[1]\,, & \rho \,=\, \rho_j \ \text{special}\,,\ j \neq 0\,, \\ \mathcal{O}_Z\,, & \rho \,=\, \rho_0\,, \\ 0\,, & \rho \ \text{nonspecial}\,. \end{cases}$$

He then explicitly constructs a *right adjoint* Φ to Ψ. The resulting isomorphism

$$\mathrm{Hom}_{D_c(Y)}\,(\Psi\,(\Delta),\,\nabla) \;\cong\; \mathrm{Hom}_{D_c^\Gamma(\mathbb{C}^2)}\,(\Delta,\,\Phi\,(\nabla))$$

leads to the desired result when applied to $\Delta := \mathcal{O}_0 \otimes V_{\rho^*}$, $\nabla := \mathcal{O}_y$, $y \in Y$.

6. A characterization of special reflexive modules and special representations

In this section we present a proof of Theorem 5 which is taken from [23] and [19]. More precisely, we show the following

Lemma 9 *Let* M *be a reflexive module on the rational singularity* X. *Then the following conditions are equivalent*:

a) M *is special, i.e.* $R^1\pi_*\,(\widetilde{M})^* = 0$;

b) *the canonical map* $\widetilde{M} \otimes \omega_{\widetilde{X}} \longrightarrow \tau(M)^\sim$ *is bijective*;

c) $M \otimes \omega_X/\,\mathrm{torsion} \,\cong\, \tau(M)$.

Proof. The equivalence of b) and c) will be deduced from a more general lemma that is proven below. The equivalence of a) and b) is obviously the same as the claim that $\widetilde{M} \otimes \omega_{\widetilde{X}}$ is full if and only if $R^1\pi_*\widetilde{M}^* = 0$. But $\widetilde{M} \otimes \omega_{\widetilde{X}}$ is locally free and generated by global sections. Hence, $\widetilde{M} \otimes \omega_{\widetilde{X}}$ is full if and only if $R^1\pi_*((\widetilde{M} \otimes \omega_{\widetilde{X}})^* \otimes \omega_{\widetilde{X}}) = 0$. □

The rest is a consequence of the next

Lemma 10 *Let M and N be reflexive modules on the rational singularity X. Then the following are equivalent*:

i) $\widetilde{M} \otimes \widetilde{N} \xrightarrow{\sim} [(M \otimes N)^{**}]^{\sim}$;

ii) $\pi_*(\widetilde{M} \otimes \widetilde{N}) \xrightarrow{\sim} (M \otimes N)^{**}$;

iii) $M \otimes N \longrightarrow (M \otimes N)^{**}$ *is surjective*;

iv) $M \otimes N/$ torsion *is reflexive*.

Proof. i) $\Rightarrow$ ii) is trivial due to the last sentence in Theorem 4. Since $\pi_*(\widetilde{M} \otimes \widetilde{N})$ has no torsion and $\alpha : \pi_*(\widetilde{M} \otimes \widetilde{N}) \to (M \otimes N)^{**}$ is an isomorphism on $X' = X \setminus \{x_0\}$, α is always a monomorphism. Hence, ii) is equivalent to the surjectivity of α which follows from iii) using the factorizations

$$(*)\ \ M \otimes N \longrightarrow \pi_*\pi^*(M \otimes N) \xrightarrow{\sim} \pi_*(\pi^*M \otimes \pi^*N) \longrightarrow \pi_*(\widetilde{M} \otimes \widetilde{N}) \longrightarrow (M \otimes N)^{**}.$$

Notice that all maps in $(*)$ are isomorphisms outside x_0. The equivalence of iii) and iv) is easily deduced from the commutative diagram

$$\begin{tikzcd} M \otimes N \arrow[rr] \arrow[dr] & & (M \otimes N)^{**} \\ & (M \otimes N)/\text{torsion}. \arrow[ur] & \end{tikzcd}$$

The implication ii) $\Rightarrow$ i) is a consequence of the fact that the injective map in i) is dominated by the epimorphism

$$\pi^*\pi_*(\widetilde{M} \otimes \widetilde{N}) \xrightarrow{\sim} \pi^*((M \otimes N)^{**}) \longrightarrow [(M \otimes N)^{**}]^{\sim}.$$

For the remaining implication ii) $\Rightarrow$ iii) to be true, we must show that the map $M \otimes N \to \pi_*(\widetilde{M} \otimes \widetilde{N})$ is surjective. We split this statement up into two parts:

v) the morphism $M \otimes N \longrightarrow \pi_*(\pi^*M \otimes \widetilde{N})$ is surjective,

vi) the morphism $\pi_*(\pi^*M \otimes \widetilde{N}) \longrightarrow \pi_*(\widetilde{M} \otimes \widetilde{N})$ is surjective.

ad v). Take a local presentation

$$\mathcal{O}_X^q \longrightarrow \mathcal{O}_X^p \longrightarrow M \longrightarrow 0$$

to get the exact sequence

$$\mathcal{O}_{\widetilde{X}}^q \longrightarrow \mathcal{O}_{\widetilde{X}}^p \longrightarrow \pi^* M \longrightarrow 0\,.$$

Tensorizing with $\widetilde{N}$, we find exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & \widetilde{N}^q & \longrightarrow & Q & \longrightarrow & 0 \quad , \\ 0 & \longrightarrow & Q & \longrightarrow & \widetilde{N}^p & \longrightarrow & \pi^* M \otimes \widetilde{N} & \longrightarrow & 0 \quad , \end{array}$$

which imply, because of $R^1 \pi_* \widetilde{N} = 0$, $R^2 \pi_* R = 0$, that $R^1 \pi_* Q = 0$ and

$$0 \longrightarrow \pi_* Q \longrightarrow \pi_* \widetilde{N}^p \longrightarrow \pi_* (\pi^* M \otimes \widetilde{N}) \longrightarrow 0$$

is an exact sequence. We also have a commutative diagram

$$\begin{array}{ccc} (\pi^* N)^p & \longrightarrow & \pi^*(M \otimes N) \\ \downarrow & & \downarrow \\ \widetilde{N}^p & \longrightarrow & \pi^* M \otimes \widetilde{N} \end{array}$$

with a surjective upper row. Taking direct images, adjoining to the new diagram the canonical morphism $N^p \to M \otimes N$ from above, and remembering that the composite map $N^p \to \pi_* \widetilde{N}^p$ is an isomorphism and the last row is an epimorphism, the proof of v) is accomplished.

Note that the proof of v) is correct for an arbitrary coherent module M. So, we have

Corollary 11 *The canonical morphism $M \to \pi_* \pi^* M$ is surjective for an arbitrary coherent module on a rational singularity X.*

ad vi) We regard the exact sequence

$$0 \longrightarrow \widetilde{T}_M \longrightarrow \pi^* M \longrightarrow \widetilde{M} \longrightarrow 0$$

and form direct images:

$$0 \longrightarrow \pi_* \widetilde{T}_M \longrightarrow \pi_* \pi^* M \longrightarrow \pi_* \widetilde{M} \longrightarrow R^1 \pi_* \widetilde{T}_M \longrightarrow R^1 \pi_* \pi^* M \;=\; 0\,.$$

Since the composite map $M \to \pi_* \pi^* M \to \pi_* \widetilde{M}$ is an isomorphism, the morphism $M \to \pi_* \pi^* M$ is injective and, by Corollary 11, bijective. Consequently,

$$\pi_* \widetilde{T}_M \;\cong\; R^1 \pi_* \widetilde{T}_M \;=\; 0\,,$$

and, due to the fact that $\widetilde{N}$ is generated by global sections,

$$R^1\pi_*(\widetilde{T}_M \otimes \widetilde{N}) \,=\, 0\,.$$

Tensorizing the first exact sequence by $\widetilde{N}$ and forming the long exact direct image sequence gives the result. $\square$

It remains to show that for quotient singularities a representation ρ is special if and only if 3) in Theorem 5 is satisfied. To see this, let us start with a reflexive module M on X, form the pull back μ^*M with respect to the finite covering $\mu:\, \mathbb{C}^2 \to \mathbb{C}^2/\,\Gamma = X$ and denote its *reflexive hull* by $\widehat{M}$. This is a (locally) *free* sheaf on $W := \mathbb{C}^2$ which carries a natural Γ–action: Starting with a local presentation

$$\mathcal{O}_X^q \longrightarrow \mathcal{O}_X^p \longrightarrow M \longrightarrow 0\,,$$

we get the exact sequences

$$\mathcal{O}_W^q \longrightarrow \mathcal{O}_W^p \longrightarrow \mu^*M \longrightarrow 0$$

and

$$0 \longrightarrow (\mu^*M)^* \longrightarrow \mathcal{O}_W^p \longrightarrow \mathcal{O}_W^q\,.$$

In particular, the sheaf $(\mu^*M)^*$ is already reflexive on W and thus locally free. The map $\mathcal{O}_W^p \to \mathcal{O}_W^q$ is Γ–equivariant since it is defined by a matrix with entries in the invariant ring under Γ. Thus, the sheaf $(\mu^*M)^*$ and its dual $\widehat{M}$ carry canonical Γ–actions. We furthermore associate to the reflexive module M the natural representation of Γ on the finite dimensional vector space $(\mu^*M)_0^*/\,\mathfrak{m}(\mu^*M)_0^*$, $\mathfrak{m} := \mathfrak{m}_{\mathbb{C}^2,0}$. This action is dual to the one on $\widehat{M}_0/\,\mathfrak{m}\widehat{M}_0$ and hence the same as the action on the fiber over the origin of the vector bundle associated to $\widehat{M}$.

Theorem 12 (H. Esnault [5]) *There exists a one–to–one correspondence between*

$$\{\ \text{(indecomposable) reflexive modules } M \text{ on } (X,\, x_0)\ \}\ ,$$

$$\{\ (\Gamma\text{–indecomposable) free modules } \widehat{M} \text{ on } (\mathbb{C}^2,\, 0) \text{ with a } \Gamma\text{–action}\ \}$$

and

$$\{\ \text{(irreducible) representations } \rho \text{ of } \Gamma\ \}\ .$$

If we write W for $\mathbb{C}^2$ as above, then the canonical representation $c:\, \Gamma \to \operatorname{Aut} W$ induces a character

$$\chi:\, \Gamma \longrightarrow \operatorname{Aut}\, \wedge^2\, W^*\,,$$

and, as Γ–modules,

$$\mathcal{O}_{\mathbb{C}^2} \,\otimes\, \wedge^2\, W^* \,\cong\, \Omega^2_{\mathbb{C}^2}\,.$$

For each representation $\rho:\, \Gamma \to \operatorname{Aut} V$, there exists an obvious morphism

$$(**)\qquad \mu_*(\mathcal{O}_{\mathbb{C}^2} \,\otimes\, V_{\rho^*})^\Gamma \otimes \mu_*(\Omega^2_{\mathbb{C}^2})^\Gamma \longrightarrow \mu_*(\Omega^2_{\mathbb{C}^2} \,\otimes\, V_{\rho^*})^\Gamma$$

where $\mu_*(\Omega^2_{\mathbb{C}^2} \,\otimes\, V_{\rho^*})^\Gamma$ is the reflexive hull of $\omega_X \otimes M$. So, the map $(**)$ coincides with the homomorphism $M \otimes \omega_X \to \tau(M)$ in item 3. of Theorem 5. Criterion iii) in Lemma 10 translates into the desired proof. $\square$

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