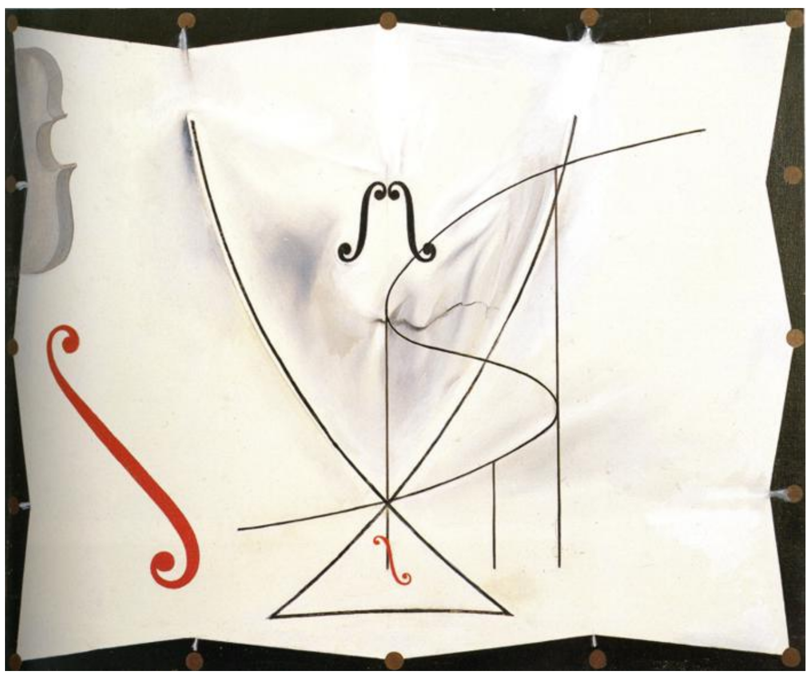
Chapter 7

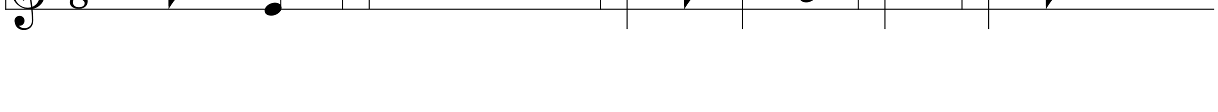
Gignit autem artificiosam lusorum gentem Cella Silvestris.
Zu deutsch: Waldzell aber bringt das kunstreiche Völkchen der Glasperlenspieler hervor.

(Hermann Hesse, *Das Glasperlenspiel*)

Und jedem Anfang wohnt ein Zauber inne,
der uns beschützt und der uns hilft zu leben.

(Hermann Hesse, *Stufen*)





Chapter 7

Jung singularities and resolutions of normal surface singularities

Jung singularities can be realized as coverings of $\mathbb{C}^2$ that are branched at most along the coordinate axes. We shall prove that (normal) Jung singularities have $\mathbb{C}^2$ as a covering space with a cyclic group of deck transformations. Normalizing the action of the cyclic group, we are able to construct a resolution in finitely many steps.

In this Chapter, we use freely some general results from the local theory of complex spaces and the theory of quotients which will be developed in more detail in later Chapters.

7.1 Jung singularities

To fix the ideas, we always work in the following situation: X is a (closed) analytic subset of the unit polydisk $D = D_{m-2} \times D_2 \subset \mathbb{C}^m$ about the origin, such that the following holds true:

(a) The projection $\rho : X \to D_2 = \{ (x, y) \in \mathbb{C}^2 : |x| < 1, |y| < 1 \}$ is a *finite map* (i.e. ρ is closed and has finite fibers) with $\rho^{-1}(0) = \{0\}$;

(b) let $\Sigma = \{ (x, y) \in D_2 : xy = 0 \}$ denote the union of the coordinate axes in D_2, let D_2^- be $D_2 \setminus \Sigma$ and set $X^- = \rho^{-1}(D_2^-)$. Then X^- is connected and dense in X, and the restriction

$$\rho : X^- \longrightarrow D_2^-$$

is a (finite) unbranched covering (i.e. X^- is a complex manifold, and ρ is surjective and locally biholomorphic).

Under these assumptions, we call the germ of X at 0 a *Jung singularity*. For *Example*, the functions

$$j_{nq}(x, y, z) = z^n - x^{n-q} y \, , \quad 1 \le q < n \, , \quad \gcd(n, q) = 1 \, ,$$

define Jung singularities which we denote by J_{nq}.

Since the map $\rho : X \to D_2$ is finite, the preimage $\rho^{-1}(\Sigma) = X \setminus X^-$ is a nowhere dense analytic subset of X which contains the singular set $\operatorname{sing} X$ of X. For the J_{nq}, we have

$$\operatorname{sing} J_{nq} = \begin{cases} \rho^{-1}(0) \, , & q = n - 1 \, , \\ \rho^{-1}(\{ x = 0 \}) \, , & q < n - 1 \, . \end{cases}$$

7.2 The classification of unbranched coverings of D_2^-

Let us first collect a few facts about (unbranched) coverings of topological manifolds. A continuous map $\rho : M_1 \to M$ between connected topological manifolds M_1 and M is called an *unbranched (and* *unbounded) covering map* (or a *covering* of M, for short) if for each point $x^{(0)} \in M$ there exists an open neighborhood U such that $\rho^{-1}(U)$ is a disjoint union $\cup V_j$ of open subsets $V_j \subset M_1$ that are homeomorphic to U under ρ. In particular, ρ is necessarily surjective and locally a topological map. These conditions are also sufficient for ρ to be a covering, if ρ is a finite map.

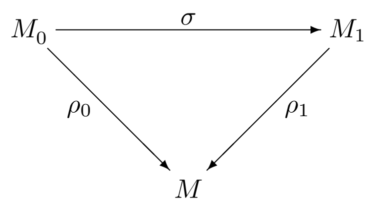
Given a covering $\rho : M_1 \to M$ of a complex analytic manifold M, there is a unique complex structure on M_1 making ρ into a locally biholomorphic map.

Since manifolds are locally pathwise connected, M (and M_1) are also globally pathwise connected such that the notion of the *fundamental group*

$$\pi_1(M)$$

(up to noncanonical isomorphism) makes sense.

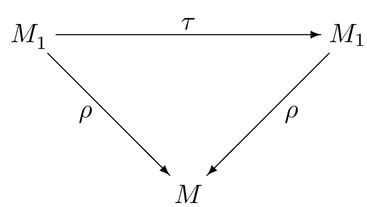
A covering $\rho_0 : M_0 \to M$ is called a *universal covering* of M, if it factors through every other covering $\rho_1 : M_1 \to M$:



Such universal coverings exist and are uniquely determined by M up to canonical isomorphisms of coverings; a covering $M_0 \to M$ is universal, if and only if M_0 is *simply connected*, i. e. if the fundamental group of M_0 is trivial:

$$\pi_1(M_0) = 1 .$$

For a covering $\rho : M_1 \to M$, a homeomorphism $\tau : M_1 \to M_1$ is called a *deck–transformation*, if it preserves the fibers of ρ, i. e. if



is a commutative diagram. We denote by $\mathrm{Deck}\,(M_1/M)$ the group of all deck–transformations; it acts in a natural way on M_1 (see also the next Section). The covering $\rho : M_1 \to M$ is called a *Galois covering*, if $\mathrm{Deck}\,(M_1/M)$ acts transitively on the fibers of ρ, i. e. if to every pair $y^{(1)}, y^{(2)} \in M_1$ with $\rho\,(y^{(1)}) = \rho\,(y^{(2)})$ there exists a deck–transformation τ with $y^{(2)} = \tau\,(y^{(1)})$.

The *Main Theorem* of the theory of coverings can then be stated as follows:

***Theorem 7.1** *Let $\rho_0 : M_0 \to M$ be the universal, $\rho_1 : M_1 \to M$ an arbitrary covering of M, and denote by $\sigma : M_0 \to M_1$ a map making the defining diagram commutative. Then $\sigma : M_0 \to M_1$ is a Galois covering of M_1. The group $G = \mathrm{Deck}\,(M_0/M_1)$ is a subgroup of $G_0 = \mathrm{Deck}\,(M_0/M)$. Moreover, $G \cong \pi_1(M_1)$ and the fibers of ρ_1 have the same cardinality as the space G_0/G of (left) cosets; in other terms:*

$$\mathrm{card}\ \rho_1^{-1}(x^{(0)}) = [\,G : G_0\,] ,$$

where the last symbol denotes the index *of G in G_0.*

Especially, taking the trivial covering $\mathrm{id} : M \to M$ implies that the universal covering $M_0 \to M$ is Galois with group $\mathrm{Deck}\,(M_0/M) \cong \pi_1(M)$, and, in the situation of the Theorem, $\pi_1(M_1)$ can be identified with those (homotopy classes of) loops in M which lift to (closed) loops in M_1.

Returning to the classification of Jung singularities, we are obviously led by Theorem 1 to the determination of the fundamental group of D_2^- and its subgroups of finite index (up to conjugacy). The first point can easily be worked out:

$$D_2^- = \{ x \in \mathbb{C} : 0 < |x| < 1 \} \times \{ y \in \mathbb{C} : 0 < |y| < 1 \}$$

can be retracted onto $S_\varepsilon^1 \times S_\varepsilon^1$, $0 < \varepsilon < 1$, where

$$S_\varepsilon^1 = \{ x \in \mathbb{C} : |x| = \varepsilon \} .$$

Thus, D_2^- is homotopy equivalent to $S^1 \times S^1$, $S^1 := S_1^1$, and therefore

$$\pi_1(D_2^-) \cong \pi_1(S^1 \times S^1) = \pi_1(S^1) \oplus \pi_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z} .$$

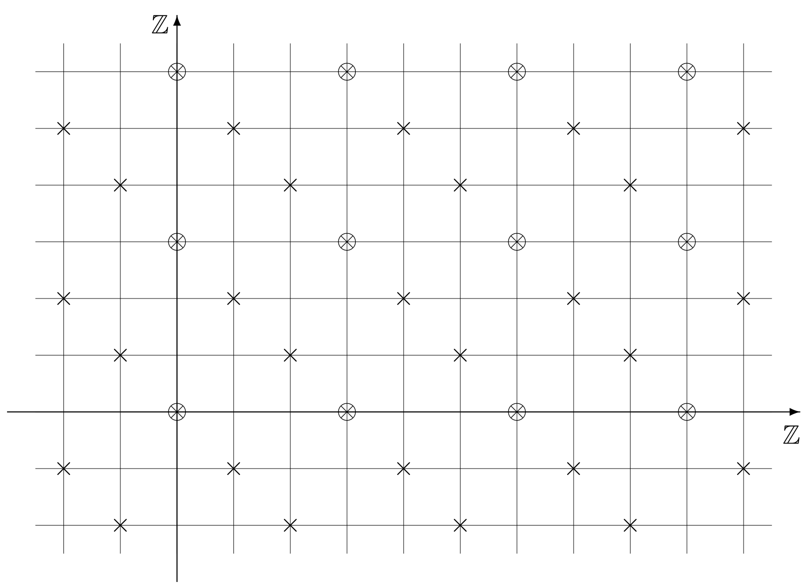
Of course, a class $[\alpha] \in \pi_1(D_2^-)$ of a loop $\alpha : S^1 \to D_2^-$ is represented by a pair $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$, if the winding number of α with respect to the y–axis and the x–axis equals a and b, respectively.

All we need with respect to the second point is the following

Lemma 7.2 *To each subgroup $G \subset \mathbb{Z} \oplus \mathbb{Z}$ of finite index there exists a diagonal subgroup $G_1 \subset G$ such that G/G_1 is cyclic of finite order.*

Proof. Denote by e_1, e_2 the canonical basis of $\mathbb{Z} \oplus \mathbb{Z}$. A simple exercise supplies us with numbers $a, b, d \in \mathbb{N}$ such that

$$G = \mathbb{Z}(ae_2) + \mathbb{Z}(be_1 + de_2) :$$



$$G = \{ \times, \otimes \}, \ G_1 = \{ \otimes \}$$

Figure 7.1

For $d = 0$, we may take $G = G_1$. If $d \neq 0$, we put $\alpha = \gcd(a, d)$ and

$$G_1 = \mathbb{Z}(ae_2) + \mathbb{Z}((ab/\alpha)e_1) .$$

G_1 is contained in G because of the identity

$$\frac{ab}{\alpha} e_1 = \frac{a}{\alpha}(be_1 + de_2) - \frac{d}{\alpha}(ae_2) .$$

The factor group G/G_1 is generated by the single element

$$\overline{x} = (be_1 + de_2) \bmod G_1 ,$$

and $n\overline{x} = 0$ for $n = a/\alpha$.

7.3 Group actions and topological quotients

The aim of the following Section is to show that (isolated) Jung singularities have the topological structure of a quotient of an open set in $\mathbb{C}^2$ by a linear action of the cyclic group G/G_1 constructed in Section 2. The purpose of the present Section is to fix our notions with respect to group actions (which we met already several times in this text) and to state some results for the topological category.

By a *group action* of a group G on a set X we always understand a map

$$(*) \qquad \begin{cases} X \times G \longrightarrow X \\ (x \ , \ g) \longmapsto x^g \end{cases}$$

with the properties

$$\begin{cases} x^{gh} = (x^g)^h & \text{for all } g, h \in G, \ x \in X, \\ x^e = x & \text{for the identity } e \in G \text{ and all } x \in X. \end{cases}$$

This is usually called more accurately an action of G on X *from the right*. There is a similar notion of a *left action* which we would like to avoid in the sequel. We also say that G *operates* via $(*)$.

Given a group action $(*)$, there exists a canonical map from G to the group $\mathrm{Aut}\ X$ of all bijective maps of X onto itself, viz.

$$G \ni g \longmapsto \alpha_g \in \mathrm{Aut}\ X \ ,$$

defined by $\alpha_g(x) = x^g$. Obviously, there are the following relations

$$\begin{cases} \alpha_{gh} = \alpha_h \circ \alpha_g \ , \\ \alpha_e = \mathrm{id} \ . \end{cases}$$

These imply that α_g is indeed bijective for all $g \in G$ and that the map

$$\begin{cases} G \longrightarrow \mathrm{Aut}\ X \\ g \longmapsto \alpha_g \end{cases}$$

is a group homomorphism, if the group structure on $\mathrm{Aut}\ X$ is defined by the map

$$\begin{cases} \mathrm{Aut}\ X \times \mathrm{Aut}\ X \longrightarrow \mathrm{Aut}\ X \\ (\alpha, \beta) \longmapsto \beta \circ \alpha \ , \end{cases}$$

where $\circ$ denotes the usual composition. This is the reason why we sometimes prefer to write

$$\alpha * \beta \text{ instead of } \beta \circ \alpha \ .$$

It is clear that $\mathrm{Aut}\ X$ operates on X from the right by

$$\begin{cases} X \times \mathrm{Aut}\ X \longrightarrow X \\ (x \ , \ \alpha) \longmapsto x^\alpha = \alpha(x) \ , \end{cases}$$

and so does every subgroup of $\mathrm{Aut}\ X$. In fact, all *effective* operations can be described this way, as we will see below.

Such operations abound throughout mathematics. For *Example*, each group G acts on itself by *right multiplication*

$$\begin{cases} G \times G \longrightarrow G \ , \\ (\gamma, g) \longmapsto \gamma g \ , \end{cases}$$

and also by *conjugation*

$$\begin{cases} G \times G \longrightarrow G \ , \\ (\gamma, g) \longmapsto g^{-1} \gamma g \ . \end{cases}$$

From the beginning of this manuscript we used the action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$. In Chapter 1.7 we emphasized the importance of the action of $\mathrm{Aut}\ \mathcal{O}_0^{(n)}$ on $\mathcal{O}_0^{(n)}$. In this Chapter, we are mainly concerned with actions of subgroups $G \subset \mathrm{Deck}(M_1/M)$ for a covering $M_1 \to M$ and related questions.

Two elements $x^{(1)}, x^{(2)} \in X$ are called *equivalent* with respect to a given G–action on the set X:

$$x^{(1)} \sim x^{(2)} \iff \text{there exists an element } g \in G \text{ with } \alpha_g(x^{(2)}) = x^{(1)} .$$

This is clearly an equivalence relation; the equivalence class

$$[x] = \{ x^{(1)} \in X : x^{(1)} \sim x \}$$

is usually called the *orbit* of x under the action of G or a G–orbit for short. For the action of a subgroup $H \subset G$ on G by *right multiplication*, these equivalence classes are the *left cosets* γH; for the action of G on itself by *conjugation*, these are the *conjugacy classes* $\{ g^{-1}\gamma g : g \in G \}$.

We always denote by X/G the set of all G–orbits in X; we call X/G the *quotient* of X by (the action of) G. The natural map $X \to X/G$ sending x to its G–orbit $[x]$ is usually denoted by ρ.

If X carries more structure, we are often compelled to equip the quotient X/G with a comparable structure. So, for instance, if G is a group acted on via right multiplication by a subgroup $H \subset G$, we would like to give the quotient G/H a group structure making the quotient map

$$\rho : G \longrightarrow G/H$$

to a group homomorphism. This task can be accomplished only by putting

$$[g_1] \cdot [g_2] = [g_1 \cdot g_2] , \quad g_1, g_2 \in G ,$$

which, however, makes no sense in general. The product is well–defined in G/H, if and only if H is a *normal subgroup* of G, i.e. if $\gamma \in H$, $g \in G$ implies $g^{-1}\gamma g \in H$.

Let us return to the homomorphism $G \to \mathrm{Aut}\ X$ associated to a G–action on X. Obviously, the kernel H of this map consists of all $h \in G$ with

$$x^h = x \text{ for all } x \in X ;$$

we call the action *effective*, if H is the trivial subgroup $\langle e \rangle$ of G. In general, the quotient group $\overline{G} = G/H$ acts on X because of

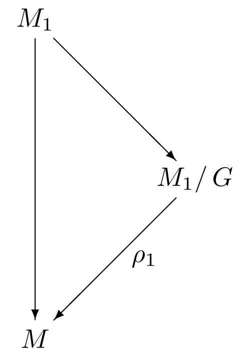
$$x^{\gamma h} = (x^\gamma)^h = x^\gamma , \quad \gamma \in G,\ h \in H,\ x \in X ,$$

and this action is effective, since the homomorphism $G \to \mathrm{Aut}\ X$ factorizes over the monomorphism $G/H \hookrightarrow \mathrm{Aut}\ X$. Thus, we conclude that the effective actions on a set X are classified by the subgroups of $\mathrm{Aut}\ X$.

We now concentrate on quotients of *topological spaces* X. Since analytic sets inherit a locally compact Hausdorff structure with countable basis from ambient space, we shall assume that all topological spaces in the present text have these properties, at least locally. However, while patching local models together, we also want to avoid the creation of new pathologies. Therefore, we always assume X to be globally Hausdorff and to have a countable basis; in particular, all topological spaces in this book are paracompact (and even metrizable).

We consider only groups G acting *topologically* on the space X; by this we mean that the group homomorphism $G \to \mathrm{Aut}\ X$ factorizes over the subgroup of all homeomorphisms of X, which - by abuse of notation - we denote again by $\mathrm{Aut}\ X$. (As a general rule, $\mathrm{Aut}\ X$ refers to the automorphisms of an object X in a category which will sometimes not be mentioned explicitly, if in the given context there is no ambiguity). Thus, each map $\alpha_g : X \to X$, $g \in G$, is a homeomorphism such that the group action G on X induces also an action on the set of all continuous maps from X to a fixed topological space Z by $(\varphi, g) \mapsto \varphi \circ \alpha_g^{-1}$.

Since $\mathrm{Deck}(M_1/M) \subset \mathrm{Aut}\ M_1$ for all coverings $\rho_1 : M_1 \to M$, each subgroup $G \subset \mathrm{Deck}(M_1/M)$ acts topologically on M_1 in a canonical manner. Moreover, by construction, there is a set–theoretical factorization



The covering ρ_1 is Galois, if the canonical map $M_1/\operatorname{Deck}(M_1/M) \to M$ is bijective. Theorem 1 says that each covering space M_1 of M is (set–theoretically) the quotient of the universal covering M_0 of M by the action of a subgroup G of the fundamental group $\pi_1(M)$, where $\pi_1(M)$ acts on M_0 by "lifting loops".

To make the quotient map $\rho : X \to X/G =: Y$ continuous for an arbitrary topological G–action on X, we have to affix to Y the *quotient topology*: $V \subset Y$ is open, if and only if $\rho^{-1}(V)$ is open in X. (Notice that for a covering $\rho_1 : M_1 \to M$ the topological space M carries automatically the quotient topology). Then a map $\overline{\varphi} : Y \to Z$ is continuous, if and only if $\overline{\varphi} \circ \rho : X \to Z$ is continuous. Clearly, $\overline{\varphi} \circ \rho$ is invariant under the G–action mentioned above:

$$\overline{\varphi} \circ \rho \circ \alpha_g^{-1} = \overline{\varphi} \circ \rho \text{ for all } g \in G .$$

On the other hand, each continuous map $\varphi : X \to Z$ that is invariant under G gives rise to a continuous map $\overline{\varphi} : Y \to Z$ with $\overline{\varphi} \circ \rho = \varphi$.

For an open set $U \subset X$, the image $\alpha_g(U)$ is open for all $g \in G$. Hence,

$$\rho^{-1}(\rho(U)) = \bigcup_{g \in G} \alpha_g(U)$$

is an open set, i.e. $\rho(U)$ is open in Y. In other words: $\rho : X \to Y$ is an *open* map. In particular, if ρ is (locally) bijective, it is (locally) a homeomorphism.

Nevertheless, the quotient Y need not be a Hausdorff space. Take, for instance, $X = \mathbb{C}$ and the multiplicative group $G = \mathbb{C}^*$ acting on $\mathbb{C}$ by multiplication. Then we have the closed orbit $\{0\}$ and the dense orbit $\mathbb{C}^*$, and the quotient $Y = \mathbb{C}/\mathbb{C}^*$ consists of two points and is not Hausdorff.

For a *finite* group G, this unpleasant behaviour of the quotient X/G can be excluded: For two G–invariant points $x^{(1)}, x^{(2)} \in X$ choose open neighborhoods U_1 of $x^{(1)}$ and U_2^g of $(x^{(2)})^g$, $g \in G$, with $U_1 \cap U_2^g = \emptyset$, and put

$$U_2 = \bigcap_{g \in G} \alpha_g^{-1}(U_2^g) .$$

U_2 is an open neighborhood of $x^{(1)}$ which does not intersect U_1. In order to show that X/G is a Hausdorff space, we have to make sure that the images $\rho(U_1)$ and $\rho(U_2)$ are distinct: otherwise, there would exist elements $z^{(1)} \in U_1$, $z^{(2)} \in U_2$ and a group element h such that $z^{(1)} = (z^{(2)})^h$, implying

$$z^{(1)} \in \alpha_h(U_2) = \alpha_h\Big(\bigcap_{g \in G} \alpha_g^{-1}(U_2^g)\Big) \subset \bigcap_{g \in G} \alpha_h \circ \alpha_g^{-1}(U_2^g) \subset U_2^h$$

which contradicts our assumption $U_1 \cap U_2^g = \emptyset$ for all $g \in G$.

Let us close this Section by proving that in this situation the quotient map $\rho : X \to Y$ is a finite map in the sense of Section 1. Only the closedness of ρ needs verification. So, assume that $A \subset X$ is a closed subset. By the finiteness of G, the set

$$\rho^{-1}\rho(A) = \bigcup_{g \in G} \alpha_g(A)$$

is closed in X, too, and the claim follows from the identity

$$Y \setminus \rho(A) = \rho(X \setminus \rho^{-1}(\rho(A))) .$$

Moreover, if the action of G on X is *free* at a point x, i.e. if $x = x^g$ implies $g = e$, then ρ is locally a homeomorphism near x. To show this, it is enough to find a neighborhood U of x with $x^{(1)}, x^{(2)} \in U$, $x^{(1)} \sim x^{(2)} \Longrightarrow x^{(1)} = x^{(2)}$. If such a neighborhood would not exist, we could construct sequences $x_j^{(1)}$ and $x_j^{(2)}$ converging to x with

$$x_j^{(1)} = (x_j^{(2)})^{g_j} \text{ for } g_j \in G , \quad g_j \neq e .$$

Since G is finite, we may assume that $g_j = g$ for all j. But then

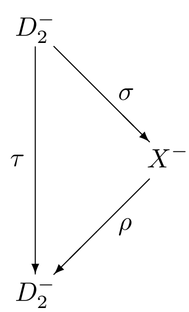
$$x = \lim_j x_j^{(1)} = \lim_j \alpha_g(x_j^{(2)}) = \alpha_g(\lim_j x_j^{(2)}) = \alpha_g(x) = x^g .$$

7.4 The topological structure of isolated Jung singularities

We call X as in Section 1 an isolated Jung singularity, if all the points of $\Delta^- = (X \setminus X^-) \setminus \{0\}$ are smooth points of the analytic set X. Hence, the singular set $\operatorname{sing} X$ of X is either empty or it consists of the point 0 only. Let $G \subset \mathbb{Z} \oplus \mathbb{Z} = \pi_1(D^-)$ be the abelian group belonging to the covering $\rho : X^- \to D^-$, and denote by $G_1 = \mathbb{Z}(nbe_1) + \mathbb{Z}(ae_2)$ the subgroup of G with G/G_1 cyclic of order n (see Lemma 2). The group G_1 can easily be realized as the group belonging to the covering

$$\tau : \begin{cases} D_2^- \longrightarrow D_2^- , \\ (u, v) \longmapsto (u^{nb}, v^a) . \end{cases}$$

Since X^- is the quotient of the universal covering of D_2^- by the larger group G, we have a commutative diagram



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where σ is given by the canonical action of the quotient G/G_1 on D_2^-. σ is a holomorphic map, since τ and ρ are locally biholomorphic; therefore, the composition

$$D_2^- \xrightarrow{\sigma} X^- \hookrightarrow D = D_{m-2} \times D_2 \hookrightarrow \mathbb{C}^m$$

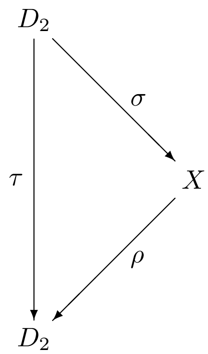
is holomorphic. Thus, writing $\sigma = (\sigma_1, \ldots, \sigma_m)$, the functions σ_j are holomorphic on D_2^- and bounded in absolute value by 1, and we are in a position to apply the *Riemann Extension Theorem* which we would like to state in the following more general form:

***Theorem 7.3** *Let M be a complex analytic manifold and $A \subset M$ be a nowhere dense analytic subset. Then each function*

$$f \in H^0(U \setminus A, \mathcal{O}_M) , \quad U \subset M \text{ open} ,$$

which is bounded locally at each point $x^{(0)} \in U \cap A$, can uniquely be extended to a holomorphic function on U.

So, there is a holomorphic extension $\sigma : D_2 \to \mathbb{C}^m$ of the map above which factorizes over X because of the closedness of X in D and the Maximum Principle for holomorphic functions. Since τ has a holomorphic extension to D_2 - given simply by $\tau(u, v) = (u^{nb}, v^a)$ - we get an extended diagram



which is commutative, since σ, ρ, τ are continuous and D_2^- is dense in D_2.

The main result of the present Section is

Theorem 7.4 *X is canonically homeomorphic to the quotient of D_2 by the action of the cyclic group $\overline{G} = G/G_1$. $(D_2 \setminus \{0\})/\overline{G}$ carries a natural structure of a complex analytic manifold, and the restricted map $X \setminus \{0\} \to (D_2 \setminus \{0\})/\overline{G}$ is biholomorphic.*

Before we present the *proof*, we compute explicitly the action of the group $\overline{G}$ on D_2. Of course, τ is the quotient map with respect to the Galois group

$$(\mathbb{Z} \oplus \mathbb{Z})/G_1$$

whose two canonical generators act by

$$(u, v) \longmapsto (\zeta_{nb} u, v) ,$$

respectively by

$$(u, v) \longmapsto (u, \zeta_a v) .$$

(From now on, ζ_ℓ denotes always a primitive ℓ–th root of unity). Thus, the generator $\overline{x} = (be_1 + de_2) \bmod G_1$ acts linearly by

$$(u, v) \longmapsto (\zeta_{nb}^b u, \zeta_a^d v) = (\zeta_n u, \zeta_n^q v) ,$$

where $q = d/\alpha$ is relatively prime to $n = a/\alpha$. (See the proof of Lemma 2). Since this action is completely determined by the natural numbers n, q, we replace the symbol $\overline{G}$ by C_{nq} (where C stands for "cyclic") and denote the quotient by X_{nq}.

It is evident from the explicit form of the action of $\overline{x}$ that the origin $0 \in \mathbb{C}^2$ is the only fixed point and that the action of C_{nq} on $\mathbb{C}^2 \setminus \{0\}$ is free. Thus, the map

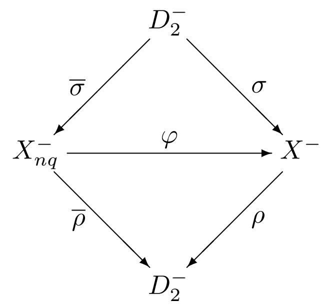
$$D_2 \setminus \{0\} \xrightarrow{\overline{\sigma}} X_{nq} \setminus \overline{\sigma}(0)$$

is an unbranched covering. The action of C_{nq} being linear, it can also be considered as a holomorphic action such that the system of charts

$$(U, \overline{\sigma}_{|U}, \overline{\sigma}(U)) , \quad U \subset D_2 \setminus \{0\} \text{ open} , \quad \overline{\sigma}_{|U} : U \longrightarrow \overline{\sigma}(U) \text{ a homeomorphism} ,$$

gives $X_{nq} \setminus \overline{\sigma}(0)$ the natural structure of a complex analytic manifold, making $\overline{\sigma}$ into a locally biholomorphic map, i.e. into an unbranched *holomorphic* covering. Clearly, a function $f : X_{nq} \setminus \overline{\sigma}(0) \to \mathbb{C}$ is holomorphic, if and only if $f \circ \sigma \in H^0(D_2 \setminus \{0\}, \mathcal{O}_{\mathbb{C}_2})$.

Since the fibers of τ are invariant under C_{nq}, the map σ factorizes over $\overline{\sigma} : \tau = \overline{\rho} \circ \overline{\sigma}$, and by the finiteness of τ and $\overline{\sigma}$, it is easily derived that $\overline{\rho}$ is finite, too, with $\overline{\rho}^{-1}(0) = \overline{\sigma}(0)$. We denote by X_{nq}^- the preimage of D_2^- under $\overline{\rho}$; by construction, there exists a homeomorphism $\varphi : X_{nq}^- \to X^-$ making the diagram



commutative. $\overline{\sigma}$ being on $D_2 \setminus \{ 0 \}$ a holomorphic covering, it is plain that φ is indeed a holomorphic map, and invoking the Riemann Extension Theorem again gives us a holomorphic extension

$$\varphi : X_{nq} \setminus \overline{\rho}^{-1}(0) \longrightarrow X \setminus \{ 0 \} .$$

In order to apply the same reasoning to the inverse $\psi = \varphi^{-1} : X^- \to X_{nq}^-$, we need a representation of this map by *bounded* holomorphic functions. In other words: we must be able to embed $X_{nq} \setminus \overline{\rho}^{-1}(0)$ into (an open subset of) a bounded polydisk in some number space $\mathbb{C}^e$. In fact, the entire space X_{nq} can be realized as an analytic subset of such a polydisk (with $\overline{\rho}^{-1}(0) = 0 \in \mathbb{C}^e$). This follows from the general theory of complex analytic quotients which we begin to study in the next Section. Accepting this result for the moment, we find a holomorphic extension

$$\psi : X \setminus \{ 0 \} \longrightarrow X_{nq} \setminus \{ 0 \}$$

which inverts φ since $\psi \circ \varphi = \mathrm{id}$ and $\varphi \circ \psi = \mathrm{id}$ by continuity.

All that remains to complete the proof of Theorem 4 is to prove the existence of *continuous* extensions

$$X_{nq} \longleftrightarrow X ,$$

or, in other terms, to ascertain the implications

$$\lim_{j \to \infty} x'_j = 0 = \overline{\rho}^{-1}(0) , \quad x'_j \in X_{nq} \setminus \{ 0 \} \iff \lim_{j \to \infty} x_j = 0 , \quad x_j = \varphi(x'_j) .$$

Since our argument applies to both directions, we restrict to one of them. So, assume that $\lim x'_j = 0$. Then $\lim \rho(x'_j) = 0$, and we can use the following purely topological fact, whose proof is left to the reader:

Lemma 7.5 *Let $\rho : Y \to Z$ be a finite continuous map between topological spaces Y and Z, let $z^{(0)} \in Z$ be a point and $\rho^{-1}(z^{(0)}) = \{ y^{(1)}, \ldots, y^{(\ell)} \}$. Then to each pair of neighborhoods U_0 of $\rho^{-1}(z^{(0)})$ and V_0 of $z^{(0)}$ there exists a neighborhood V of $z^{(0)}$ such that:*

(i) $V \subset V_0$,

(ii) $U := \rho^{-1}(V) \subset U_0$,

(iii) $U = \bigcup_{\lambda=1}^{\ell} U_\lambda$, $y_\lambda \in U_\lambda$ *open* , $\overline{U}_\lambda \cap \overline{U}_\mu = \emptyset$, $\lambda \neq \mu$,

(iv) $\rho_\lambda := \rho_{|U_\lambda} : U_\lambda \to V$ *is a finite map for all* λ .

7.5 The analytic structure of cyclic quotients and invariant theory

We claimed in the preceding Section that the quotient $X_{nq} = D_2 / C_{nq}$ (together with its complex analytic manifold structure outside the possibly singular point $\overline{\rho}^{-1}(0)$) can be realized as an analytic subset of a polydisk $D_e \subset \mathbb{C}^e$ about the origin. Let us first try to figure out the meaning of this statement by assuming that $X_{nq} \subset D_e$ is already known to be analytic. Then, for instance, choosing any coordinate functions $x_1, \ldots, x_e$ in $\mathbb{C}^e$, the restrictions $\overline{g}_j = x_{j\,|X_{nq}}$ are continuous on X_{nq} and holomorphic outside $0 = \overline{\rho}^{-1}(0)$. As we remarked earlier, the lifted functions $g_j = \overline{g}_j \circ \overline{\sigma}$ are holomorphic on $D_2 \setminus \{0\}$ and continuous at 0, hence holomorphic on D_2 by Riemann's Extension Theorem. It is also clear that these functions are invariant under the action of the group C_{nq}, in symbols:

$$g_j \in H^0(D_2, \mathcal{O}_{\mathbb{C}^2})^{C_{nq}} .$$

They are bounded and satisfy $g(z) = (g_1(z), \ldots, g_e(z)) = 0 \in \mathbb{C}^e$ if and only if $z = 0$. Moreover, g separates all C_{nq}–orbits:

$$g(z) = g(z') \implies z \sim z' \text{ with respect to } C_{nq} ,$$

and induces a holomorphic embedding of $(D_2 \setminus \{0\})/C_{nq}$ into some punctured polydisk.

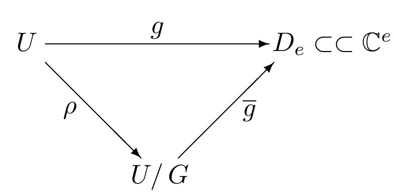
So, loosely speaking, we need a lot of invariant holomorphic functions on D_2. That these really exist, is a consequence of the fact that, for any finite group G acting linearly on $\mathbb{C}^2$, the *invariant algebra*

$$S^G , \text{ where } S = \mathbb{C}[u, v] ,$$

is sufficiently large (see Chapter 8 for more details). To be more precise: S is finitely generated over S^G (as an S^G–module), and S^G is finitely generated as an algebra, i.e. there exists an algebra epimorphism

$$\varepsilon : \mathbb{C}[x_1, \ldots, x_e] \longrightarrow S^G ,$$

such that each element in S^G is a complex polynomial in the images g_j of the x_j. Of course, the g_j can be chosen as homogeneous polynomials in S of positive degree. Then, restricting $g = (g_1, \ldots, g_e)$ to any G–invariant bounded open neighborhood U of $0 \in \mathbb{C}^2$, gives a commutative diagram



D_e a bounded polydisk in $\mathbb{C}^e$, where $\overline{g}$ is a (closed) topological embedding (i.e. $\overline{g}$ is injective and defines a homeomorphism between U/G and the (closed) image $\overline{g}(U/G)$ equipped with the relative topology coming from D_e). Further, if $f_1, \ldots, f_r$ denotes a basis of the ideal $\ker \varepsilon$, then

$$\overline{g}(U/G) = \{ x \in \mathbb{C}^e : f_1(x) = \cdots = f_r(x) = 0 \} \cap D_e ,$$

and $g : U \to \overline{g}(U/G)$ is a locally biholomorphic map of complex analytic manifolds at points in U, where G acts freely, so that $\overline{g}$ is biholomorphic if restricted to the image under ρ of this set of points.

By a classical result of E. Noether, it is possible, at least in principle, to determine a finite set of generators for the invariant algebra S^G with respect to a finite group $G \subset \mathrm{GL}(2, \mathbb{C})$. First of all, there exists a canonical projection

$$\mu : S \longrightarrow S^G$$

by taking the *average*

$$\mu(P) = \frac{1}{\text{ord } G} \sum_{\gamma \in G} P \circ \gamma^{-1}$$

for any polynomial $P \in S$. Since μ is obviously an S^G–module homomorphism, the algebra S^G is generated as a $\mathbb{C}$–algebra by the elements

$$\mu(u^j v^k) , \quad j + k \geq 1 .$$

Actually, it suffices to take the elements with $j + k \leq \text{ord } G$.

It is an easy étude to perform these calculations for the cyclic groups C_{nq}, acting via the *generator* $\gamma := \operatorname{diag}(\zeta_n, \zeta_n^q)$ by

$$(u^j v^k) \circ \gamma = \zeta_n^{j+qk} u^j v^k .$$

Therefore,

$$\mu(u^j v^k) = \frac{1}{n} \sum_{\ell=0}^{n-1} \zeta_n^{(j+qk)\ell} u^j v^k ,$$

which is equal to

$$\begin{cases} \dfrac{1}{n} \cdot \dfrac{1 - \zeta_n^{(j+qk)n}}{1 - \zeta_n^{(j+qk)}} u^j v^k = 0 , & \text{if } \zeta_n^{j+qk} \neq 1 , \quad \text{i.e. } j + qk \not\equiv 0 \bmod n \\ \dfrac{1}{n} \cdot n u^j v^k = u^j v^k , & \text{if } \zeta_n^{j+qk} = 1 , \quad \text{i.e. } j + qk \equiv 0 \bmod n , \end{cases}$$

such that $S^{C_{nq}}$ is generated by the elements

$$(*) \qquad u^j v^k , \quad 1 \leq j + k \leq n , \ j + qk \equiv 0 \bmod n .$$

So, for *Example*, taking $q = 1$, yields $n + 1$ generators

$$u^n, u^{n-1}v, \ldots, uv^{n-1}, v^n ,$$

which are, in fact, independent. For $q = n - 1 \geq 2$ however, we find the set of generators

$$u^n, v^n, uv, (uv)^2, \ldots, (uv)^\ell , \quad 2\ell \leq n ,$$

which obviously contains redundant elements.

For general n and q with $\gcd(n, q) = 1$, $1 \leq q < n$, the *Hirzebruch–Jung algorithm* allows us to select a minimal set of generators from $(*)$. Let us explain this method geometrically before we convert it into an arithmetical device in the following Section.

Regard the (additive) semigroup

$$\Gamma_{nq} = \{ (j, k) \in \mathbb{N}^2 : j + qk \equiv 0 \bmod n \} \subset \mathbb{Z} \oplus \mathbb{Z}$$

as a subset of $\mathbb{R}_+^2$, where $\mathbb{R}_+ = \{ r \in \mathbb{R} : r \geq 0 \}$, and form the sets

$$L_{nq} = \bigcup_{(j,k) \in \Gamma_{nq}^*} ((j, k) + \mathbb{R}_+^2) , \quad \Gamma_{nq}^* = \Gamma_{nq} \setminus \{ (0, 0) \} ,$$

$$\operatorname{conv} L_{nq} = \textit{convex hull of } L_{nq} ,$$

$$B_{nq}^\infty = \textit{boundary of } \operatorname{conv} L_{nq} :$$

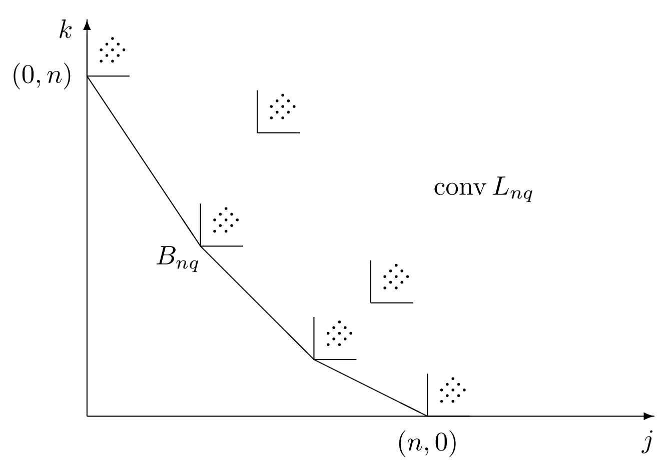


Figure 7.2

B_{nq}^{∞} contains two unbounded parts

$$\{\,(r,\,0) : r > n\,\} \text{ and } \{\,(0,\,r) : r > n\,\}$$

which we remove from B_{nq}^{∞} to get the *essential boundary* (or the *Newton boundary*) B_{nq} of the semigroup Γ_{nq}.

To find a minimal set of generators for the algebra $S^{C_{nq}}$ is evidently the same as to find a minimal set of generators for the semigroup Γ_{nq}. We claim:

Theorem 7.6 *A minimal set of generators for the semigroup* Γ_{nq} *is given by* $B_{nq} \cap \Gamma_{nq}$.

Proof. It is plain due to convexity that the system $B_{n,q} \cap \Gamma_{n,q}$ cannot be shortened. Thus, it is sufficient to show that it generates the semigroup $\Gamma_{n,q}$. Ordering the elements γ_ε of $B_{n,q} \cap \Gamma_{n,q}$ from right to left, starting with $\gamma_1 = (n,\,0)$, $\gamma_2 = (n - q,\,1)$, we get with $\gamma_\varepsilon = (j_\varepsilon,\,k_\varepsilon)$ the finite sequences $(j_\varepsilon,\,k_\varepsilon)$ satisfying

$$n = j_1 > j_2 > \cdots > j_e = 0,$$

$$0 = k_1 < k_2 < \cdots < k_e = n\,.$$

We will show below that these sequences are easily computable by the numbers n and q and determine a concrete minimal set of generators of $\mathbb{C}\,\langle\, u,\, v\,\rangle^{C_{n,q}}$.

Let now γ_ε and $\gamma_{\varepsilon+1}$ be two neighboring elements. We claim:

$$\gamma_\varepsilon \text{ and } \gamma_{\varepsilon+1} \text{ form a } \mathbb{Z}\text{–basis for } \Gamma_{n,q}\,.$$

Suppose to the contrary that there is an element $\rho \in \Gamma_{n,q}$ which is no $\mathbb{Z}$–linear combination of γ_ε and $\gamma_{\varepsilon+1}$. Since the closed parallelogram Π with edges $0,\, \gamma_\varepsilon,\, \gamma_{\varepsilon+1}$ and $\gamma_\varepsilon + \gamma_{\varepsilon+1}$ covers together with its translates under the group $\mathbb{Z}\,\gamma_\varepsilon \oplus \mathbb{Z}\,\gamma_{\varepsilon+1}$ the whole plane $\mathbb{R}^2$ we can assume that $\rho \in \Pi$. Because of the choice of γ_ε and $\gamma_{\varepsilon+1}$, ρ does not lie on the diagonal of Π from γ_ε to $\gamma_{\varepsilon+1}$. It cannot lie below the diagonal either since γ_ε and $\gamma_{\varepsilon+1}$ are elements of the convex hull of $\Gamma_{n,q} \setminus \{\,0\,\}$. If it lies above the diagonal, $\gamma_\varepsilon + \gamma_{\varepsilon+1} - \rho$ is below the diagonal and must be zero for the same reason. Contradiction !

Next, denote by S_ε the *sector* between the lines in $\mathbb{R}_+ \times \mathbb{R}_+$ generated by γ_ε and $\gamma_{\varepsilon+1}$, $\varepsilon = 1, \ldots, e - 1$. From the preceding claim we conclude that

$$\Gamma_{n,q} \cap S_\varepsilon = \{\,\alpha\,\gamma_\varepsilon + \beta\,\gamma_{\varepsilon+1} : \alpha,\,\beta \in \mathbb{N}\,\}\,.$$

Since $\mathbb{R}_+ \times \mathbb{R}_+$ is covered by these sectors, the Theorem follows. $\square$

Example. We illustrate the situation in the preceding Theorem by the case $(n,\,q) = (7,\,4)$. Here, we have $n/(n - q) = 7/3 = 3 - \underline{1\,\rceil\,2} - \underline{1\,\rceil\,2}$.

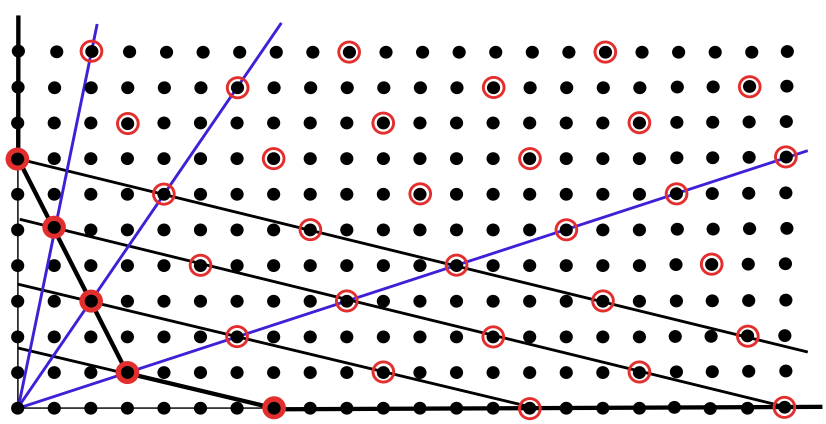


Figure 7.3

7.6 The Hirzebruch - Jung algorithm and equations for cyclic quotients

As promised in the previous Section, we are now going to develop a numerical algorithm for computing the numbers j_ε and k_ε.

Clearly, for each triple $(\varepsilon - 1, \varepsilon, \varepsilon + 1)$, $\varepsilon = 2, \ldots, e-1$, we have $\gamma_{\varepsilon-1} + \gamma_{\varepsilon+1} \in S_{\varepsilon-1} \cup S_\varepsilon$. Let us assume that

$$\gamma_{\varepsilon-1} + \gamma_{\varepsilon+1} \in S_{\varepsilon-1} \text{ and } \gamma_{\varepsilon-1} + \gamma_{\varepsilon+1} = \alpha\,\gamma_{\varepsilon-1} + \beta\,\gamma_\varepsilon \text{ with } \alpha > 0\,,\ \beta \geq 0\,.$$

Then, $\gamma_{\varepsilon+1} = (\alpha - 1)\,\gamma_{\varepsilon-1} + \beta\,\gamma_\varepsilon \in S_{\varepsilon-1}$ which is nonsense. Similarly, the assumption $\gamma_{\varepsilon-1} + \gamma_{\varepsilon+1} = \alpha\,\gamma_\varepsilon + \beta\,\gamma_{\varepsilon+1}$, $\alpha \geq 0$, $\beta > 0$, leads to a contradiction. Henceforth, we get

$$(+) \qquad \gamma_{\varepsilon-1} + \gamma_{\varepsilon+1} = a_\varepsilon\,\gamma_\varepsilon \ (\text{with } a_\varepsilon \geq 2 \text{ due to convexity})\,.$$

The sequence a_ε is easily computed from the numbers n and q. Since the sequence j_ε is strictly decreasing and nonnegative by assumption, we immediately see that the numbers j_ε and $a_\varepsilon \geq 2$ are uniquely determined by the following modification of Euclid's algorithm (recall that $j_1 = n$, $j_2 = n - q$):

$$\begin{cases} j_1 = n & \\ j_2 = n - q & \\ j_3 = a_2 j_2 - j_1\,, & 0 < j_3 < j_2 \\ j_4 = a_3 j_3 - j_2\,, & 0 < j_4 < j_3 \\ \quad\vdots & \quad\vdots \\ j_e = a_{e-1} j_{e-1} - j_{e-2}\,, & 0 = j_e < j_{e-1}\,. \end{cases}$$

This is the *Hirzebruch–Jung algorithm* alluded to in the headline of this Section.

This algorithm leads immediately to a continued fraction expansion of $n/(n-q)$:

$$\frac{n}{n-q} = \frac{j_1}{j_2} = a_2 - \frac{j_3}{j_2} = a_2 - \frac{1}{\left(\dfrac{j_2}{j_3}\right)} = a_2 - \frac{1}{a_3 - \left(\dfrac{j_3}{j_4}\right)}$$

$$= a_2 - \cfrac{1}{a_3 - \cfrac{1}{\ddots \, a_{e-2} - \cfrac{1}{a_{e-1}}}}$$

which we also write in the simpler form

$$(*) \qquad a_2 - 1 \lceil a_3 - \cdots - 1 \lceil a_{e-1}\,.$$

On the other hand, developing $n/(n-q)$ into an expansion $(*)$ with $a_\varepsilon \geq 2$ gives us the numbers j_ε back via the equations

$$(**) \qquad j_1 = n\,,\ j_2 = n - q\,,\ j_{\varepsilon+1} = a_\varepsilon j_\varepsilon - j_{\varepsilon-1}\,, \quad \varepsilon = 2, \ldots, e-1\,.$$

The numbers k_ε are then easily found as well by

$$(***) \qquad k_1 = 0\,,\ k_2 = 1\,,\ k_{\varepsilon+1} = a_\varepsilon k_\varepsilon - k_{\varepsilon-1}\,, \quad \varepsilon = 2, \ldots, e-1\,.$$

Moreover, putting

$$\ell_1 = 1\,,\ \ell_2 = 1\,,\ \ell_{\varepsilon+1} = a_\varepsilon\,\ell_\varepsilon - \ell_{\varepsilon-1}\,,\quad \varepsilon = 2, \ldots, e-1\,,$$

we check by induction that

$$j_\varepsilon + q k_\varepsilon = n\ell_\varepsilon\ ,\qquad \varepsilon = 1, \ldots, e$$

$$\left.\begin{aligned} k_{\varepsilon+1} j_\varepsilon - k_\varepsilon j_{\varepsilon+1} &= n \\ \ell_{\varepsilon+1} j_\varepsilon - \ell_\varepsilon j_{\varepsilon+1} &= q \\ k_{\varepsilon+1}\ell_\varepsilon - k_\varepsilon \ell_{\varepsilon+1} &= 1 \end{aligned}\right\}\quad \varepsilon = 1, \ldots, e-1\,.$$

In particular, $\gcd(k_\varepsilon, \ell_\varepsilon) = 1$ for $\varepsilon = 1, \ldots, e$. Since $j_e = 0$, $k_e = n$ it follows automatically that $\ell_e = q$. Remark also that

$$\ell_1 \le \ell_2 \le \cdots \le \ell_{e-1} \le \ell_e\,.$$

In the theory of continued fractions it is shown (which the reader can easily check by himself) that $k_{\varepsilon+1}$ is equal to the uniquely determined *reduced* numerator of $a_2 - \underline{1\,\big|}\overline{\,a_3\,} - \cdots - \underline{1\,\big|}\overline{\,a_\varepsilon\,}$.

In conclusion, we have proven:

Theorem 7.7 *Let n, q be natural numbers satisfying $1 \le q < n$, $\gcd(n, q) = 1$, and denote by*

$$a_2 - \underline{1\,\big|}\overline{\,a_3\,} - \cdots - \underline{1\,\big|}\overline{\,a_{e-1}\,}\ ,\quad a_\varepsilon \ge 2\,,$$

the Hirzebruch–Jung continued fraction expansion for $n/(n-q)$. Then the invariant algebra $S^{C_{nq}}$ is minimally generated by the monomials

$$u^{j_\varepsilon} v^{k_\varepsilon}\ ,\quad \varepsilon = 1, \ldots, e\,,$$

where the sequences j_ε and k_ε are given by $(**)$ and $(***)$, respectively.

Because of the completely analogous laws for the formation of the j_ε and k_ε, we can write down at once a bunch of algebraic relations for the functions

$$x_\varepsilon = g_\varepsilon(u, v) = u^{j_\varepsilon} v^{k_\varepsilon}\ ,\quad \varepsilon = 1, \ldots, e\,,$$

namely

$$x_{\varepsilon-1} x_{\varepsilon+1} = x_\varepsilon^{a_\varepsilon}\ ,\quad \varepsilon = 2, \ldots, e-1\,.$$

There are still other obvious relations: Remark that

$$\begin{aligned} j_\delta &= a_{\delta+1} j_{\delta+1} - j_{\delta+2} \\ &= (a_{\delta+1} - 1) j_{\delta+1} + (a_{\delta+2} - 1) j_{\delta+2} - j_{\delta+3}\,,\quad \text{etc.} \end{aligned}$$

and similarly for the numbers k_δ, and therefore

$$x_\delta x_\varepsilon = x_{\delta+1}^{a_{\delta+1}-1} x_{\delta+2}^{a_{\delta+2}-2} \cdot \ldots \cdot x_{\varepsilon-2}^{a_{\varepsilon-2}-2} x_{\varepsilon-1}^{a_{\varepsilon-1}-1}\ ,\quad 2 \le \delta + 1 < \varepsilon - 1 \le e - 1\,.$$

Due to the general theory of analytic quotients sketched in the previous Section, it follows that X_{nq} is contained in the analytic set

$$\{\, x = (x_1, \ldots, x_e) \in D_e \subset \mathbb{C}^e : f_{\delta\varepsilon}(x) = 0\,,\quad 2 \le \delta + 1 < \varepsilon - 1 \le e - 1 \}\,,$$

where

$$f_{\delta\varepsilon}(x) = \begin{cases} x_\delta x_\varepsilon - x_{\delta+1}^{a_{\delta+1}}\,, & \delta + 1 = \varepsilon - 1 \\[2ex] x_\delta x_\varepsilon - x_{\delta+1}^{a_{\delta+1}-1} \cdot \ldots \cdot x_{\varepsilon-1}^{a_{\varepsilon-1}-1}\,, & 2 \le \delta + 1 < \varepsilon - 1 \le e - 1\,. \end{cases}$$

It is in fact not difficult to show that the kernel of the algebra homomorphism

$$\begin{cases} \mathbb{C}[x_1, \ldots, x_e] \longrightarrow \mathbb{C}[u, v]^{C_{nq}} \\ \quad x_\varepsilon \quad \longmapsto \quad g_\varepsilon \end{cases}$$

is minimally generated by the $(e-1)(e-2)/2$ polynomials $f_{\delta\varepsilon}$. In particular, the minimal number of equations does only depend on the minimal number of generators for the invariant algebra. This is no coincidence; the natural explanation for this phenomenon shall be gained from the general theory of *rational singularities* in Chapter 12. For this reason, we resist the temptation to handle the case of cyclic quotients here with more elementary tools.

Let us, however, propose a form of these equations that seems to be the easiest to memorize and has, indeed, some conceptual advantage. We look at *generalized* $2 \times n$ *matrices* (with entries in an arbitrary ring R) of type

$$M = \begin{pmatrix} x_1 & & x_2 & & x_3 & \cdots & & & x_n \\ & c_{12} & & c_{23} & & & \cdots & c_{n-1,n} & \\ y_1 & & y_2 & & y_3 & \cdots & & & y_n \end{pmatrix}$$

and form all possible *generalized maximal minors*

$$f_{ij} = x_i y_j - y_i c_{i,i+1} \cdot \ldots \cdot c_{j-1,j} x_j , \quad 1 \le i < j \le n .$$

We then call the ideal generated by these elements an ideal of *quasi–determinantal type* in R. If R is a polynomial ring over a field k in m variables, we call the algebraic set

$$\{ x \in k^m : f_{ij}(x) = 0 , \quad 1 \le i < j \le n \}$$

the variety of *quasi–determinantal type* associated to the matrix M. The case $c_{12} = \cdots = c_{n-1,n} = 1$ is referred to as the *determinantal type*. For an ideal (or a variety) to *be* (quasi-) determinantal (and not merely to be of this *type*), it has to satisfy an extra purely algebraic condition (see Chapter 11).

Using this notion, we can summarize the result on the equations for the cyclic quotients X_{nq} by stating that they form an ideal of quasi–determinantal type associated to the matrix

$$M_{nq} = \begin{pmatrix} x_1 & & x_2 & & x_3 & \cdots & & & x_{e-1} \\ & x_2^{a_2-2} & & x_3^{a_3-2} & & & \cdots & x_{e-1}^{a_{e-1}-2} & \\ x_2 & & x_3 & & x_4 & \cdots & & & x_e \end{pmatrix}$$

Observe that it is of determinantal type, if $a_2 = \cdots = a_{e-1} = 2$, that is if

$$q = n - 1$$

(which implies $e = n + 1$). But these equations are already known to us: they define *the cone over the rational normal curve* of degree n in $\mathbb{P}_n$. Hence, the singularity at the vertex of this cone is realizable (at least topologically) as the singular point of the quotient $\mathbb{C}^2 / C_{n,n-1}$.

Of course, we should remark here that for $e = 3, 4$ the equations can always be presented in determinantal form looking at the matrices

$$\begin{pmatrix} x_1 & x_2^{a_2-1} \\ x_2 & x_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 & x_2 & x_3^{a_3-1} \\ x_2^{a_2-1} & x_3 & x_4 \end{pmatrix} .$$

More generally, such a representation is possible, if $a_3 = \cdots = a_{e-2} = 2$, and these cases exhaust indeed the list of all *determinantal* cyclic quotients.

7.7 The classification of normal Jung singularities

In the following, we would like to emphasize the importance of the notion of *normality* for singular points of complex analytic sets by showing that

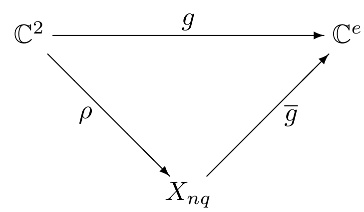
(a) *the cyclic quotients* X_{nq} *are normal*

and

(b) *each normal Jung singularity is not only homeomorphically, but also complex analytically equivalent to such a quotient.*

Moreover, we will establish the first example of a *normalization* by studying the special Jung singularities J_{nq}.

Deviating slightly from our previous notations, we write X_{nq} for the global quotient $\mathbb{C}^2 / C_{nq}$ and have a closer look to the diagram



where $g = (g_1, \ldots, g_e)$ is composed by the generating polynomials $g_\varepsilon(u, v)$ of the invariant algebra $S^{C_{nq}}$. Of course, we would prefer to view the maps ρ and $\overline{g}$ as being holomorphic rather than being continuous only. Now, for a continuous map $\varphi : M_1 \to M_2$ between abstract complex manifolds M_1 and M_2, it is almost immediate from the definition that φ is holomorphic, if and only if $\overline{f} \circ \varphi \in H^0(\varphi^{-1}(V), \mathcal{O}_{M_1})$ for all $\overline{f} \in H^0(V, \mathcal{O}_{M_2})$, $V \subset M_2$ open. Thus, there is only one way to define holomorphy on the abstract quotient X_{nq}:

A function $\overline{f} : V \longrightarrow \mathbb{C}$, V *open in* X_{nq}, *is called* holomorphic, *if* $\overline{f} \circ \rho \in H^0(\rho^{-1}(V), \mathcal{O}_{\mathbb{C}^2})$.

Similarly, we call a map $\overline{f} = (\overline{f}_1, \ldots, \overline{f}_m) : V \to \mathbb{C}^m$ *holomorphic*, if all coordinate functions $\overline{f}_\mu$ are holomorphic.

As in the case of continuous functions, we see at once that $\overline{f} \circ \rho$ is invariant under the action of the group C_{nq} on $H^0(\rho^{-1}(V), \mathcal{O}_{\mathbb{C}^2})$. On the other hand, each invariant holomorphic function $f \in H^0(\rho^{-1}(V), \mathcal{O}_{\mathbb{C}^2})^{C_{nq}}$ is the lifting of a function $\overline{f}$ on V. Hence, denoting by $H^0(V, \mathcal{O}_{X_{nq}})$ as usual the algebra of holomorphic functions on $V \subset X_{nq}$, there exists a canonical algebra–isomorphism

$$H^0(V, \mathcal{O}_{X_{nq}}) \xrightarrow{\sim} H^0(\rho^{-1}(V), \mathcal{O}_{\mathbb{C}^2})^{C_{nq}} .$$

Since we have already introduced a complex analytic manifold structure on $X_{nq} \setminus \{ 0 \}$, $0 = \rho(0)$, we have to convince ourselves that our definition is correct for open sets V not containing 0. But this is clear, since the holomorphic coordinate charts of $X_{nq} \setminus \{ 0 \}$ are built up by localizing ρ.

So, we have a priori some sort of analytic structure on the quotient X_{nq} making ρ and $\overline{g}$ into holomorphic maps, where topologically ρ is a finite map and $\overline{g}$ is (as we already claimed in Section 5) a closed immersion such that X_{nq} can be topologically identified with a closed subset (in fact, an algebraic subset) of $\mathbb{C}^e$. The main point to be proven later is the fact that each invariant holomorphic function on $\mathbb{C}^2$ can be approximated by invariant polynomials. Therefore, the holomorphic functions on X_{nq} are precisely the restrictions of holomorphic functions on $\mathbb{C}^e$.

Having the notion of holomorphic functions on X_{nq} at our disposal, we can also introduce the concept of *analytic subsets* as in the case of manifolds. If

$$A = \{ x \in V \subset X_{nq} : \overline{f}_1(x) = \cdots = \overline{f}_r(x) = 0 \} ,$$

then $\rho^{-1}(A)$ is the set of points in $\rho^{-1}(V)$, where the functions $f_\rho = \overline{f}_\rho \circ \rho$ vanish simultaneously. In particular, if A is analytic and nowhere dense then so is $\rho^{-1}(A)$, ρ being finite. This implies the following:

If $A \subset V$ is a nowhere dense closed analytic subset of the open set $V \subset X_{nq}$, then each everywhere in V locally bounded function $f \in H^0(V \setminus A, \mathcal{O}_{X_{nq}})$ can uniquely be extended to a holomorphic function on V.

Indeed: $\overline{f} \circ \rho = f$ is holomorphic on $\rho^{-1}(V) \setminus \rho^{-1}(A)$ and holomorphically extendable to $\rho^{-1}(V)$ by Riemann's Extension Theorem. Since f is invariant under C_{nq}, the same holds for the extension (just by continuity).

For short, we say that Riemann's Extension Theorem holds for X_{nq}. In general, for spaces X with a suitable notion of holomorphic functions, we call a point $x^{(0)} \in X$ a *normal* point, if Riemann's Extension Theorem is true in an open neighborhood U of $x^{(0)}$ in X. Of course, all points of a complex *manifold* are normal.

It is quite obvious that the normality of the quotients X_{nq} is of great significance for the problem to classify all Jung singularities. Notice first that we constructed a continuous map

$$\varphi : X_{nq} \longrightarrow X$$

for *each* Jung singularity $X \subset \mathbb{C}^m$, the pair (n, q) of course depending on X (and X_{nq} suitably localized near 0). Writing $\varphi = (\varphi_1, \ldots, \varphi_m)$, the functions φ_μ are holomorphic on $X_{nq} \setminus \{0\}$ and continuous at the origin. Thus, φ can be viewed as a holomorphic map $\varphi : X_{nq} \to \mathbb{C}^m$ that factorizes over X, and hence as a holomorphic map $\varphi : X_{nq} \to X$, if we endow X with holomorphic functions by restricting holomorphic functions on $\mathbb{C}^m$ to X.

This procedure works also in the opposite direction. We extended the inverse $\psi : X^- \to X^-_{nq}$ holomorphically to $X \setminus \{0\} \to X_{nq} \setminus \{0\}$ by assuming that $X \setminus \{0\}$ was a manifold. Now, we see that it is sufficient to assume all points of $X \setminus \{0\}$ to be *normal*. Finally, if the origin $0 \in X$ is a normal point, then the extension $\psi : X \to X_{nq}$ is holomorphic in the same sense as above.

So, let us call a Jung singularity $X \subset D_m \subset \mathbb{C}^m$ *normal*, if all points $x^{(0)} \in X$ are normal (with respect to restrictions of holomorphic functions in $\mathbb{C}^m$ to X). Then we have:

Theorem 7.8 *Let $X \subset D \subset \mathbb{C}^m$ be a normal Jung singularity. Then there exist numbers n, q such that the quotient X_{nq} is biholomorphically equivalent to X near 0. More precisely: if a neighborhood of $0 \in X_{nq}$ is represented by an analytic subset of a polydisk $D_e \subset \mathbb{C}^e$, then there exist holomorphic maps*

$$D_e \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} D$$

inducing the homeomorphism $\varphi : X_{nq} \to X$ and its inverse ψ, respectively.

In particular, it follows that X is smooth outside the origin, i.e. X represents automatically an isolated Jung singularity. To some extent, this is no surprise, since normal surface singularities are always isolated (see Chapter 5). But since Theorem 8 is obviously correct for any polydisk D (no matter how large), it suggests the following

Corollary 7.9 *If the branch locus of a normal Jung singularity X is contained in a line, then X is a manifold.*

Indeed: In this case, we have the fundamental group

$$\pi_1(\mathbb{C}^* \times \mathbb{C}) \cong \pi_1(\mathbb{C}^*) \oplus \pi_1(\mathbb{C}) \cong \mathbb{Z} \oplus (0)$$

whose subgroups of finite index are necessarily of type $H = b\,\mathbb{Z} \oplus (0)$, $b \in \mathbb{N}^*$. In order to connect this degenerate situation with the one studied previously, we must restrict coverings over $\mathbb{C}^* \times \mathbb{C}$ to $\mathbb{C}^* \times \mathbb{C}^*$, that is we must take the preimage G of the group H under the natural homomorphism

$$\mathbb{Z} \oplus \mathbb{Z} = \pi_1(\mathbb{C}^* \times \mathbb{C}^*) \longrightarrow \mathbb{Z} \oplus (0)$$

induced by the inclusion $\mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}$, which, of course, is just the quotient by the second summand. Consequently,

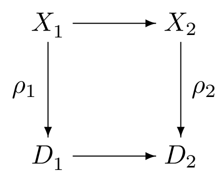
$$G = b\,\mathbb{Z} \oplus 1\,\mathbb{Z}\,,$$

and Corollary 9 is a special case of

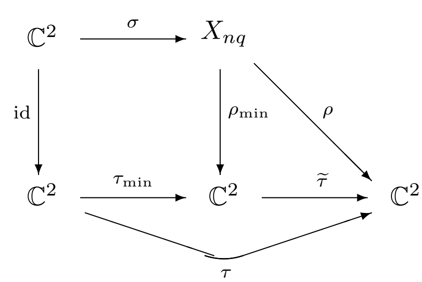
Corollary 7.10 *If for a normal Jung singularity X the group $G \subset \mathbb{Z} \oplus \mathbb{Z}$ describing the covering $X^- \to D_2^-$ is diagonal, then X is smooth.*

In fact, under this condition we may take $n = 1$ and $q = 1$, and the group C_{nq} is trivial such that $D_2 / C_{nq} = D_2$. Moreover, the finite (branched) covering $X = D_2 \to D_2$ is explicitly given by a map of type $(u, v) \mapsto (u^b, v^a)$. This statement reflects the well–known one–dimensional result that each (normal, branched or unbranched) covering of a smooth curve is free of singularities and locally given by $u \mapsto u^a$, $a \in \mathbb{N}^*$. □

Although we introduced the concept of Jung singularities via embeddings and projections, we only want to classify them up to abstract biholomorphic equivalence in the sense explained in Theorem 8. This is a very strong relation; for instance, all *smooth Jung singularities* are isomorphic, but in general such a biholomorphic map between isomorphic Jung singularities cannot be factored to give a commutative diagram



E.g., the abstract quotients X_{nq} can be realized by different branched coverings $\rho : X_{nq} \to \mathbb{C}^2$, depending on the numbers b and α (see Section 4). However, there is always a minimal realization with $b = 1 = \alpha$, since the quotient map $\sigma : \mathbb{C}^2 \to X_{nq}$ factorizes over the map $\tau_{\min}(u, v) = (u^n, v^n)$. Thus, the (global or local) diagram in Section 4 can be factored into



where $\widetilde{\tau}(x, y) = (x^b, y^\alpha)$. It is easily checked that, if X_{nq} is identified with the algebraic subset in $\mathbb{C}^e$ given by the equations $f_{\delta\varepsilon}(x_1, \ldots, x_e) = 0$ of Section 6, the projection $\rho_{\min}$ is induced by $x = x_1$, $y = x_e$. In the following, we assume that $\rho = \rho_{\min}$.

Excluding the uninteresting smooth case $n = 1$, we may assume - as we already did before - that $n \geq 2$. The following is then (together with Theorem 8) the final answer to the classification problem for normal Jung singularities.

Theorem 7.11 *Assume that $n \geq 2$, $1 \leq q < n$, $\gcd(n, q) = 1$. Then X_{nq} is a non–smooth normal Jung singularity which normalizes J_{nq}. Two such quotients X_{nq} and $X_{n'q'}$ are biholomorphically equivalent, if and only if*

$$n = n' \text{ and } q = q'$$

or

$$n = n' \text{ and } qq' \equiv 1 \bmod n\,.$$

Proof. X_{nq} is already known to be a normal Jung singularity. Further, it is easily seen that there exists a commutative diagram

$$\begin{array}{ccc}
\mathbb{C}^* \times \mathbb{C}^* & & \\
 & \searrow^{\sigma'} & \\
\tau \downarrow & & J_{nq}^- = \{ (x,\, y,\, z) \in (\mathbb{C}^*)^3 : z^n = x^{n-q} y \} \\
 & \swarrow_{\rho'} & \\
\mathbb{C}^* \times \mathbb{C}^* & &
\end{array}$$

where $\sigma'(u,\, v) = (u^n,\, v^n,\, u^{n-q} v)$, $\rho'(x,\, y,\, z) = (x,\, y)$ and $\tau\,(u,\, v) = (u^n,\, v^n)$. Moreover, σ' is surjective and

$$\begin{aligned}
\sigma'(u,\, v) = \sigma'(u',\, v') &\iff u' = \zeta_n^\beta u\,,\ v' = \zeta_n^\gamma v\,, \qquad \zeta_n^{\beta(n-q)+\gamma} = 1 \\
&\iff u' = \zeta_n^\beta u\,,\ v' = \zeta_n^\gamma v\,, \qquad \gamma \equiv \beta\, q \bmod n \\
&\iff u' = \zeta_n^\beta u\,,\ v' = (\zeta_n^q)^\beta v\,.
\end{aligned}$$

Therefore, J_{nq}^- is the quotient of $\mathbb{C}^* \times \mathbb{C}^*$ by C_{nq} (in particular, it is connected), and by the normality of X_{nq}, we can extend this isomorphism to get a commutative diagram

$$\begin{array}{ccc}
X_{nq} & \xrightarrow{\ \nu\ } & J_{nq} \\
 {}_{\rho}\searrow & & \swarrow_{\rho'} \\
 & \mathbb{C}^2 &
\end{array}$$

where ν is automatically surjective and finite. But over

$$\Sigma = \{ (x,\, y) \in \mathbb{C}^2 : xy = 0 \}\,,$$

both maps ρ and ρ' in this diagram are obviously one–to–one, such that ν is bijective and consequently a homeomorphism. Using the explicit realization of X_{nq} in $\mathbb{C}^e$, it is clear that ν is the map induced by

$$\begin{cases} \mathbb{C}^e & \longrightarrow \quad \mathbb{C}^3 \\ (x_1, \ldots, x_e) & \longmapsto \quad (x_1,\, x_e,\, x_2)\,. \end{cases}$$

For $q = n - 1$, this map is the identity (up to permutation of the coordinates), such that ν^{-1} is holomorphic, too. But, if $q \neq n-1$, the inverse ν^{-1} cannot be holomorphic, since J_{nq} has nonisolated singularities (and, therefore, J_{nq} is not normal). So, in some definite sense, X_{nq} is a normal complex–analytic structure (via ν) on the topological space J_{nq}. This is what we mean by a normalization of J_{nq} (see also the remarks below).

That X_{nq} is not smooth at the origin for $n \geq 2$ can be proved in many different ways. The first one is of topological nature: it is easy to see that the fundamental group of the manifold $X_{nq} \setminus \{ 0 \}$ is cyclic of order n, hence not trivial. In fact, this is true for a fundamental set of punctured neighborhoods of the origin, such that X_{nq} is even not a topological manifold near 0. The second way is to use the equations for $X_{nq} \subset \mathbb{C}^e$ and to prove that X_{nq} cannot abstractly be realized as an analytic subset of some $\mathbb{C}^m$ with $m < e$ (locally near 0). This follows from the fact that all the germs of the functions $f_{\delta\varepsilon}$ at the origin lie in $\mathfrak{m}_e^2$ (see also the next Chapter). The third way shall be explained in Chapter 8: A quotient $\mathbb{C}^2 / G$, G a finite subgroup of $\mathrm{GL}\,(2,\, \mathbb{C})$, is smooth at the origin, if and only if G is generated by reflections (which leave, by definition, a line through 0 pointwise fixed). But, for the groups C_{nq}, only the identity has this property.

The last assertion is also a consequence of the general theory of quotients (see Theorem 8.15): X_{nq} and $X_{n'q'}$ are biholomorphically equivalent, if and only if the groups C_{nq} and $C_{n'q'}$ are conjugate in $\mathrm{GL}(2, \mathbb{C})$. This, of course, implies that necessarily $n = n'$, and from

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^q \end{pmatrix} = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{q'} \end{pmatrix}^r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

it is easily deduced that $q = q'$ (for $r = 1$) or $r = q$, $q\,q' \equiv 1 \bmod n$ (for $r \neq 1$). □

We would like to add here a few remarks in connection with Theorem 11. First of all, the proof shows that *the complex analytic singularities of J_{nq} outside the origin (which exist in case $q \neq n-1$) are invisible from the abstract topological point of view!* For two–dimensional *normal* singularities this situation is not possible (see Chapter 15). (Observe also that locally near such a point, J_{nq} is a product of a smooth curve and a locally irreducible curve, and it is a classical fact that all (germs of) irreducible curves are homeomorphic to each other). That $\nu : X_{nq} \to J_{nq}$ is not holomorphically invertible in general is reflected by the induced algebra homomorphism

$$\mathcal{O}_{\mathbb{C}^3,0}/\, j_{nq,0}\mathcal{O}_{\mathbb{C}^3,0} =: \mathcal{O}_{J_{nq},0} \longrightarrow \mathcal{O}_{X_{nq},0}$$

which is always injective, but not surjective for $q \neq n-1$. However, one can prove directly (or conclude later from the general theory), that the integral domains $\mathcal{O}_{J_{nq},0}$ and $\mathcal{O}_{X_{nq},0}$ have the same field of fractions, say Q, and that $\mathcal{O}_{X_{nq},0}$ is the *integral closure* of $\mathcal{O}_{J_{nq},0}$ in Q, i.e. $\mathcal{O}_{X_{nq},0}$ consists of all elements in Q that are algebraic over $\mathcal{O}_{J_{nq},0}$.

The reader may also amuse himself by computing the continued fraction

$$\frac{n}{n-q'} = a'_2 - \underline{1 \rfloor}\, \overline{a'_3} - \cdots - \underline{1 \rfloor}\, \overline{a'_{e'-1}}$$

for $q\,q' \equiv 1 \bmod n$. He will find that $e = e'$, $a'_\varepsilon = a_{e+1-\varepsilon}$, $\varepsilon = 2, \ldots, e-1$, if

$$\frac{n}{n-q} = a_2 - \underline{1 \rfloor}\, \overline{a_3} - \cdots - \underline{1 \rfloor}\, \overline{a_{e-1}}$$

denotes the corresponding continued fraction for the pair (n, q).

We want to close this Section with the following complementary result to Theorem 8:

Theorem 7.12 *If C is any finite cyclic subgroup of* $\mathrm{GL}(2, \mathbb{C})$, *then the quotient $\mathbb{C}^2/C$ is biholomorphic to a quotient of type X_{nq}.*

Remark. In Chapter 8, we will prove that the action of any finite group G acting holomorphically on $\mathbb{C}^2$ can be linearized near the origin by introducing new local holomorphic coordinates on $\mathbb{C}^2$ at 0. As a consequence of Theorem 12, we thus can state that *any quotient of $\mathbb{C}^2$ by a finite cyclic group of (local) biholomorphic maps is (locally) biholomorphically equivalent to one of the singularities X_{nq} (or smooth).*

Proof of Theorem 12. Take a generator ψ of C and bring it into Jordan normal form. Since matrices of type

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \quad a \neq 0\,,$$

are not of finite order, ψ must be diagonalizable. So, if n denotes the order of C, we may assume that

$$\psi = \begin{pmatrix} \zeta_n^a & 0 \\ 0 & \zeta_n^b \end{pmatrix}, \quad \gcd(a, b) = 1\,.$$

If $\gcd(a, n) = 1$, it is possible to solve the congruence $b \equiv aq \bmod n$. Hence, $\psi = \gamma^a$, where, as above,

$$\gamma = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^q \end{pmatrix}$$

and $C = \langle \psi \rangle = \langle \gamma \rangle$. If $\gcd(q, n) = 1$, we have $C = C_{nq}$. Otherwise, C has a generator ψ of the form

$$\psi = \begin{pmatrix} \zeta_{rm} & 0 \\ 0 & \zeta_m^s \end{pmatrix}$$

with $1 \le m < n = rm$, such that

$$\psi^m = \begin{pmatrix} \zeta_r & 0 \\ 0 & 1 \end{pmatrix} .$$

If, on the other hand, $\gcd(a, n) \ne 1$, then there exists a number m dividing n with

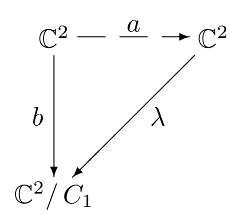
$$\psi^m = \begin{pmatrix} 1 & 0 \\ 0 & \zeta_n^\beta \end{pmatrix}$$

where $\beta \not\equiv 0 \bmod n$.

Hence, in both cases (interchanging coordinates in the second one) we find a generator ψ of our cyclic group C of order n such that for some m dividing n we have

$$\psi = \begin{pmatrix} \zeta_m^\beta & 0 \\ 0 & 1 \end{pmatrix} , \quad \beta \not\equiv 0 \bmod m .$$

By Corollary 9, the quotient $\mathbb{C}^2/C_1$ of $\mathbb{C}^2$ by the subgroup $C_1 = \langle \psi \rangle \subset C$ is isomorphic to $\mathbb{C}^2$, where the biholomorphic map $\lambda : \mathbb{C}^2/C_1 \to \mathbb{C}^2$ is induced by the dotted arrow



given by $(x, y) = (u^\ell, v)$, where $\beta^\ell \equiv 0 \bmod m$.

It is obvious that the quotient group $\overline{C} = C/C_1$ acts on $\mathbb{C}^2/C_1$ by the generator $\overline{\psi} = \psi \bmod C_1$. If we transfer this action to $\mathbb{C}^2$ via the isomorphism λ, we immediately see that $\overline{\psi}$ acts linearly on $\mathbb{C}^2$ (with the coordinates (x, y)) by a matrix of type

$$\begin{pmatrix} \zeta_\nu^\alpha & 0 \\ 0 & \zeta_\nu^\beta \end{pmatrix}$$

where ν is a proper divisor of n.

So, we can keep on playing this game until we reach the trivial group or a group of type C_{nq} generated by an element of the correct form (with new n and q). □

The reader may have noticed that the subgroup $C_1 \subset C$ is generated by a reflection. So, the proof illustrates in a special case the general principle concerning reflection groups we already mentioned in the course of the proof for Theorem 11. Moreover, it suggests as a general method to divide out first the subgroup H generated by all reflections in a given finite subgroup $G \subset \mathrm{GL}(2, \mathbb{C})$ and to study the action of the factor group G/H on the manifold $\mathbb{C}^2/H$. (In fact, as it will turn out, H is always a normal subgroup of G).

7.8 The toroidal group structure of X_{nq}^-

As in the previous Section, we write $X_{nq} = \mathbb{C}^2/C_{nq}$ and $J_{nq} = \{(x, y, z) \in \mathbb{C}^3 : z^n = x^{n-q}y\}$, respectively, and we identify $X_{nq}^- = (\mathbb{C}^*)^2/C_{nq}$ with $J_{nq}^- = \{(x, y, z) \in (\mathbb{C}^*)^3 : z^n = x^{n-q}y\}$ by means of the map $(u, v) \mapsto (u^n, v^n, u^{n-q}v)$. Our goal in the present Section consists in providing X_{nq}^- with a natural group structure of an algebraic torus T_2 which acts canonically on X_{nq} by extending the action of T_2 on itself (via multiplication from the right). Since T_2 is isomorphic, as a complex–analytic manifold, to $\mathbb{C}^* \times \mathbb{C}^*$, we may and will replace the open and dense part X_{nq}^- of X_{nq} by $\mathbb{C}^* \times \mathbb{C}^*$. The resolution of X_{nq} is then achieved in the following Section by a *partial compactification* of $\mathbb{C}^* \times \mathbb{C}^*$.

Let us start with a simple remark on *commuting actions* of two groups G and H on a set X, i.e. (right) actions satisfying $(x^g)^h = (x^h)^g$ for all $g \in G$, $h \in H$. We claim that, under this hypothesis, the group H acts canonically on the quotient X/G. This, of course, amounts to showing that the map $\alpha_h : X \to X$ belonging to an element $h \in H$ maps G–orbits $[x]_G$ onto orbits of the same kind. In other words, we must show that

$$x \sim_G x_1 \iff x^h \sim_G x_1^h \text{ for all } h \in H .$$

One of these implications is trivial; the other one follows from the sequence of implications

$$\begin{aligned} x \sim_G x_1 &\Longrightarrow \text{it exists } g \in G \text{ with } x^g = x_1 \\ &\Longrightarrow \text{it exists } g \in G \text{ with } (x^h)^g = (x^g)^h = x_1^h \text{ for all } h \in H \\ &\Longrightarrow x^h \sim_G x_1^h \text{ for all } h \in H . \end{aligned}$$

Returning to the group $C_{nq} \subset \mathrm{GL}(2, \mathbb{C})$, we try to exhibit a (maximal) subgroup in $\mathrm{GL}(2, \mathbb{C})$ commuting with C_{nq}. Such a group is evidently the *maximal torus*

$$T_2 = \left\{ \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} : s, t \in \mathbb{C}^* \right\}$$

of $\mathrm{GL}(2, \mathbb{C})$. By acting in the canonical way on $\mathbb{C}^2$, T_2 produces the four orbits

$$\begin{aligned} (1, 1) \cdot T_2 &= \mathbb{C}^* \times \mathbb{C}^* , \\ (1, 0) \cdot T_2 &= \mathbb{C}^* \times \{0\} , \\ (0, 1) \cdot T_2 &= \{0\} \times \mathbb{C}^* , \\ (0, 0) \cdot T_2 &= \{(0, 0)\} . \end{aligned}$$

Due to the first identity, which in fact establishes a bijection between T_2 and $\mathbb{C}^* \times \mathbb{C}^*$, we sometimes identify T_2 with $\mathbb{C}^* \times \mathbb{C}^*$. By the remarks made before, there exists a canonical action of T_2 on the quotient

$$X_{nq} = (\mathbb{C} \times \mathbb{C})/C_{nq} ,$$

and the T_2–orbit of the image $(1, 1)$ in X_{nq} is equal to $(\mathbb{C}^* \times \mathbb{C}^*)/C_{nq} = X_{nq}^-$. However, it is clear that T_2 does not act effectively on X_{nq}, whereas the quotient T_2/C_{nq} does. Again, we identify the group T_2/C_{nq} with its orbit X_{nq}^-. Obviously, the map

$$\begin{cases} \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}^* \times \mathbb{C}^* \\ (u, v) \longmapsto (u^n, u^{-q}v) \end{cases}$$

is a surjective group homomorphism with kernel isomorphic to C_{nq} such that

$$X_{nq}^- \cong T_2/C_{nq} \cong \mathbb{C}^* \times \mathbb{C}^*$$

is a *torus* acting canonically on X_{nq}. Identifying X_{nq}^- with J_{nq}^-, the bijection $\mathbb{C}^* \times \mathbb{C}^* \to J_{nq}^-$ is induced by

$$(s, t) \longmapsto (s, s^q t^n, st) .$$

In other words: associated to the cyclic quotient X_{nq} there is an action of T_2 on $\mathbb{C}^3$, namely (multiplicatively written):

$$(x,\, y,\, z) \cdot (s,\, t) \longmapsto (xs,\, ys^q t^n,\, zst)\ ,$$

that induces an action on $J_{nq} \subset \mathbb{C}^3$ and a bijection $T_2 \stackrel{\sim}{\to} J_{nq}^- = X_{nq}^-$ by taking the T_2–orbit of the point $(1,\, 1,\, 1)$.

Now recall that the normalization map from $X_{nq} \subset \mathbb{C}^e$ onto J_{nq} was explicitly presented in the form of a projection $x = x_1$, $y = x_e$, $z = x_2$. Thus, if we want to extend the action of T_2 on X_{nq} to $\mathbb{C}^e$, we have to do it by the rules $x_1 \mapsto x_1 s$, $x_2 \mapsto x_2 st$, $x_e \mapsto x_e s^q t^n$. But on X_{nq}^-, the relations $x_1 x_3 = x_2^{a_2}$, $x_2 x_4 = x_3^{a_3}, \ldots$ can be solved successively for $x_3,\, x_4$ and so on, such that there is only one possible extension, viz.

$$(x_1, \ldots, x_e) \cdot (s,\, t) \longmapsto (\ldots, x_\varepsilon s^{\ell_\varepsilon} t^{k_\varepsilon}, \ldots)_{\varepsilon = 1, \ldots, e}\ ,$$

where the numbers $(k_\varepsilon,\, \ell_\varepsilon)$ are determined by the Hirzebruch–Jung algorithm for $n/(n-q)$. Notice that this leads to the correct action on x_e, since $\ell_e = q$, $k_e = n$, and that the action on $\mathbb{C}^e$ constructed this way is compatible with *all* equations $f_{\delta\varepsilon}$, that is: if the functions $f_{\delta\varepsilon}$ vanish at a point $x \in \mathbb{C}^e$, then they vanish at all points of the T_2–orbit of x. To be more precise, there is the following identity

$$f_{\delta\varepsilon}((x_1, \ldots, x_e) \cdot (s,\, t)) = s^{\ell_\delta + \ell_\varepsilon} t^{k_\delta + k_\varepsilon} f_{\delta\varepsilon}(x_1, \ldots, x_e)$$

for all $s,\, t \in \mathbb{C}^*$, $(x_1, \ldots, x_e) \in \mathbb{C}^e$, $2 \le \delta + 1 \le \varepsilon - 1 \le e - 1$. So, accepting all yet unproven details, we can summarize our considerations in the following form:

Theorem 7.13 *Let n and q be given with $1 \le q < n$, $\gcd(n,\, q) = 1$ and denote by k_ε, $\varepsilon = 1, \ldots, e$, the numbers associated to the Hirzebruch–Jung continued fraction expansion for $n/(n-q)$. Then, embedding the torus $T_2 = \mathbb{C}^* \times \mathbb{C}^*$ into $(\mathbb{C}^*)^e$ via*

$$(s,\, t) \longmapsto (s^{\ell_\varepsilon} t^{k_\varepsilon})_{\varepsilon = 1, \ldots, e}$$

and projecting down to the $(x_1,\, x_e)$–plane gives a concrete realization of the unbranched covering of $\mathbb{C}^* \times \mathbb{C}^*$ with group C_{nq} such that the topological closure of the image of T_2 in $\mathbb{C}^e$ is biholomorphically equivalent to the analytic quotient $X_{nq} = \mathbb{C}^2/C_{nq}$.

The main point here is the (implicit) assertion that the closure of the immersed torus has the structure of a *normal* variety; all other statements are direct or indirect implications of earlier results. We will come back to the question of normality in Appendix B to this Chapter that presents an outline of the general theory of such *torus embeddings*.

7.9 Resolution of J_{nq} by partial compactifications of $\mathbb{C}^* \times \mathbb{C}^*$

According to the previous Section, we have found an open dense part $T_2 \cong \mathbb{C}^* \times \mathbb{C}^*$ in J_{nq} and in X_{nq} such that the restrictions of the canonical projections $\rho : J_{nq} \to \mathbb{C}^2$ and $X_{nq} \to \mathbb{C}^2$ to T_2 are of the form

$$(*) \qquad (s,\, t) \longmapsto (s,\, s^q t^n)\ .$$

Hence, for any resolution $\pi : \widetilde{X}_{nq} \to X_{nq}$ of the Jung singularity X_{nq}, $T_2 \cong \pi^{-1}(T_2)$ is an open and dense subset of $\widetilde{X}_{nq}$ having the property that the functions s and $s^q t^n$ (existing on T_2) can be holomorphically extended to $\widetilde{X}_{nq}$ such that the map $\widetilde{X}_{nq} \to \mathbb{C}^2$ defined by these extensions is proper (as the composition of the resolution π with the finite branched covering ρ) and, in particular, surjective.

We are therefore led to the idea to closing up $\mathbb{C}^* \times \mathbb{C}^*$ in some manifold in such a way that $(*)$ extends to a surjective holomorphic map onto $\mathbb{C}^2$. It is clear that just taking $\mathbb{C} \times \mathbb{C}$ as a closure is not the ultimate choice, since the image of the preimage of the origin under the extension is the t–axis and, therefore, not a compact subset of $\mathbb{C} \times \mathbb{C}$. Consequently, it seems to be reasonable to close up the t–axis to a *projective* line and the whole set $\mathbb{C}^* \times \mathbb{C}^*$ to a holomorphic line bundle over $\mathbb{P}_1$ (since this works at least for the cones over the rational normal curves). To be more precise, we regard the line bundle $\mathcal{O}_{\mathbb{P}_1}(-b)$ on $\mathbb{P}_1$ given by the patching rules

$$u_0 = \frac{1}{u_1}\ , \quad v_0 = u_1^b v_1\ ,$$

and identify $\mathbb{C}^* \times \mathbb{C}^*$ with an open dense part of the total space of this line bundle via $s = v_0$, $t = u_0$. Then the functions $\widetilde{g}_1 = s$, $\widetilde{g}_e = s^q t^n$ (the number e will be identified later with the number that already appeared in Section 6) extend to the functions

$$\widetilde{g}_1 = v_0 = u_1^b v_1\ ,\ \widetilde{g}_e = u_0^n v_0^q = u_1^{bq-n} v_1^q\ ,$$

which are holomorphic everywhere, if and only if $b \geq 0$ and $bq - n \geq 0$.

For the cones mentioned above, i.e. for arbitrary $n \geq 2$ and $q = 1$, we can take $b = n$ and see that we get the missing axis in $\mathbb{C}^2$ by the image of the fiber of $\mathcal{O}(-n)$ over ∞. In fact, as we already know, the singularity X_{n1} can be resolved by the total space of the line bundle $\mathcal{O}(-n)$. In all other cases, $bq - n$ will always be different from zero, such that the new fiber is still mapped to the origin (if $bq - n > 0$). But, taking $b_1 > 0$ minimally with $b_1 q - n > 0$, it is trivial that

$$0 < b_1 q - n =: q_2 < n_2 := q =: q_1 < n_1 := n\ ,$$

and $\widetilde{g}_e = u_1^{q_2} v_2^{n_2}$ behaves better in the new variables with respect to the variable u_1 compared to the original behaviour with respect to v_0.

The moral to be drawn from these considerations is easy: Starting with $n_1 = n$ and $q_1 = q = n_2$ as above, define numbers $b_i \geq 2$, n_i and q_i inductively by

$$n_i = b_i q_i - q_{i+1}\ ,\ 0 \leq q_{i+1} < q_i\ ,\ n_{i+1} = q_i\ ,$$

and stop when $q_{r+1} = 0$. In other words, determine the numbers $b_1 \geq 2, \ldots, b_r \geq 2$ for the Hirzebruch–Jung continued fraction of n/q instead of $n/(n-q)$ (see Section 6):

$$\frac{n}{q} = b_1 - \underline{1 \rceil b_2} - \cdots - \underline{1 \rceil b_r}\ .$$

Then the resolution of X_{nq} should consist of a manifold which can be constructed by patching the total spaces of the line bundles $\mathcal{O}_{\mathbb{P}_1}(-b_1), \ldots, \mathcal{O}_{\mathbb{P}_1}(-b_r)$ together in a specific manner.

To fill in all the details, let us begin with the r copies $\mathcal{O}(-b_1), \ldots, \mathcal{O}(-b_r)$ given in coordinates by

$$\mathcal{O}(-b_1):\qquad u_0 = \frac{1}{u_1}\ ,\ v_0 = u_1^{b_1} v_1$$

$$\mathcal{O}(-b_2):\qquad \widetilde{v}_1 = \frac{1}{v_2}\ ,\ \widetilde{u}_1 = v_2^{b_2} u_2$$

$$\vdots \qquad\qquad \vdots$$

$$\mathcal{O}(-b_r):\qquad \begin{cases} \widetilde{u}_{r-1} = \dfrac{1}{u_r}\ ,\ \widetilde{v}_{r-1} = u_r^{b_r} v_r\ , \text{ if } r \text{ is odd,} \\ \widetilde{v}_{r-1} = \dfrac{1}{v_r}\ ,\ \widetilde{u}_{r-1} = v_r^{b_r} u_r\ , \text{ if } r \text{ is even.} \end{cases}$$

It is easily shown by induction on r that by successively identifying

$$\mathcal{O}(-b_{i-1}) \supset \mathbb{C} \times \mathbb{C} \ni (u_{i-1}, v_{i-1}) \cong (\widetilde{u}_{i-1}, \widetilde{v}_{i-1}) \in \mathbb{C} \times \mathbb{C} \subset \mathcal{O}(-b_i)\ ,\ i = 2, \ldots, r\ ,$$

we get a topological Hausdorff space which we will denote by

$$\widetilde{X}_{nq} \text{ resp. by } \widetilde{X}(b_1, \ldots, b_r)\ .$$

Due to the construction, $\widetilde{X}_{nq}$ is a complex–analytic manifold covered by the $r + 1$ open dense sets

$$U_i \cong \mathbb{C} \times \mathbb{C} = \{ (u_1, v_i) : u_i, v_i \in \mathbb{C} \} , \quad i = 1, \dots, r ,$$

where U_{i-1} and U_i always form a covering of $\mathcal{O}(-b_i)$. Therefore, $\widetilde{X}_{nq}$ contains the union

$$E = \{ v_0 = v_1 = 0 \} \cup \{ u_1 = u_2 = 0 \} \cup \dots$$

of r copies E_i of the rational curve $\mathbb{P}_1$ which intersect each other schematically in the following manner:

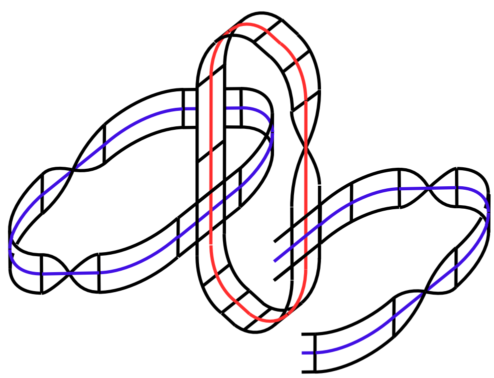


Figure 7.4

It is an immediate consequence of the construction of the manifolds $\widetilde{X}_{nq}$ that all monomials $u_0^j v_0^k$, $j, k \geq 0$, extend to *meromorphic* functions on all of $\widetilde{X}_{nq}$. The first step in determining all holomorphic functions on $\widetilde{X}_{nq}$ (along E) is provided by

Theorem 7.14 *The functions* $\widetilde{g}_1 = v_0$, $\widetilde{g}_2 = u_0 v_0$ *and* $\widetilde{g}_e = u_0^n v_0^q$ *extend holomorphically to* $\widetilde{X}_{nq}$.

Proof. We write $\widetilde{g}_\varepsilon = u_0^{k_\varepsilon} v_0^{\ell_\varepsilon}$, $\varepsilon = 1, 2, e$, and define the integers $k_\varepsilon^{(i)}$, $\ell_\varepsilon^{(i)}$, $i = 0, \dots, r$ inductively by

$$\begin{cases} k_\varepsilon^{(0)} = k_\varepsilon , \quad k_\varepsilon^{(i)} = \ell_\varepsilon^{(i-1)} , & i = 1, \dots, r \\ \ell_\varepsilon^{(0)} = \ell_\varepsilon , \quad \ell_\varepsilon^{(i)} = b_i \ell_\varepsilon^{(i-1)} - k_\varepsilon^{(i-1)} , & i = 1, \dots, r . \end{cases}$$

This choice is obviously made for having the expansions

$$\widetilde{g}_\varepsilon = u_0^{k_\varepsilon} v_0^{\ell_\varepsilon} = u_0^{k_\varepsilon^{(0)}} v_0^{\ell_\varepsilon^{(0)}} = v_1^{k_\varepsilon^{(1)}} u_1^{\ell_\varepsilon^{(1)}} = u_2^{k_\varepsilon^{(2)}} v_2^{\ell_\varepsilon^{(2)}} = \cdots ,$$

and our claim is equivalent to saying that

$$\ell_\varepsilon^{(i)} \geq 0 \text{ for } \varepsilon = 1, 2, e \text{ and } i = 0, \dots, r .$$

But, putting $\ell_\varepsilon^{(-1)} = k_\varepsilon^{(0)} = k_\varepsilon$, the three series $\ell_\varepsilon^{(i)}$, $i = -1, \dots, r$, are generated by

$$\begin{cases} \ell_1^{(i)} : \quad 0, 1 , & \\ \ell_2^{(i)} : \quad 1, 1 , & \ell_\varepsilon^{(i+1)} = b_{i+1} \ell_\varepsilon^{(i)} - \ell_\varepsilon^{(i-1)} , \quad \varepsilon = 1, 2, e . \\ \ell_e^{(i)} : \quad n, q . & \end{cases}$$

Hence, replacing q by $n - q$ in Section 6, we get

$$\begin{aligned} 0 &= \ell_1^{(-1)} < \ell_1^{(0)} < \cdots < \ell_1^{(r)} = n\,, \\ 1 &= \ell_2^{(-1)} \le \ell_2^{(0)} \le \cdots \le \ell_2^{(r)} = n - q\,, \\ n &= \ell_e^{(-1)} > \ell_e^{(0)} > \cdots > \ell_e^{(r)} = 0\,. \end{aligned}$$

Following the proof of Theorem 14, we have in the last coordinate system

$$\widetilde{g}_1(u_r, v_r) = \begin{cases} v_r^{k_1^{(r)}} u_r^n\,, & r \text{ odd}\,, \\ u_r^{k_1^{(r)}} v_r^n\,, & r \text{ even} \end{cases}$$

with a positive number $k_1^{(r)}$ - in fact, it is clear that $k_1^{(r)} = \ell_1^{(r-1)}$ is the uniquely determined k satisfying $0 < k < n$ and $k(n-q) \equiv -1 \bmod n$ - whereas the last function $\widetilde{g}_e$ is simply given by

$$\widetilde{g}_e(u_r, v_r) = \begin{cases} v_r\,, & r \text{ odd}\,, \\ u_r\,, & r \text{ even}\,. \end{cases}$$

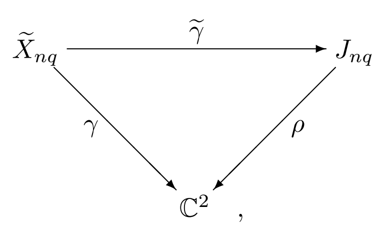
Therefore, the holomorphic map $\gamma : \widetilde{X}_{nq} \to \mathbb{C}^2$ defined by $(\widetilde{g}_1, \widetilde{g}_e)$ is surjective, as we wanted. Moreover, the set

$$\{\, \widetilde{x} \in \widetilde{X}_{nq} : \max(|\,\widetilde{g}_1(x)\,|\,,\, |\,\widetilde{g}_e(x)\,|) \le 1 \,\}$$

is easily seen to be a compact neighborhood of E which can also be described by

$$\{\,(u_0, v_0) : |\,v_0\,| \le 1\,\} \cup \{\,(u_r, v_r) : |\,v_r\,| \le 1\,\}\,.$$

(From now on, we always assume r to be odd. For the case r even, one must replace in all arguments v_r by u_r and vice versa). Thus, γ is a proper map. Since $\widetilde{g}_2$ is holomorphic on $\widetilde{X}_{nq}$ and satisfies the relation $\widetilde{g}_2^n = \widetilde{g}_1^{n-q} \widetilde{g}_e$ on U_0, this identity must hold on all of $\widetilde{X}_{nq}$ due to the Identity Theorem. Consequently, γ factorizes over $\rho : J_{nq} \to \mathbb{C}^2$:



,

$\widetilde{\gamma}\,(\widetilde{x}) = (\widetilde{g}_1(\widetilde{x}), \widetilde{g}_e(\widetilde{x}))\,,\ J_{nq} = \{\,(x_1, x_2, x_e) \in \mathbb{C}^3 : x_2^n = x_1^{n-q} x_e\,\}\,.$

Since ρ is a finite map, $\widetilde{\gamma}$ is automatically proper (which also could be checked directly). Of course

$$(\widetilde{X}_{nq})^- := \gamma^{-1}((\mathbb{C}^2)^-) \cong \{\,(u_0, v_0) \in \mathbb{C}^2 : u_0 v_0 \ne 0\,\} = \mathbb{C}^* \times \mathbb{C}^*\,,$$

and, γ being the correct covering of $(\mathbb{C}^2)^-$, $\widetilde{\gamma}$ is a biholomorphic map when restricted to $(\widetilde{X}_{nq})^-$:

$$\widetilde{\gamma}^- : (\widetilde{X}_{nq})^- \xrightarrow{\sim} J_{nq}^-\,.$$

Now, recall that sing J_{nq} equals $\{\,0\,\}$ for $q = n - 1$ and the set $\{\,x_1 = 0\,\}$ otherwise. In both cases, one easily checks that

$$\widetilde{\gamma}^{-1}(J_{nq} \setminus \operatorname{sing} J_{nq}) = \begin{cases} \{\,(u_0, v_0) \in \mathbb{C}^2 : v_0 \ne 0\,\}\,, & q < n - 1 \\ \{\,(u_0, v_0) : v_0 \ne 0\,\} \cup \{\,(u_r, v_r) : v_r \ne 0\,\}\,, & q = n - 1\,, \end{cases}$$

and that

$$\widetilde{\gamma} : \widetilde{\gamma}^{-1}(J_{nq} \setminus \mathrm{sing}\ J_{nq}) \longrightarrow J_{nq} \setminus \mathrm{sing}\ J_{nq}$$

is bijective. But since, in the first case, the composition

$$\mathbb{C} \times \mathbb{C}^* \cong \widetilde{\gamma}^{-1}(J_{nq} \setminus \mathrm{sing}\ J_{nq}) \longrightarrow J_{nq} \setminus \mathrm{sing}\ J_{nq} \hookrightarrow \mathbb{C}^3$$

is defined by $(x_1, x_2, x_e) = (v_0, u_0 v_0, u_0^n v_0^q)$ which has maximal Jacobi rank, the restriction is biholomorphic. In the second case, one has to compute the restriction of $\widetilde{\gamma}$ to the part U_r, too. But since $q = n - 1$, all $b_i = 2$ and $r = n - 1$, and thus

$$\widetilde{g}_1(u_r, v_r) = v_r^{n-1} u_r^n , \ \widetilde{g}_3(u_r, v_r) = v_r ,$$

such that

$$\widetilde{g}_2(u_r, v_r) = v_r u_r ,$$

and we are in the same situation as before.

Summarizing the results obtained so far in this Section, we state

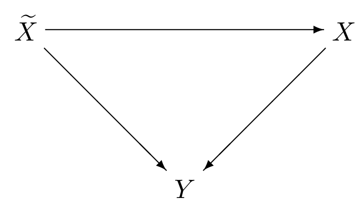
Theorem 7.15 *The proper holomorphic map* $\widetilde{\gamma} = (\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_e) : \widetilde{X}_{nq} \to J_{nq}$ *has the following properties* :

(i) $\widetilde{\gamma}$ *is surjective, and* $\widetilde{\gamma}^{-1}(\mathrm{sing}\ J_{nq})$ *is nowhere dense in* $\widetilde{X}_{nq}$;

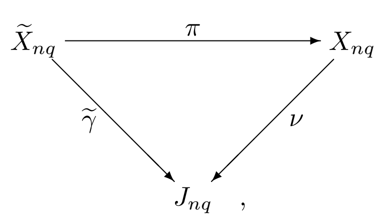
(ii) *the restriction of* $\widetilde{\gamma}$ *to* $\widetilde{X}_{nq} \setminus \widetilde{\gamma}^{-1}(\mathrm{sing}\ J_{nq})$ *maps this open dense part of* $\widetilde{X}_{nq}$ *biholomorphically onto the regular part of* J_{nq} .

7.10 The resolution of normal Jung singularities

It is a general fact that any resolution $\widetilde{X} \to Y$ of a (reduced) complex–analytic singularity Y factorizes uniquely over the normalization X of Y :



and induces a resolution of X (see Chapter 5). So, in our special case, we must be able to construct a holomorphic map π making the following diagram commutative:



It is the purpose of the present Section to construct π explicitly, anticipating some of the general arguments. But we will refrain from carrying out all steps needed for showing that π actually resolves the singularity X_{nq} .

Recall that X_{nq} can be embedded into $\mathbb{C}^e$ via the map $\overline{g}$ induced by $g : \mathbb{C}^2 \to \mathbb{C}^e$, where $g = (g_1, \ldots, g_e)$, $g_\varepsilon(u, v) = u^{j_\varepsilon} v^{k_\varepsilon}$. In the following, we will identify the variable x_ε of $\mathbb{C}^e$ (or more precisely, its restriction as a function to X_{nq}) with the corresponding function $\overline{g}_\varepsilon$. Recall moreover, that the normalization map $\nu : X_{nq} \to J_{nq}$ is induced by the projection $(x_1, \ldots, x_e) \mapsto (x_1, x_e, x_2)$.

Since ν is a biholomorphic map outside the discriminant set, the functions $x_\varepsilon = \overline{g}_\varepsilon$ can be thought of being holomorphic functions on J_{nq}. Of course, x_1, x_2 and x_e extend to holomorphic functions on J_{nq}. But, due to the non–normality of X_{nq} for $q < n - 1$ (i.e. for $e > 3$), it is impossible to extend all the functions x_ε holomorphically to J_{nq}. However, the fact that X_{nq} normalizes J_{nq} implies that all functions x_ε are at least meromorphic on J_{nq}, that is quotients of holomorphic functions whose numerators vanish on nowhere dense analytic subsets only. In our concrete example, such a representation is easily derived: Using the relations

$$(*) \qquad x_1 x_3 = x_2^{a_2}\,,\ x_2 x_4 = x_3^{a_3}, \ldots$$

on X_{nq}, it follows immediately that each x_ε is a rational function in x_1 and x_2 and lifts therefore under $\widetilde{\gamma}$ to a meromorphic function $\widetilde{g}_\varepsilon$ on $\widetilde{X}_{nq}$. So, our problem is reduced to showing that these functions are indeed holomorphic.

Before we give the general argument for the last step, let us calculate the functions $\widetilde{g}_\varepsilon$ in the case of the singularities X_{nq}. By the construction of $\widetilde{X}_{nq}$, it is obvious that $\widetilde{g}_1, \widetilde{g}_2$ and $\widetilde{g}_e$ coincide with the functions which were defined in Section 9 and denoted by the same symbols. In particular, they are of the form

$$\widetilde{g}_\varepsilon(u_0, v_0) = u_0^{j_\varepsilon} v_0^{k_\varepsilon}\ , \quad \varepsilon = 1, 2, e$$

in the first coordinate system U_0 of $\widetilde{X}_{nq}$. Invoking the relations $(*)$ once more and the definition of the series j_ε, k_ε, $\varepsilon = 1, 2, \ldots, e$, in Section 6 immediately yields

$$\widetilde{g}_\varepsilon(u_0, v_0) = u_0^{j_\varepsilon} v_0^{k_\varepsilon} \text{ for all } \varepsilon = 1, \ldots, e\,.$$

Thus, we could prove our claim directly by showing as in the previous Section that these monomials extend holomorphically to $\widetilde{X}_{nq}$. But this can also be achieved by applying the structure of the local ring $A = \mathcal{O}_{X_{nq},0}$ over the ring $B = \mathcal{O}_{J_{nq},0}$ as the algebraic closure of B in its field Q of fractions. In fact, what we only need to know is that the elements $x_\varepsilon \in Q$ are algebraic over B which follows easily by showing inductively the relations

$$x_{\varepsilon+1}^{j_\varepsilon} = x_\varepsilon^{j_{\varepsilon+1}} x_e\ , \quad \varepsilon = 1, \ldots, e - 2\,.$$

Hence, the a priori meromorphic functions $\widetilde{g}_\varepsilon$ satisfy on $\widetilde{X}_{nq}$ the same relations

$$\widetilde{g}_{\varepsilon+1}^{j_\varepsilon} = \widetilde{g}_\varepsilon^{j_{\varepsilon+1}} \widetilde{g}_e\,.$$

Since $\widetilde{X}_{nq}$ is a manifold, its local rings $\mathcal{O}_{\widetilde{X}_{nq},\widetilde{x}}$ are factorial and hence algebraically closed in their resp. fields of fractions. Consequently, all functions $\widetilde{g}_\varepsilon$ are holomorphic on $\widetilde{X}_{nq}$.

Summarizing the content of the previous paragraphs, we have the following concrete description of the resolution $\widetilde{X}_{nq} \to X_{nq}$:

Theorem 7.16 *The functions* $\widetilde{g}_\varepsilon(u_0, v_0) = u_0^{j_\varepsilon} v_0^{k_\varepsilon}$, $\varepsilon = 1, \ldots, e$, *extend holomorphically to the manifold* $\widetilde{X}_{nq}$. *The resolution* $\pi : \widetilde{X}_{nq} \to X_{nq}$ *can explicitly be given by* $\pi(\widetilde{x}) = (\widetilde{g}_1(\widetilde{x}), \ldots, \widetilde{g}_e(\widetilde{x}))$, $\widetilde{x} \in \widetilde{X}_{nq}$.

It may be amusing that the embedding dimension e can also be calculated from the numbers $b_1, \ldots, b_r$: It exists always a diagram of the form

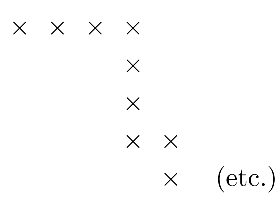


Figure 7.5

such that, if the numbers of crosses in the rows are equal to the $b_i - 1$, then the numbers in the columns are the $a_\varepsilon - 1$; for instance, in the Example above,

$$(b_i) = (5,\, 2,\, 2,\, 3,\, 2)\ ,$$
$$(a_\varepsilon) = (2,\, 2,\, 2,\, 5,\, 3)\ ,$$

and in fact,

$$5 - \underline{1\,|}\overline{\,2\,} - \underline{1\,|}\overline{\,2\,} - \underline{1\,|}\overline{\,3\,} - \underline{1\,|}\overline{\,2\,} = 47/11\ ,$$
$$2 - \underline{1\,|}\overline{\,2\,} - \underline{1\,|}\overline{\,2\,} - \underline{1\,|}\overline{\,5\,} - \underline{1\,|}\overline{\,3\,} = 47/36\ .$$

Moreover, one has the obvious relations

$$\sum_{i=1}^{r} (b_i - 1) = \sum_{\varepsilon=2}^{e-1} (a_\varepsilon - 1)$$

and

$$e - 2 = 1 + \sum_{i=1}^{r} (b_i - 2)\ .$$

Thus,

$$e = 3 + \sum_{i=1}^{r} (b_i - 2)\ .$$

The last formula should be considered as a special case of a more general result applying to the much wider class of *rational singularities* which includes *all* quotient surface singularities (see Chapters 8 and 12).

7.11 The description of the resolution by an invariant - theoretical approach

Inside the manifold $\widetilde{X}_{nq} = \widetilde{X}\,(b_1, \ldots, b_r)$, the open part $U_1 \cup \ldots \cup U_r$ resolves the singularity $X_{n_2 q_2}$, where the coprime pair $(n_2,\, q_2)$ is determined by the continued fraction

$$\frac{n_2}{q_2} = b_2 - \underline{1\,|}\overline{\,b_3\,} - \cdots - \underline{1\,|}\overline{\,b_r\,}$$

(if $r \geq 2$). Therefore, we can modify $\widetilde{X}_{nq}$ in replacing a neighborhood of the union $E_2 \cup \ldots \cup E_r$ by the singularity $X_{n_2 q_2}$ (i.e. by *blowing down* the configuration $E_2 \cup \ldots \cup E_r$; the general theory of this process shall be developed in Chapter 9). The new space - which automatically lies properly and holomorphically over X_{nq} - contains a compact curve (which is rational) and a somewhat simpler singularity and can be considered to be a *partial* resolution of the singularity X_{nq}. So, it appears to be more natural, instead of partially compactifying $\mathbb{C}^* \times \mathbb{C}^*$ to the line bundle $\mathcal{O}_{\mathbb{P}_1}(-b_1)$, a regular object, to rather insert a singularity at infinity. In what follows, we will outline an invariant theoretical description of these partial resolutions that has the advantage to explain why the singularity $X_{n_2 q_2}$ comes into the play.

We first have to treat the case of cones separately, thus trying to find another reason for the appearance of the bundles $\mathcal{O}_{\mathbb{P}_1}(-b)$. We start with Y, a copy of $\mathbb{C}^2$, and let C_n be the group generated by the element

$$g = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n \end{pmatrix}\ ,$$

acting on Y as usual. Denote by $\widetilde{Y}$ the σ–modification of Y at the origin with the standard covering $\widetilde{Y} = U_0 \cup U_1$ so that the projection $\sigma : \widetilde{Y} \to Y$ will be described by

$$(u,\, v) = (v_0,\, u_0 v_0) = (u_1 v_1,\, v_1)\ .$$

Obviously, there exists a unique C_n–action on $\widetilde{Y}$ making σ into a C_n–equivariant holomorphic map; namely

$$\begin{cases} (u_0,\, v_0)^g \;=\; (u_0,\, \zeta_n v_0) \\ (u_1,\, v_1)^g \;=\; (u_1,\, \zeta_n v_1)\,. \end{cases}$$

The quotient $\widetilde{Y}/\,C_n$ lies over $X_{n1} \,=\, Y/\,C_n$:

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{Y}/\,C_n \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ Y & \longrightarrow & X_{n1} \,=\, Y/\,C_n\,, \end{array}$$

and π is easily seen to be a proper holomorphic map inducing a biholomorphic isomorphism outside the (nowhere dense) preimage of the singular point of X_{n1}. (Such maps are special cases of *modifications*; see the next Chapter). In fact, π is already a resolution of X_{n1}, since C_n acts on the manifold $\widetilde{Y}$ as a reflection group forcing the quotient $\widetilde{Y}/\,C_n$ to be smooth. Of course, the quotient can explicitly be calculated: clearly,

$$\begin{cases} U_0/\,C_n \,\cong\, \mathbb{C}^2 \ \text{with variables}\ u_0,\, w_0 \,=\, v_0^n\,, \\ U_1/\,C_n \,\cong\, \mathbb{C}^2 \ \text{with variables}\ u_1,\, w_1 \,=\, v_1^n\,, \end{cases}$$

and these open subsets of $Y/\,C_n$ are patched together according to the rule

$$u_0 \,=\, \frac{1}{u_1}\,, \quad w_0 \,=\, v_0^n \,=\, (u_1 v_1)^n \,=\, u_1^n w_1\,,$$

i.e. the quotient is isomorphic to the total space of the bundle $\mathcal{O}_{\mathbb{P}_1}(-n)$.

In the general case $0 \,<\, q \,<\, n$, $\gcd\,(n,\, q) \,=\, 1$, we return to our standard notation C_{nq} for the cyclic group generated by

$$g \,=\, \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^q \end{pmatrix}$$

acting on $\mathbb{C}^2$ (with variables $\xi,\, \eta$). Let Y be another copy of $\mathbb{C}^2$ (with coordinates $u,\, v$) on which the reflection group G_q generated by

$$h \,=\, \begin{pmatrix} 1 & 0 \\ 0 & \zeta_q \end{pmatrix}$$

acts. We shall identify $Y/\,G_q$ and $\mathbb{C}^2$ via the map

$$(u,\, v) \longmapsto (\xi,\, \eta) \,=\, (u,\, v^q)\,.$$

The action of C_{nq} on $\mathbb{C}^2$ may be lifted to Y such that this map is C_{nq}–equivariant by setting

$$(u,\, v)^g \,=\, (\zeta_n u,\, \zeta_n v)\,.$$

Therefore, we are back to the situation described at the beginning of the present Section. Using moreover that the actions of g and h on $\mathbb{C}^2$ commute, we get the following commutative diagram

$$
\begin{array}{ccccc}
\widetilde{Y} & \xrightarrow{/C_{nq}} & \widetilde{Y}/C_n = \widetilde{X}_{n1} & \xrightarrow{/G_q} & \widetilde{X}_{n1}/G_q \\
\downarrow & & \downarrow & & \downarrow \\
Y & \xrightarrow{/C_{nq}} & Y/C_n = X_{n1} & \xrightarrow{/G_q} & X_{nq} \\
\downarrow \mathrm{id} & & & & \downarrow \mathrm{id} \\
Y & \xrightarrow{/G_q} & \mathbb{C}^2 & \xrightarrow{/C_{nq}} & X_{nq}
\end{array}
$$

in which we would like to understand the upper right square. In order to do so, we must compute the action of the generator $h \in G_q$ on the line bundle $\widetilde{X}_{n1} = \mathcal{O}_{\mathbb{P}_1}(-n)$ which is a simple exercise: It is evident that h acts on $\widetilde{Y}$ via

$$
\left\{ \begin{array}{rcl} (u_0,\, v_0)^h &=& (\zeta_q u_0,\, v_0) \\ (u_1,\, v_1)^h &=& (\zeta_q^{-1} u_1,\, \zeta_q v_1) \end{array} \right.
$$

such that the induced action on $\widetilde{X}_{n1}$ - having the local coordinates $(u_0,\, w_0 = v_0^n)$ and $(u_1,\, w_1 = u_1^n)$ - is of the form

$$
\left\{ \begin{array}{rcl} (u_0,\, w_0)^h &=& (\zeta_q u_0,\, w_0) \\ (u_1,\, w_1)^h &=& (\zeta_q^{-1} u_1,\, \zeta_q^n w_1)\,. \end{array} \right.
$$

In particular, h acts on the zero section of $\widetilde{X}_{n1}$, when identified with the Riemann sphere $S^2 \cong \mathbb{P}_1 \cong \overline{\mathbb{C}}$, by a rotation around the axis through 0 and ∞ with an angle of $2\pi/q$. Hence, X_{n1}/G_q contains the compact analytic subset $\mathbb{P}_1/G_q$ which again is a Riemann sphere $\mathbb{P}_1$ (with homogeneous coordinates u_1^q and v_1^q). Moreover, h acts as a reflection on the first coordinate system. Therefore, there can only be a singularity in $\widetilde{X}_{n1}/G_q$ at the image of $\infty \in \mathbb{P}_1 \subset \widetilde{X}_{n1}$; it must be a cyclic quotient singularity whose type remains to be computed. Of course, we may replace h by its inverse, thus finding a cyclic action of the form

$$
(u_1,\, w_1) \longmapsto (\zeta_q u_1,\, \zeta_q^{-n} w_1) = (\zeta_{n_2} u_1,\, \zeta_{n_2}^{q_2} w_1)
$$

with $n_2 = q$ and $0 \le q_2 < q$ the uniquely determined natural number satisfying $-n \equiv q_2 \bmod q$. So, we meet the same pair of numbers $(n_2,\, q_2)$ as in the preceding Section.

Setting $X_{nq}^{(0)} = X_{nq}$, $X_{nq}^{(1)} = \widetilde{X}_{n1}/G_q$, we find the beginning of a tower of modifications

$$
X_{nq}^{(r)} \xrightarrow{\pi_r} X_{nq}^{(r-1)} \longrightarrow \cdots \longrightarrow X_{nq}^{(1)} \xrightarrow{\pi_1} X_{nq}^{(0)}\,,
$$

where $X_{nq}^{(r)}$ is smooth, r the length of the Hirzebruch–Jung continued fraction of n/q, and $\pi = \pi_r \circ \cdots \circ \pi_1$ is a resolution of the singularity X_{nq}. In order to show that $X_{nq}^{(r)}$ coincides with the manifold $\widetilde{X}_{nq}$ we may assume that $q > 1$, hence $r > 1$. Then, in the tower above, we consider the part

$$
X_{nq}^{(2)} \xrightarrow{\pi_2} X_{nq}^{(1)} \xrightarrow{\pi_1} X_{nq}^{(0)}\,.
$$

Denote by E_i' the preimage under π_i of the only singular point in $X_{nq}^{(i-1)}$, $i = 1, 2$, and by E_1 the strict transform of E_1' in $X_{nq}^{(2)}$. The proof can be finished by induction if the following is true:

a) $X_{nq}^{(2)}$ is smooth near $E_1 \cong \mathbb{P}_1$ and (near E_1) isomorphic to the total space of the line bundle $\mathcal{O}_{\mathbb{P}_1}(-b_1)$;

b) E_2' intersects E_1 as a fiber of $\mathcal{O}_{\mathbb{P}_1}(-b_1)$.

Here, of course, the number b_1 is determined by $n = b_1 q - q_2$.

Let us first have a closer look to $X_{nq}^{(1)}$: It is easy to check that E_1' is described by $\{ w_0 = 0 \}$ in the first (smooth) resp. by the image of $\{ w_1 = 0 \}$ in the second (non smooth) open part which together constitute $X_{nq}^{(1)}$. Let us further check what happens to (the image of) the curve $\{ v = 0 \}$ in X_{nq} under π_1. The strict transform of this curve in $\widetilde{Y}$ being described by the equation $\{ u_0 = 0 \}$ in the first coordinate patch, its strict transform under π_1 is the set given by the equation $z^{z_0} := u_0^q = 0$ in $X_{nq}^{(1)}$.

Hence, by performing the same process once more, the curve $E_2' \cong \mathbb{P}_1$ coming from $X_{n_2 q_2}$ must intersect the strict transform E_1 of E_1' transversely. Thus, $X_{nq}^{(1)}$ is smooth near E_1 and E_2' intersects E_1 transversely, and it remains to show that between z_0', w_0 and z_0 the correct identity holds. To this end we first run again through all the coordinate systems in the construction of $X_{nq}^{(1)}$. For the composite map

$$\widetilde{Y} \longrightarrow Y \longrightarrow \mathbb{C}^2 ,$$

we have in the first coordinate system

$$(\xi, \eta) = (u, v^q) = (v_0, (u_0 v_0)^q) .$$

Hence, the preimage of the meromorphic function η / ξ^q on $\mathbb{C}^2$ can be extended on $\widetilde{Y}$ to the meromorphic function u_0^q:

$$\frac{\eta}{\xi^q} = u_0^q .$$

If we apply this formula to $X_{n_2 q_2}^{(1)}$, we have to take into consideration that

$$\xi' = u_1 , \quad \eta' = w_1 , \quad q' = q_2 = b_1 q - n ,$$

such that we finally get

$$z_0' = u_0'^{q_2} = \frac{w_1}{u_1^{q_2}} = \frac{u_0^n w_0}{u_1^{q_2}} = (u_0^q)^{b_1} w_0 = z_0^{b_1} w_0 . \qquad \square$$

7.12 Resolutions of normal surface singularities

We are now in the position to prove the existence of resolutions of normal two–dimensional singularities in the spirit of H. W. E. Jung. Another approach due to Zariski shall be discussed in Chapter 17.

Theorem 7.17 *Any normal surface singularity X admits a resolution $\pi : \widetilde{X} \to X$.*

Proof. We consider a (small) representative $\rho : X \to S$, S a disk in $\mathbb{C}^2$ with center the origin, of a Noether normalization $R_2 = \mathcal{O}_{\mathbb{C}^2,0} \hookrightarrow \mathcal{O}_{X,x}$ such that the *branch locus* $B \subset S$ has only one singularity at 0, if any. In fact, if 0 is a smooth point of B, we have seen above that X is smooth at the origin such that nothing has to be done. In any case, due to the existence of embedded resolutions for plane curve singularities, we can perform a finite iteration of blowing ups of S, say $\sigma : \widehat{S} \to S$, such that the preimage $\sigma^{-1}(B) =: \widehat{B}$ has only normal crossings. We denote by $\widehat{X}$ the *normalized reduction* of the fiber product[1] $X \times_S \widehat{S}$ which fits into a commutative diagram

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\ \widehat{\pi}\ } & X \\ {\scriptstyle \widehat{\rho}}\downarrow & & \downarrow{\scriptstyle \rho} \\ \widehat{S} & \xrightarrow[\ \sigma\ ]{} & S \end{array}$$

[1] More precisely: of the component of the fiber product that maps <u>onto</u> $\widehat{S}$.

It is clear that $\widehat{\pi}$ is a proper modification, and $\widehat{\rho}$ is a finite covering of $\widehat{S}$ branched along $\widehat{B}$. Hence, $\widehat{X}$ has only finitely many normal Jung singularities which can be resolved separately thus yielding a resolution $\pi : \widetilde{X} \longrightarrow X$ factorizing over $\widehat{X}$. The composition of $\widetilde{X} \longrightarrow \widehat{X}$ with $\widehat{\rho}$ will sometimes be denoted by $\widetilde{\rho}$. □

Remark. To find the concrete resolution data for a given singularity is not as easy as it may sound by the formulation of Theorem 17. First of all, it is not immediately clear how to effectively normalize the Jung singularities that appear during the process. Even in the case of *hypersurfaces* they will in general not show up in the standard form $z^n = x^{n-q}y$. We will demonstrate in the remaining part of this Section how this can be achieved constructively. Second, the exceptional curves - whose genera may take on arbitrary values - are realized as *finite branched coverings* of the rational curve such that we need a device for computing the genera from the branching data. This is the *Riemann–Hurwitz formula* we present in the following Section. How to compute the self–intersection numbers will be clarified in the last Section of the present Chapter in which we compute the resolution of some specific examples. (See also Appendix A). They show among others that Jung's method does not in general yield the *minimal* resolution. (What this actually means and how to handle the problem will be discussed in Chapter 9).

To be precise we assume for the rest of this Section that the Jung singularity is locally given in the concrete form

$$z^N = x^a y^b .$$

We have to find the *normalization* of the corresponding Jung singularity. Clearly, it may happen that the numbers N, a, b have a common (maximal) divisor d. Then, due to the identity

$$P^d - Q^d = \prod_{\delta=0,\ldots,d-1} (P - \zeta_d^\delta Q)$$

the normalization is isomorphic to the disjoint union of normalizations of exactly d isomorphic such singularities for which the corresponding parameters N, a, b have *no common divisor*.

So, suppose that we are in this special situation. Then put $d_{a,b} := \gcd(a, b)$, $d_a := \gcd(a, N)$, $d_b := \gcd(b, N)$ and

$$a_0 := \frac{a}{d_{a,b}d_a} , \quad b_0 := \frac{b}{d_{a,b}d_b} , \quad n := \frac{N}{d_a d_b} .$$

It is not difficult to prove with our former considerations in this Chapter the following

Lemma 7.18 *The normalization of the Jung singularity* $z^N = x^a y^b$ *is in the case* $\gcd(a, b, N) = 1$ *isomorphic to the singularity* $A_{n,q}$ *where* q *is the unique nonnegative integer solution of the congruence* $a_0 + b_0 q \equiv 0 \bmod n$ *with* $q < n$.

Proof. The (finite) map $\mathbb{C}^2 \to \mathbb{C}^3$ defined by

$$(s, t) \longmapsto (s^{nd_b}, t^{nd_a}, s^{a_0 d_{a,b}} t^{b_0 d_{a,b}}) = (s^{N/d_a}, t^{N/d_b}, s^{a/d_a} t^{b/d_b})$$

factorizes obviously over $Y_{N;a,b} := \{ z^N = x^a y^b \} \subset \mathbb{C}^3$. Moreover, it is $C_{n,q}$–equivariant, the cyclic group $C_{n,q}$ acting in the usual way on $\mathbb{C}^2$ and trivially on $\mathbb{C}^3$ because of $a_0 d_{a,b} + q b_0 d_{a,b} = d_{a,b}(a_0 + b_0 q) \equiv 0 \bmod n$, and thus it induces a finite mapping $X_{n,q} \to Y$ that factorizes over the normalization $\widehat{Y}$ of Y. By Riemann's first removable singularities theorem, we conclude that $X_{n,q} \to \widehat{Y}$ is biholomorphic. □

Remarks. 1. In the case $a_0 \equiv 0 \bmod n$ - which is equivalent to $q = 0$ - the normalization is *smooth*, hence isomorphic to the "regular singularity" A_0.

2. In case $\gcd(n, q) = 1$ and $N = n$, $a = n - q$, $b = 1$, we have $a_0 = n - q$, $b_0 = 1$ such that Lemma 18 implies the well–known fact that

$$X_{n,q} = \widehat{Y}_{n;n-q,1} .$$

For later use we need an explicit description of the composition

$$\widetilde{X}_{n,q} \longrightarrow X_{n,q} \longrightarrow Y_{N;a,b}$$

in terms of x, y, z as functions on $\widetilde{X}_{n,q}$. For this, it is sufficient to specify their power series expansion in the standard first coordinate system (u_0, v_0) of $\widetilde{X}_{n,q}$. As an extra new constant we introduce c_0 that is defined via $a_0 + b_0\, q = c_0\, n$.

Lemma 7.19 *In the standard first coordinate system* (u_0, v_0) *of* $\widetilde{X}_{n,q}$ *the functions* x, y, z *are given by the polynomials*

$$x = v_0^{d_b} , \quad y = (u_0^n v_0^q)^{d_a} , \quad z = (u_0^{b_0} v_0^{c_0})^{d_{a,b}} .$$

Proof. This is immediate for x and y since we have on $X_{n,q}$ the relations

$$x = s^{nd_b} = x_1^{d_b} \text{ and } y = t^{nd_a} = x_e^{d_a} .$$

Using Theorem 18, we find the first two results. For the last one we have to show that

$$(u_0^{b_0} v_0^{c_0})^{Nd_{a,b}} = v_0^{ad_b}\, (u_0^n v_0^q)^{bd_a} ,$$

that is

$$N d_{a,b}\, b_0 = n\, b\, d_a , \quad N d_{a,b}\, c_0 = a\, d_b + q\, b\, d_a ,$$

and these relations follow immediately from the definitions. □

Remark. If the normalization of $Y_{N;a,b}$ is smooth then one has to replace the resolution $\widetilde{X}_{n,q}$ by the affine plane $\mathbb{C}^2$ with coordinates u_0 and v_0.

Finally, it will be necessary to have a concrete description of the branched coverings of the x–axis and the y–axis, resp., in $\mathbb{C}^2$ by their preimages in $X_{n,q}$ under the composition

$$X_{n,q} \longrightarrow Y_{N;a,b} \longrightarrow \mathbb{C}^2 .$$

By the proof of Lemma 18, we know that the canonical mapping $\mathbb{C}^2 \longrightarrow X_{n,q} \longrightarrow \mathbb{C}^2$ is of the form

$$(s, t) \longmapsto (s^{nd_b}, t^{nd_a}) ,$$

hence the preimage of the x–axis in $\mathbb{C}^2$ is the s–axis $\{(s, t) \in \mathbb{C}^2 : t = 0\}$ on which the group $C_{n,q}$ acts via $(s, 0) \longmapsto (\zeta_n s, 0)$ such that the covering curve we are looking for is smooth with a local parameter $\varsigma = s^n$, and the covering mapping is just of the form $x = \varsigma^{d_b}$. Needless to say that over the y–axis the corresponding covering is $y = \tau^{d_a}$.

7.13 Genera of Riemann surfaces and the Riemann–Hurwitz formula

Each (connected) compact Riemann surface C carries a nontrivial meromorphic function and can thus be realized as a finite branched covering $\rho : C \to \mathbb{P}_1$ of the Riemann sphere. From the concrete "branching data" one can calculate the genus $g\,(C)$. Even more is true: We start with a finite branched covering

$$\rho : C \longrightarrow C_0$$

of arbitrary connected compact Riemann surfaces of genus g, g_0 resp. Then for each $y^{(0)} \in C_0$ the number $\operatorname{card} \rho^{-1}(y^{(0)})$ is finite, and so is the set of *branch points*

$$B_0 := \{ y^{(0)} \in C_0 : \operatorname{card} \rho^{-1}(y^{(0)}) < n := \max_{y \in C_0} \operatorname{card} \rho^{-1}(y)\} .$$

At each point $x^{(0)} \in C$, the mapping ρ has in suitable holomorphic coordinates the concrete form $x \mapsto x^{n_0}$, $n_0 = n_x \geq 1$, such that outside $x^{(0)}$ the mapping ρ is locally near $x^{(0)}$ a n_0–sheeted covering (unbranched outside $x^{(0)}$). Therefore, for all $y^{(0)} \in C_0$,

$$\sum_{x^{(0)} \in \rho^{-1}(y^{(0)})} n_0 = n ,$$

and the restriction

$$\rho : C \setminus B \longrightarrow C_0 \setminus B_0 , \quad B := \rho^{-1}(B_0) ,$$

is a connected *unbranched covering* of Riemann surfaces having n sheets. Finally, we call

$$b := b(\rho) := \sum_{x^{(0)} \in C} (n_0 - 1) = \sum_{y^{(0)} \in B_0} \sum_{x^{(0)} \in \rho^{-1}(y^{(0)})} (n_0 - 1)$$

the *total branching order* of ρ. According to the preceding formulae it follows that

$$b = n \operatorname{card} B_0 - \operatorname{card} B .$$

Theorem 7.20 (Riemann - Hurwitz formula) *Under the above assumptions,*

$$g = \frac{b}{2} + n(g_0 - 1) + 1 = \frac{n \operatorname{card} B_0 - \operatorname{card} B}{2} + n(g_0 - 1) + 1 .$$

Remarks. 1. If $g_0 = 0$ and the covering is unbranched, then necessarily $n = 1$, $g = 0$, and the covering $\rho : C \longrightarrow C_0$ is an isomorphism. Conversely, if $g = g_0 = 0$ and $n \geq 2$, the covering ρ is branched with $b/2 = n - 1$.

2. In the special case of a *two-sheeted* branched covering, i. e. $n = 2$, the total branching order b coincides with the number of branch points $\operatorname{card} B_0 = \operatorname{card} B$, and the Riemann–Hurwitz formula specializes to

$$g = \frac{b}{2} + 2g_0 - 1 .$$

The *Theorem of Riemann and Roch* (Theorem 9.16) together with *Serre duality* immediately leads for the canonical bundle K_C to the identity

$$g - 1 = \dim H^0(C, \mathcal{O}(K_C)) - \dim H^0(C, \mathcal{O}(K_C^* \otimes K_C)) = 1 - g + d(K_C) ,$$

such that the degree of K_C is equal to

$$d(K_C) = 2g - 2 .$$

The Riemann - Hurwitz formula is therefore equivalent to the following much more natural identity.

Theorem 7.21 (Riemann - Hurwitz formula, second formulation) *Under the above assumptions,*

$$d(K_C) = b + n\, d(K_{C_0}) .$$

By Theorem [**??**], the degree of a holomorphic line bundle L may be computed by the degree of a nontrivial meromorphic section in L. Therefore, the degree of K_C coincides with the degree of any nontrivial meromorphic 1–form on C.

Examples. 1. On the Riemann sphere $\mathbb{P}_1$, we have the nontrivial meromorphic function $u_1 = 1/u_0$ and thus a nontrivial meromorphic 1–form

$$\mathrm{d}u_1 = -1\, u_0^{-2} \mathrm{d}u_0 .$$

So, $d(K_{\mathbb{P}_1}) = -2$ and $g(\mathbb{P}_1) = 0$. In fact, the Riemann sphere is the unique Riemann surface of genus 0 as follows directly from Remark 1.

2. $\mathrm{d}z$ is a holomorphic 1–form over $\mathbb{C}$ which is invariant under translations such that it defines a non vanishing holomorphic 1–form on each torus $\mathbb{C}/\Omega$. Hence, $d(K_{\mathbb{C}/\Omega}) = 0$ and $g(\mathbb{C}/\Omega) = 1$. The *uniformization* theorem implies that each compact Riemann surface of genus 1 is biholomorphically equivalent to a torus $\mathbb{C}/\Omega$.

3. Compact Riemann surface of genus 1 are - as algebraic objects - also called *elliptic curves*. In generalization, 2–sheeted branched coverings $C \to \mathbb{P}_1$ are termed as *hyperelliptic curves*. Their branch locus B_0 have *even* cardinality b, and $g(C) = b/2 - 1$.

4. The meromorphic Weierstraß $\wp$–function $\wp_\Omega$ associated to a lattice $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is invariant under the action of Ω on $\mathbb{C}$ by translations and thus defines a holomorphic mapping $\overline{\wp} : \mathbb{C}/\Omega \to \mathbb{P}_1$. Since $\wp$ has a pole of order 2 at the lattice points, it takes each value on with multiplicity 2, i. e.: $\overline{\wp}$ is a 2–sheeted branched covering of $\mathbb{P}_1$. Since the derivative of $\wp$ vanishes exactly at the points $\omega_1/2, \omega_2/2$ and $(\omega_1 + \omega_2)/2$, the branch points of $\overline{\wp}$ are exactly the points $e_1, e_2, e_3, \infty \in \mathbb{P}_1$ where.

$$e_1 = \wp(\omega_1/2), \quad e_2 = \wp(\omega_2/2), \quad e_3 = \wp((\omega_1 + \omega_2)/2).$$

Hence, again, $g(\mathbb{C}/\Omega) = 1$.

Proof of Theorem 21. Let ω_0 be a non trivial meromorphic 1–form on C_0. Locally near a point $y^{(0)} \in C_0$, ω_0 is essentially of the form $\omega_0 = y^{k_0}\,\mathrm{d}y$, $k_0 \in \mathbb{Z}$, such that

$$d(K_{C_0}) = \deg \omega_0 = \sum_{y^{(0)} \in C_0} k_0 .$$

Since the covering ρ is near $x^{(0)} \in \rho^{-1}(y^{(0)})$ of the form $y = x^{n_0}$, we deduce for the lifting $\omega := \rho^*\omega_0$ the local repesentation

$$\omega = x^{n_0 k_0}\mathrm{d}(x^{n_0}) = n_0\, x^{n_0 k_0 + n_0 - 1}\,\mathrm{d}x ,$$

such that

$$\deg \omega = \sum_{x^{(0)} \in C} (n_0 k_0 + n_0 - 1) = b + \sum_{y^{(0)} \in C_0} k_0 \sum_{x^{(0)} \in \rho^{-1}(y^{(0)})} n_0 = b + n \deg \omega_0 . \qquad \square$$

Remark. It is clear that the Riemann–Hurwitz formula is an exclusively *topological* statement and should thus also have a purely topological proof. This can be achieved via the *Euler–Poincar'e characteristic* or *Euler number*

$$\chi(C) := \dim H_0(C, \mathbb{C}) - \dim H_1(C, \mathbb{C}) + \dim H_2(C, \mathbb{C})$$

of an oriented compact surface C which is related to the genus by the formula

$$\chi(C) := 2 - 2g .$$

Given any *triangulation* of C, the Euler number has a combinatorial interpretation as

$$\chi(C) := v - e + f ,$$

where v denotes the number of *vertices*, e the number of *edges* and f the number of *triangles*. (For more details, see Chapter **[??]**.**[??]**). The Riemann–Hurwitz formula follows from the lifting of a sufficiently fine triangulation of C whose set of vertices is contained in the branch locus B.

Example. The sphere S^2 inherits a triangulation from the regular *tetrahedron* with

$$v = 4, \quad e = 6, \quad f = 4 .$$

Hence, $\chi(S^2) = 2$ and $g(S^2) = 0$.

For the rest of this Section we study the following situation which will show up for some concrete branched coverings of a smooth surface M in connection with Jung's resolution method applied to the hypersurface singularities $z^N = x^2 + y^3$, $N = 2, 3, \ldots, 6$. We assume that $C \cong \mathbb{P}_1$ is a rational curve embedded in M, and that the given covering $\rho_N : M_N \longrightarrow M$ for fixed N is branched exactly at three different points in C, say $0, 1, \infty$, and has there in conveniently chosen local coordinates the following representations:

$$z^N = x\,y^6\,, \quad z^N = x^2\,y^6\,, \quad z^N = x^3\,y^6\,.$$

Our results in Section 12 enable us to calculate the genera of the preimages $C_N = \widehat{\rho}_N^{-1}(C)$, where $\widehat{\rho}_N$ denotes the composition of ρ_N with the *normalization* $\widehat{M}_N \longrightarrow M_N$.

<u>$N = 2$</u>. Because of $\gcd(2, 6) = 2$ we have a twofold cover that is branched over B_0 consisting of the two points $0, \infty$. Hence, $\operatorname{card} B_0 = \operatorname{card} B = 2$ and $g(C_2) = 0$.

<u>$N = 3$</u>. Because of $\gcd(3, 6) = 3$ we have a threefold cover that is branched over B_0 consisting of the two points $0, 1$. Hence, $\operatorname{card} B_0 = \operatorname{card} B = 2$ and $g(C_3) = 0$.

<u>$N = 4$</u>. Because of $\gcd(4, 6) = 2$ we have exactly the same situation as in the case $N = 2$. Consequently, $g(C_4) = 0$.

<u>$N = 5$</u>. Because of $\gcd(5, 6) = 1$ we have the special situation of an unbranched cover. I. e., $C_5 \longrightarrow C$ is an isomorphism, and therefore, $g(C_5) = 0$.

<u>$N = 6$</u>. This is the first really interesting case. Because of $\gcd(6, 6) = 6$ we have a sixfold cover that is branched over B_0 consisting of the three points $0, 1, \infty$. Over 0, C_6 is "fully" branched, over 1, there are 2 branch points of order $3 - 1$, and over ∞, we have 3 branch points of order $2 - 1$. Hence, $\operatorname{card} B_0 = 3$, $\operatorname{card} B = 6$ and $g(C_6) = 1$, i. e., C_6 is in fact an elliptic curve.

7.14 Some examples illuminating Jung's method

We investigate the resolutions of the hypersurface singularities given by

$$z^N = x^2 + y^3$$

for the exponents $N = 2, 3, 4, 5, 6$. Perhaps surprisingly, even in the simplest case $N = 2$ which evidently defines a A_2–singularity, the resolution obtained by Jung's method is not the standard one we found in Section 10. There are extra rational (-1)–curves that can be removed by blowing down according to Castelnuovo's criterion (see Chapter 9). In other words: His method does not yield in general the *minimal resolution* of a given singularity (loc. cit.). The same phenomenon occurs also in the other cases. We will find the minimal resolutions directly by other methods for $N = 4, 5$, i. e. the *Klein singularities* of type E_6, E_8, in Chapter 11.

The exceptional curves in such a resolution are always realized as *branched coverings* of the rational curve $\mathbb{P}_1$. Counting carefully the local branching orders enables one to determine the *genus* of the given exceptional curve (*Riemann–Hurwitz formula*). For $N = 6$ we find the first example of an *elliptic* exceptional curve. The corresponding (simple elliptic) singularity can more naturally be obtained by blowing down the zero section in a line bundle of degree -1 on an elliptic curve (Chapter 10.5 [**??**]).

The calculation of the *self–intersection numbers* will be achieved by a method which we shall fully justify in Chapter 9.

In the first Appendix we present another example showing that the resolution of a surface singularity may also contain *cycles* of exceptional curves.

7.14.1 Preparing the branching locus

We first have to blow up the branching locus $B := \{x^2 + y^3 = 0\}$ so many times until its *total transform* has only normal crossings. Recall that its proper transform is already smooth after one σ–process. Repeating the calculation in Chapter 5.6 using the σ–process described by $(x, y) = (\zeta\xi, \zeta) = (\zeta', \zeta'\xi')$ (with different notations of the coordinates compared to loc. cit.) we get as usual the "exceptional" component

$$B_1 = \{\zeta = 0, \zeta' = 0\} \cong \mathbb{P}_1$$

in the total preimage, and the strict transform of B (which we denote by B_0) will be described in the first coordinate system by the equation

$$\xi^2 + \zeta = 0 .$$

Hence, it touches the exceptional curve to first order (and there are no other singular points in the total transform). In the following pictures, the compact exceptional curves are successively denoted by $B_1, B_2, \cdots$ and drawn in black, and the strict transforms B_0 of B has a blue color.

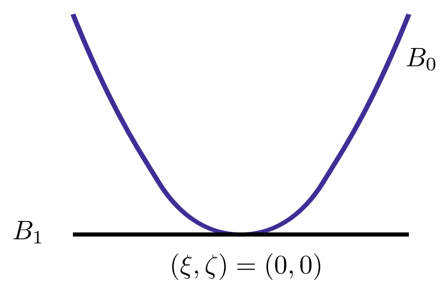


Figure 7.6

Thus, we need a second blow–up at the origin $(\xi, \zeta) = (0, 0)$. Put $(\xi, \zeta) = (\sigma\tau, \tau) = (\tau', \sigma'\tau')$. Then, B_2 is given by $\tau = \tau' = 0$, and (locally near the origin in the first coordinate system) the (strict transforms of) B_1 resp. B_0 have the equations $\sigma = 0$ resp. $\sigma + \tau = 0$ (see Figure 7.8 on the next page). Consequently, we have finally to blow up once more the origin in the (σ, τ)–plane. This will be done by setting

$$(\sigma, \tau) = (st, t) = (t', s't') .$$

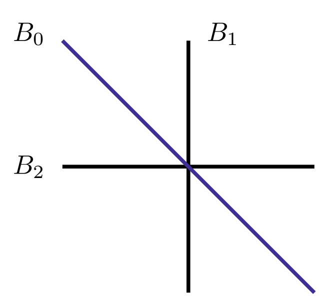


Figure 7.7

Clearly, after this third step, B_3 is defined via $t = t' = 0$, and near this curve, we have $B_1 = \{s = 0\}, B_2 = \{s' = 0\}$ and $B_0 = \{s = -1\}, .$

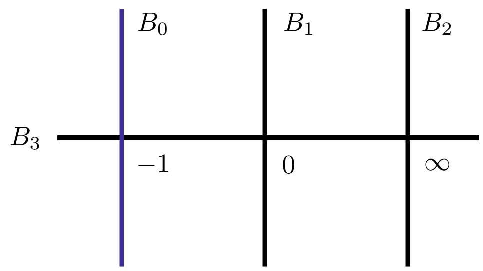


Figure 7.8

In the terminology of Section 12, the union $B_0 \cup B_1 \cup B_2 \cup B_3 = \sigma^{-1}(B) =: \widehat{B} \subset \widehat{S}$ coincides with the branching locus of $\widehat{\rho} : \widehat{X} \longrightarrow \widehat{S}$ where $\widehat{\rho}$ is the normalization of the covering $X \times_S \widehat{S} \longrightarrow \widehat{S}$ which locally has the equations

$$z^N = s^2 (s + 1) t^6 \text{ near } s = 0 \text{ and } s = -1 ,$$

resp.

$$z^N = s'^3 (s' + 1) t'^6 \text{ near } s = \infty , \text{ i.e. } s' = 0 .$$

In particular, the lifting of our given covering to $B_3 \cong \mathbb{P}_1$ is branched at three different points exactly as we have discussed at the end of Section 13.

7.14.2 Determination of the Jung singularities

In all cases in the preceding subsection we determined the branched coverings locally at the three interesting places in the concrete form

$$z^N = x^a y^b .$$

Thus, we can apply the results of the second part of Section 12. Notice that in all cases $a_0 = 1$ such that necessarily $b_0 q + 1 \equiv 0 \bmod n$.

$s = -1$, i.e. $z^N = xy^6$

N	d	N/d	a	b	a_0	b_0	n	q	c_0	sing.
2	1	2	1	6	1	3	1	0	1	A_0
3	1	3	1	6	1	2	1	0	1	A_0
4	1	4	1	6	1	3	2	1	2	A_1
5	1	5	1	6	1	6	5	4	5	A_4
6	1	6	1	6	1	1	1	0	1	A_0

$s = 0$, i.e. $z^N = x^2y^6$

N	d	N/d	a	b	a_0	b_0	n	q	c_0	sing.
2	2	1	1	3	1	3	1	0	1	$2A_0$
3	1	3	2	6	1	1	1	0	1	A_0
4	2	2	1	3	1	3	2	1	2	$2A_1$
5	1	5	2	6	1	3	5	3	2	$A_{5,3}$
6	2	3	1	3	1	1	1	0	1	$2A_0$

$s = \infty$, i.e. $z^N = x^3y^6$

N	d	N/d	a	b	a_0	b_0	n	q	c_0	sing.
2	1	2	3	6	1	1	1	0	1	A_0
3	3	1	3	6	1	2	1	0	1	$3A_0$
4	1	4	3	6	1	1	2	1	1	A_1
5	1	5	3	6	1	2	5	2	1	$A_{5,2}$
6	3	2	3	6	1	1	1	0	1	$3A_0$

7.14.3 Local determination of the divisor of the function z

In preparation for evaluating the so called *selfintersection numbers* of the exceptional components in the resolution $\widetilde{X} \longrightarrow \widehat{X}$ in our examples, it is necessary to find the *divisor* of a global holomorphic or meromorphic function on the resolution. Such a global function is the coordinate function z or, more precisely, its lifting to $\widetilde{X}$.

We start with the local situation in which the divisor of z on the resolution has been already described in Lemma 19. We write as in Section 9

$$\frac{n}{q} = b_1 - \underline{1\,|}\overline{\,b_2} - \cdots - \underline{1\,|}\overline{\,b_r}$$

and construct the resolution $\widetilde{X}_{n,q}$ by patching $r + 1$ copies of $\mathbb{C}^2$ via

$$u_0 = 1/u_1 \,, \quad v_0 = u_1^{b_1}v_1 \quad \text{etc.}\,.$$

Let $E_1, \ldots, E_r$ denote the exceptional curves $v_0 = v_1 = 0$, $u_1 = u_2 = 0$, etc., and let E_0 the line through $0 \in \mathbb{P}^1 \cong E_1$ and perpendicular to E_1, i. e. $E_0 = \{ u_0 = 0 \}$, and E_{r+1} the corresponding line through $\infty \in \mathbb{P}^1 \cong E_r$, i. e. $E_{r+1} = \{ u_r = 0 \}$ or $E_{r+1} = \{ v_r = 0 \}$ depending on whether r is odd or even.

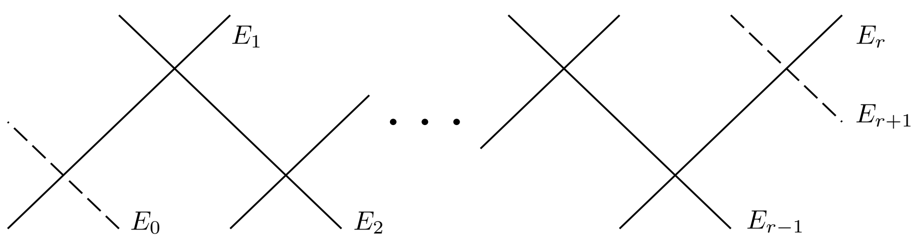


Figure 7.9

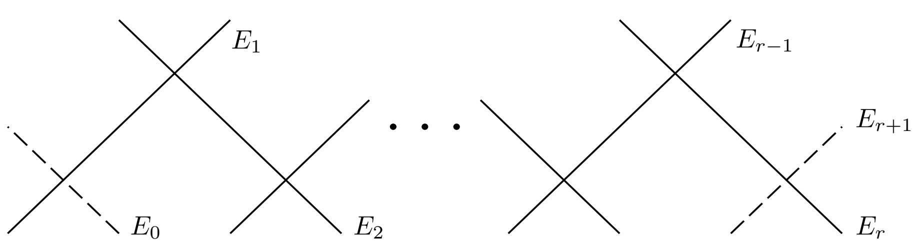


Figure 7.10

Then, one can easily determine by our standard algorithms the vanishing order of z on $E_2, \ldots, E_{r+1}$. Moreover, one should have the following observation in ones mind.

Lemma 7.22 *The strict transform of the x–axis $y = 0$ in $\widetilde{X}_{n,q}$ is the curve E_0, and correspondingly E_{r+1} for the y–axis.*

Finally, in the case that the normalization of $Y_{N;a,b}$ is *smooth* (and irreducible), one has obviously to replace $\widetilde{X}_{n,q}$ by $\mathbb{C}^2$ with coordinates u_0, v_0 (such that E_0 is the u_0–axis and E_1 the v_0–axis), and z is given directly by the formula in Lemma 19. So, after a few trivial manipulations, we can state:

Lemma 7.23 *The normalization of $Y_{N;a,b}$ is* smooth *if and only if it is smooth at at least one irreducible component. Under this condition one has on each component*

$$z = u_0^{b/d_b} v_0^{a/(d_a n)} .$$

Coming back to our examples, we compute for each N separately the divisor of z over small neighborhoods of the points $-1,\, 0,\, \infty \in \mathbb{P}_1 \cong B_3$.

<u>$N = 2$</u> Here, as in the cases $N = 3, 6$, we are at all places and for each irreducible component just dealing with the situation in Lemma 23. One easily computes for z:

$$u_0^3 v_0 \ \ (\text{over } -1) , \quad u_0^3 v_0 \ \ (2 \text{ times over } 0) , \quad u_0^3 v_0^3 \ \ (\text{over } \infty) .$$

<u>$N = 3$</u> $$u_0^2 v_0^1 \ \ (\text{over } -1) , \quad u_0^2 v_0^2 \ \ (\text{over } 0) , \quad u_0^2 v_0^1 \ \ (3 \text{ times over } \infty) .$$

<u>$N = 6$</u> $$u_0^1 v_0^1 \ \ (\text{over } -1) , \quad u_0^1 v_0^1 \ \ (2 \text{ times over } 0) , \quad u_0^1 v_0^1 \ \ (3 \text{ times over } \infty) .$$

<u>$N = 4$</u> Over -1 we find $z = u_0^3 v_0^2 = u_1^1 v_1^2$. Over 0, we have two times the singularity A_1, and since $z = u_0^3 v_0^2$, we find $z = u_1^1 v_1^2$ in the other coordinate system. Similarly, we get over ∞ the representations $z = u_0^3 v_0^3 = u_1^3 v_1^3$.

<u>$N = 5$</u> Over -1, we easily calculate $z = u_0^6 v_0^5 = u_1^4 v_1^5 = u_2^4 v_2^3 = u_3^2 v_3^3 = u_4^2 v_4^1$. Over 0, we have $z = u_0^6 v_0^4$, and since $\dfrac{5}{3} = 2 - \underline{1 \,\rceil}\, 3$, this yields $z = u_0^6 v_0^4 = u_1^2 v_1^4 = u_2^2 v_2^2$. Finally, over ∞, we have $\dfrac{5}{2} = 3 - \underline{1 \,\rceil}\, 2$ and therefore $z = u_0^6 v_0^3 = u_1^3 v_1^3 = u_2^3 v_2^3$.

7.14.4 Determination of the divisor of the function z on $\widetilde{X}$

Since $z^N = x^2 + y^3$ the function z vanishes on $\widetilde{X}$ exactly on the preimage of $\widehat{B}$ under the composition $\widetilde{\rho} : \widetilde{X} \longrightarrow \widehat{X} \longrightarrow \widehat{S}$. We decompose $\widetilde{\rho}^{-1}(\widehat{B})$ into irreducible components where the compact ones are called E_j or E_{jk} when lying over B_j. They are all *rational* curves when $j \neq 3$. Clearly, the union E of all compact components is the *exceptional set* of the resolution $\pi : \widetilde{X} \longrightarrow X$, i.e. the preimage of the singular point in X under π. Finally, the noncompact part $\widetilde{\rho}^{-1}(B_0)$ will be denoted by C.

<u>$N = 2$</u> In this case, we have already $\widetilde{X} = \widehat{X}$, and we find the following situation:

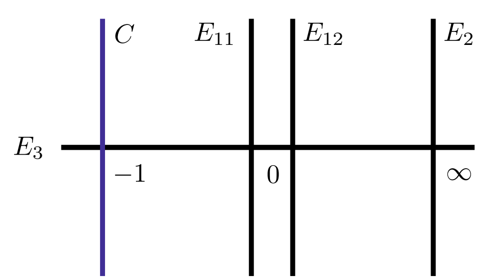


Figure 7.11

From our calculations, we can easily look up the vanishing orders of z along these curves:

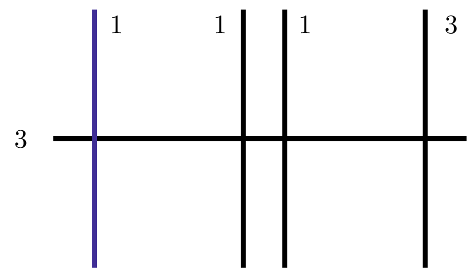


Figure 7.12

We condensate this information into a short formula for the *divisor* of the function z (c.f. Chapter 5.11):

$$\operatorname{div} z \;=\; 1\,C \;+\; 1\,E_{11} \;+\; 1\,E_{12} \;+\; 3\,E_2 \;+\; 3\,E_3 \;.$$

<u>$N = 3$</u> Again, $\widehat{X}$ is smooth, and we find the following configuration:

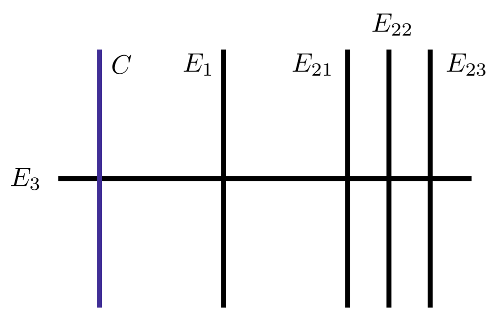


Figure 7.13

Our calculations yield as vanishing orders for the function z:

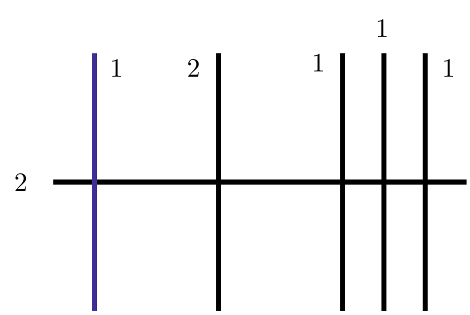


Figure 7.14

Thus, $\operatorname{div} z \;=\; 1\,C \;+\; 2\,E_1 \;+\; 1\,E_{21} \;+\; 1\,E_{22} \;+\; 1\,E_{23} \;+\; 2\,E_3 \;.$

<u>$N = 4$</u> This is the first case in our series in which we have really to resolve (four) Jung singularities (each of type A_1).

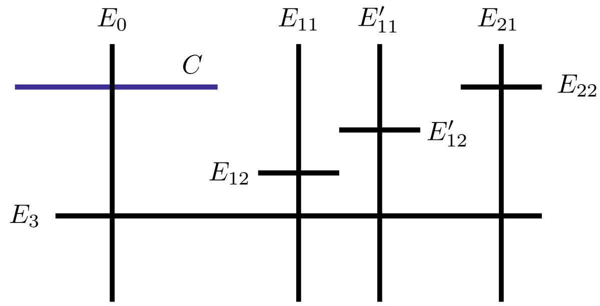


Figure 7.15

The result is:

$$\text{div } z \ = \ 1\,C \ + \ 2\,E_0 \ + \ 2\,E_{11} \ + \ 1\,E_{12} \ + \ 2\,E'_{11} \ + \ 1\,E'_{12} \ + \ 3\,E_{21} \ + \ 3\,E_{22} \ + \ 3\,E_3 \ .$$

<u>$N = 5$</u> Even more involved, we find a A_4–singularity and two others of type $A_{5,3}$ and $A_{5,2}$.

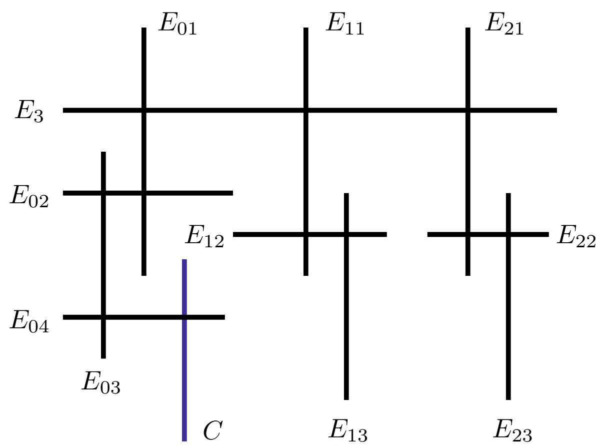


Figure 7.16

This implies:

$$\text{div } z \ = \ 1\,C + 5\,E_{01} + 4\,E_{02} + 3\,E_{03} + 2\,E_{04} + 4\,E_{11} + 2\,E_{12} + 2\,E_{13} + 3\,E_{21} + 3\,E_{22} + 3\,E_{23} + 6\,E_3 \ .$$

<u>$N = 6$</u>

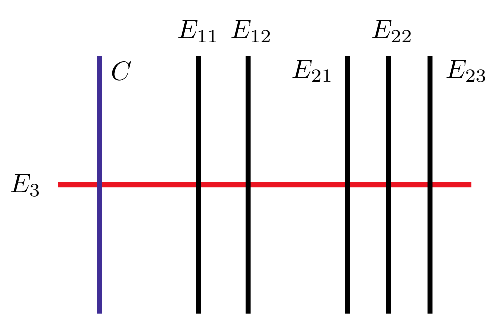


Figure 7.17

This finally yields:

$$\text{div } z \ = \ 1\,C \ + \ 1\,E_{11} \ + \ 1\,E_{12} \ + \ 1\,E_{21} \ + \ 1\,E_{22} \ + \ 1\,E_{23} \ + \ 1\,E_3 \ .$$

7.14.5 Determination of the selfintersection numbers and (dual) resolution graphs

Our next goal is the complete examination of the *intersection matrix* of the divisor $\operatorname{div} z$ and, thereby, of the exceptional set E. By construction it is immediately clear that two different components do not intersect at all or they intersect transversely at precisely one point. In other words: Their intersection number is 0 in the first and 1 in the second case. It remains to calculate the *selfintersection numbers* of the compact components. This will be achieved by using the next Theorem that shall be proven in Chapter 9 (Theorem 9.28).

***Theorem 7.24** *If $C \subset M$ is a compact Riemann surface in a two–dimensional complex manifold M and g a meromorphic function on M, then*

$$(\operatorname{div} g, C) = 0 .$$

Before we return to our series of examples, we test this criterion in some special case of cyclic quotient singularities A_{nq}. Suppose that we have 3 components E_1, E_2, E_3 in the standard resolution, and remember the patching rules

$$u_0 = 1/u_1 , \quad v_0 = u_1^{b_1} v_1 ; \quad v_1 = 1/v_2 , \quad u_1 = v_2^{b_2} u_2 ; \quad u_2 = 1/u_3 , \quad v_2 = u_3^{b_3} v_3 .$$

As we know, the function $g = v_0$ can even *holomorphically* be extended to the total resolution space $\widetilde{X}_{nq}$. Plugging the coordinate transformations in, we realize that

$$g = u_0^0 v_0^1 = v_1^1 u_1^{b_1} = u_2^{b_1} v_2^{b_1 b_2 - 1} = v_3^{b_1 b_2 - 1} u_3^{b_1 b_2 b_3 - b_1 - b_3} .$$

So, with two small discs E_0 and E_4 as in Figure 9, we get

$$\operatorname{div} g = 0\, E_0 + 1\, E_1 + b_1\, E_2 + (b_1 b_2 - 1)\, E_3 + (b_1 b_2 b_3 - b_1 - b_3)\, E_4 .$$

Theorem 24 immediately yields

$$(E_1, E_1) = -b_1 , \quad b_1 (E_2, E_2) = -1 - (b_1 b_2 - 1) = -b_1 b_2 , \quad (b_1 b_2 - 1)(E_3, E_3) = -b_3 (b_1 b_2 - 1)$$

and hence

$$(E_1, E_1) = -b_1 , \quad (E_2, E_2) = -b_2 , \quad (E_3, E_3) = -b_3 .$$

Of course, these examples can easily be generalized to

Lemma 7.25 *Let $E_1, \ldots, E_r$ be the exceptional curves in the resolution $\widetilde{X}_{nq} = \widetilde{X}(b_1, \ldots, b_r)$. Then,*

$$(E_j, E_j) = -b_j .$$

Proof. We concentrate our reasoning to the j-th curve E_j and the coordinate system (u_j, v_j), $j = 1, \ldots, r$. The monomial $g_j = u_j^{\alpha} v_j^{\beta}$, $\alpha, \beta \in \mathbb{Z}$ fixed, extends to a meromorphic function on $\widetilde{X}_{nq}$. In particular, writing the relevant coordinate transformation in the form $u_{j-1} = u_j^{-1}$, $v_{j-1} = u_j^{b_j} v_j$, we get

$$g = u_{j-1}^{\beta b_j - \alpha} v_{j-1}^{\beta} .$$

Thus, writing the part of the divisor of g on the curves E_{j-1}, E_j, E_{j+1} only, we find

$$\operatorname{div} g = \cdots + \alpha\, E_{j-1} + \beta\, E_j + (\beta\, b_j - \alpha)\, E_{j+1} + \cdots$$

and thus

$$\beta (E_j, E_j) = -\alpha - (\beta\, b_j - \alpha) = -\beta\, b_j .$$

Hence, our claim. □

Thanks to our preparatory work it is now pure counting to find the selfintersection numbers in our examples. We write E_j^2 instead of (E_j, E_j) etc.

<u>$N = 2$</u> $E_{11}^2 = -3$, $E_{12}^2 = -3$, $E_2^2 = -1$, $E_3^2 = -2$.

<u>$N = 3$</u> $E_1^2 = -1$, $E_{21}^2 = -2$, $E_{22}^2 = -2$, $E_{23}^2 = -2$, $E_3^2 = -3$.

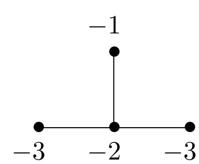
<u>$N = 4$</u> $E_0^2 = -2$, $E_{11}^2 = -2$, $E_{12}^2 = -2$, $(E'_{11})^2 = -2$, $(E'_{12})^2 = -2$, $E_{21}^2 = -2$, $E_{22}^2 = -1$, $E_3^2 = -3$.

<u>$N = 5$</u> $E_{01}^2 = -2$, $E_{02}^2 = -2$, $E_{03}^2 = -2$, $E_{04}^2 = -2$, $E_{11}^2 = -2$, $E_{12}^2 = -3$, $E_{13}^2 = -1$, $E_{21}^2 = -3$, $E_{22}^2 = -2$, $E_{23}^2 = -1$, $E_3^2 = -2$.

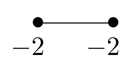
<u>$N = 6$</u> $E_{11}^2 = -1$, $E_{12}^2 = -1$, $E_{21}^2 = -1$, $E_{22}^2 = -1$, $E_{23}^1 = -1$, $E_3^2 = -6$.

Remark. Jung's method yields - as in all examples we investigated up to now - so called *good resolutions*. Under this assumption the information contained in the intersection matrix may be encoded in a *dual resolution graph*. (For precise definitions, cf. Chapter 9.26).

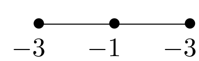
<u>$N = 2$</u> Our result is the following dual resolution graph



which is surprising since we started with an equation of an A_2–singularity such that we should expect the standard dual resolution graph

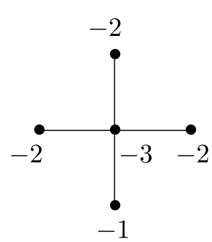


However, when blowing up the intersection point of the two curves gives a resolution

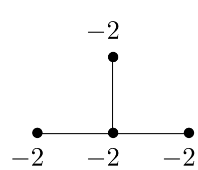


and then the above graph looks like what we get after blowing up once more any non–intersection point on the (-1)–curve. We shall later see that the process of blowing up can be reversed for rational curves with intersection number -1 (Castelnuovo criterion; see Theorem 9.38).

<u>$N = 3$</u> We found

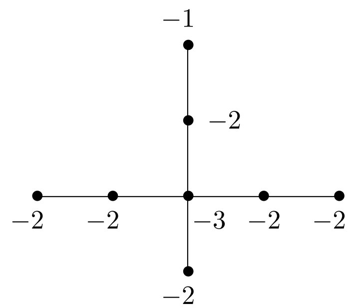


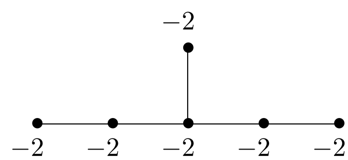
and after blowing down the (-1)–curve this gives



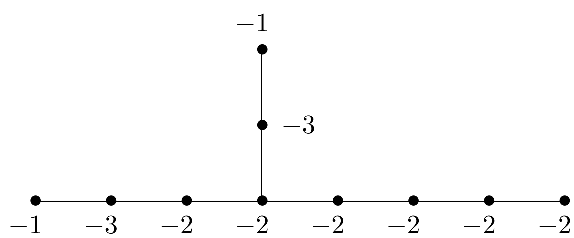
This is in fact the *Klein singularity* of type D_4. (See Chapters 8 and 11).

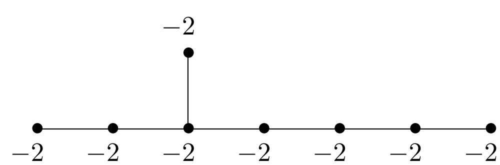
<u>$N = 4$</u> This is the Klein singularity of type E_6 with (a non–minimal and) the minimal resolution graph



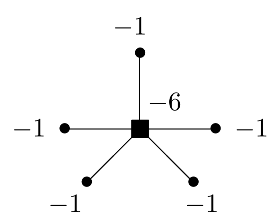


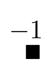
<u>$N = 5$</u> This is the Klein singularity of type E_8 with (a non–minimal and) the minimal resolution graph





<u>$N = 6$</u> If we denote by ■ a certain *elliptic* curve, the dual resolution graph looks like



which after 5 blowing downs is just  , a *simple elliptic* singularity (c.f. Chapter 10.5).

7.A Appendix A: Another example illustrating Jung's method

We study now the hypersurface singularity $z^2 = (x + y^2)(x^2 + y^7)$. As a twofold cover of the (x, y)–plane it has a branch locus B_0 which decomposes into two irreducible components, a smooth one $B_{01} = \{(x, y) : x = -y^2\}$ and a singular one $B_{02} = \{(x, y) : x^2 = -y^7\}$ with a A_4–curve singularity.

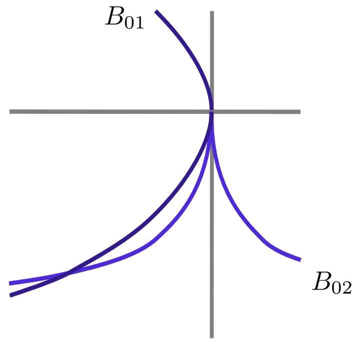


Figure 7.18

So, it's necessary to blow up 3 times to resolve this singularity, to separate the irreducible components, and even once more until we get a divisor with normal crossings.

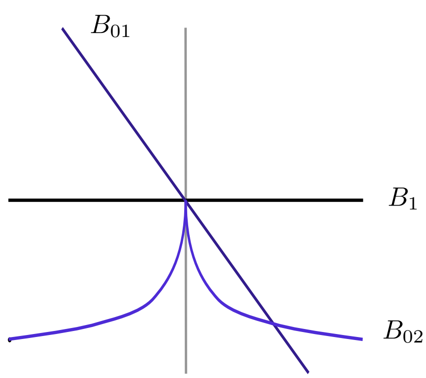


Figure 7.19

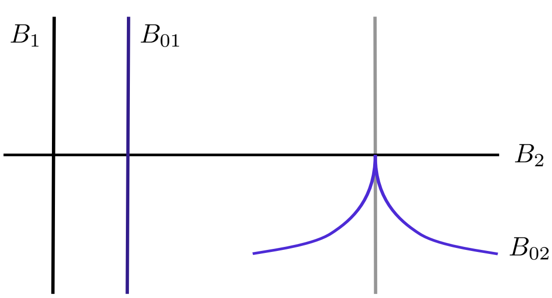


Figure 7.20

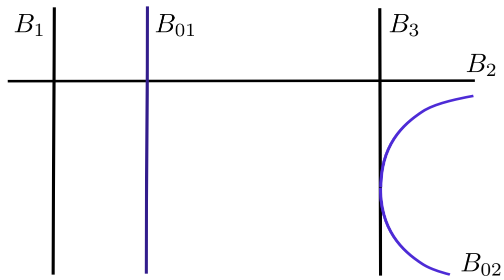


Figure 7.21

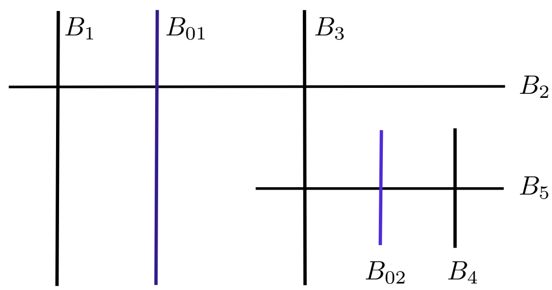


Figure 7.22

Carefully performing the σ–processes we find the equations of the Jung singularities over the intersection points:

$$\begin{aligned}
B_1 \cap B_2 :&\quad z^2 = u^3 v^6 \\
B_2 \cap B_{01} :&\quad z^2 = u^6 v^1 \\
B_2 \cap B_3 :&\quad z^2 = u^6 v^8 \qquad \text{i.e.} \quad 2 \text{ times} \quad z^1 = u^3 v^4 \\
B_3 \cap B_5 :&\quad z^2 = u^8 v^{18} \qquad \text{i.e.} \quad 2 \text{ times} \quad z^1 = u^4 v^9 \\
B_5 \cap B_{02} :&\quad z^2 = u^{18} v^1 \\
B_4 \cap B_5 :&\quad z^2 = u^9 v^{18} \quad .
\end{aligned}$$

All these singularities have a *smooth* normalization, and the twofold covering $\widetilde{\rho} : \widetilde{\rho}^{-1}(B_3) \longrightarrow B_3$ is unbranched such that $\widetilde{\rho}^{-1}(B_3) = E_{31} \cup E_{32}$ with two non–intersecting rational curves (Remark 1). Hence, the configuration in $\widetilde{X}$ looks as follows:

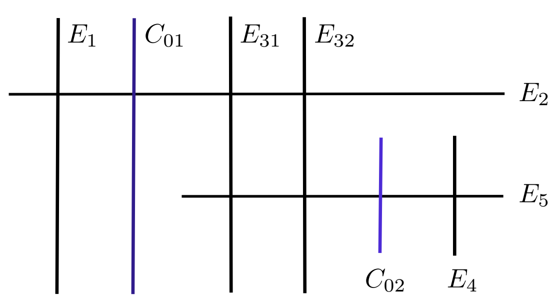


Figure 7.23

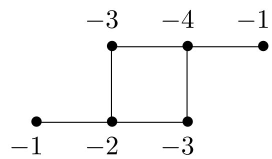
and the divisor of z can explicitly be written down:

$$\operatorname{div} z = 3E_1 + 3E_2 + 4E_{31} + 4E_{32} + 9E_4 + 9E_5 + C_{01} + C_{02} .$$

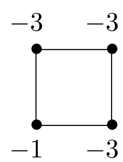
From this we deduce

$$E_1^2 = -1 , \; E_2^2 = -4 , \; E_{31}^2 = -3 , \; E_{32}^2 = -3 , \; E_4^2 = -1 , \; E_5^2 = -2 ,$$

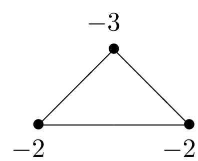
and Jung's procedure gives a resolution with the following dual graph ($\bullet$ representing rational curves):



After blowing down the exceptional curves of first kind we find



and



7.B Appendix B: Torus embeddings and toric varieties

It is the goal of this Appendix to put some of the results of Chapter 7 in the right perspective by viewing Jung singularities in the light of a general concept.

7.B.1 Strongly convex rational polyhedral cones and fans

We first describe the combinatorial "background". It consists in a free $\mathbb{Z}$–module $N \cong \mathbb{Z}^r$ of rank r and its dual $M = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ together with the canonical pairing

$$\langle \cdot , \cdot \rangle : \; M \times N \longrightarrow \mathbb{Z}$$

which extends to the canonical $\mathbb{R}$–bilinear pairing

$$M_{\mathbb{R}} \times N_{\mathbb{R}} \longrightarrow \mathbb{R} , \quad N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r , \quad M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \cong \operatorname{Hom}_{\mathbb{R}}(N_{\mathbb{R}}, \mathbb{R}) .$$

A subset σ of $N_{\mathbb{R}}$ is called a *strongly convex rational polyhedral cone* (with vertex at the origin) if

$$\sigma = \Big\{ \sum_{j=1}^{s} c_j n_j : c_j \geq 0 \text{ for all } j \Big\}$$

for some elements $n_1, \ldots, n_s \in N$, σ not containing any positive dimensional subspace of $N_{\mathbb{R}}$, i.e. $\sigma \cap (-\sigma) = \{ 0 \}$. Thus, σ is, in fact, a strongly convex polyhedral cone, and it is called *rational* since it is spanned by finitely many rational vectors with respect to the lattice N, that is by elements of $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ (this is obviously equivalent to the generation by *integral* elements, i.e. elements in N itself).

The *dual cone* of σ in $M_{\mathbb{R}}$ is denoted by $\check{\sigma}$:

$$\check{\sigma} = \{ x \in M_{\mathbb{R}} : \langle x, y \rangle \geq 0 \text{ for all } y \in \sigma \} .$$

One can show that $\check{\sigma}$ is again a convex rational polyhedral cone.

The dimension $\dim \sigma$ of an arbitrary cone σ is by definition the dimension of the smallest $\mathbb{R}$–subspace of $N_{\mathbb{R}}$ containing σ, viz. $\sigma + (-\sigma)$. Since $\check{\sigma} + (-\check{\sigma}) = M_{\mathbb{R}}$ for a strongly convex cone, it follows that always

$$\dim \check{\sigma} = r\,.$$

A subset τ of σ is called a *face*, in symbols $\tau \le \sigma$, if

$$\tau = \sigma \cap \{ m_0 \}^{\perp} = \{ y \in \sigma : \langle m_0, y \rangle = 0 \}$$

for an element $m_0 \in \check{\sigma}$ (which can be chosen from $M \cap \check{\sigma}$ such that τ is a strongly convex rational polyhedral cone, as well). Clearly, $\sigma \le \sigma$ and $\{ 0 \} \le \sigma$ (because of the strong convexity).

For a strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, we define

$$H_\sigma := M \cap \check{\sigma} = \{ m \in M : \langle m, y \rangle \ge 0 \text{ for all } y \in \sigma \}\,.$$

Clearly, H_σ is an *additive subsemigroup* of M, i.e. $0 \in H_\sigma$ and $m', m'' \in H_\sigma$ implies $m' + m'' \in H_\sigma$ (whence the symbol H for German "Halbgruppe"). Moreover, H_σ is *saturated*, that means: $cm \in H_\sigma$, $m \in M$, $c \in \mathbb{N} \setminus \{ 0 \}$ implies $m \in H_\sigma$. The following properties are more difficult to show:

***Lemma 7.26**

1. *H_σ is finitely generated as an additive subsemigroup of M: there exist $m_1, \dots, m_t \in H_\sigma$ such that*
$$H_\sigma = \Big\{ \sum_{k=1}^{t} c_k m_k : c_k \in \mathbb{Z},\ c_k \ge 0 \Big\}\,.$$
2. *H_σ generates the group M:*
$$M = H_\sigma + (-H_\sigma)\,.$$
3. *For any saturated additive semigroup H satisfying* 1. *and* 2.*, there exists a unique strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$ such that $H = H_\sigma$.*

Instead of a *proof*, we only remark that 1. is known as *Gordan's Lemma*, and 3. is roughly shown as follows: If H is generated as a semigroup by $m_1, \dots, m_t$, then $\rho := \sum_\tau \mathbb{R}_+ m_\tau$ is a convex polyhedral cone in $M_{\mathbb{R}}$, and $\sigma = \check{\rho}$ does the job. □

The basic combinatorial object, called a *fan* (or a *rational partial polyhedral decomposition*), is a collection $\Delta \ne \emptyset$ of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ satisfying the following conditions:

(i) $$\sigma \in \Delta,\ \tau \le \sigma \Longrightarrow \tau \in \Delta\,,$$

(ii) $$\sigma_1,\ \sigma_2 \in \Delta \Longrightarrow \sigma_1 \cap \sigma_2 \le \sigma_1,\ \sigma_2\,.$$

The union $\bigcup_{\sigma \in \Delta} \sigma$ is called the *support* of Δ, denoted by $|\,\Delta\,|$.

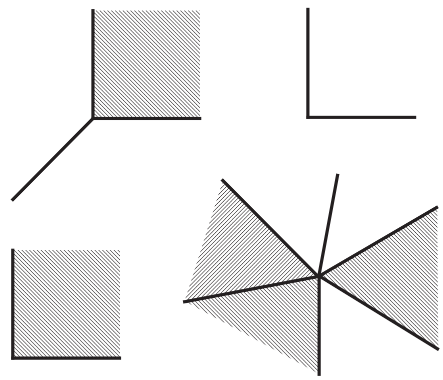


Figure 7.24

7.B.2 Construction of toric varieties

In the present Section, it is our aim to construct for each fan Δ a *toric variety*. Let us first repeat that an r–dimensional algebraic torus is just the variety

$$T = \mathbb{C}^* \times \cdots \times \mathbb{C}^* \ (r \text{ times}) ,$$

viewed as an r–dimensional commutative Lie (or algebraic) group. In a more intrinsic way, starting from the lattice N of rank r as above, we can identify T with

$$T_N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^* .$$

Hence, each element $m \in M$ gives rise, via the canonical pairing $\langle \cdot , \cdot \rangle$, to a *character* of $T = T_N$, i.e. to a group homomorphism

$$\chi_m : T_N \longrightarrow \mathbb{C}^* ;$$

more precisely,

$$\chi_m(t) = m\Big(\sum_j n_j \otimes c_j \Big) = \prod_j c_j^{\langle m, n_j \rangle} ,$$

where $t = \sum n_j \otimes c_j$, $n_j \in \mathbb{N}$, $c_j \in \mathbb{C}^*$. Obviously, $\chi_{m'+m''} = \chi_{m'} \cdot \chi_{m''}$; in particular, $\chi_0 =$ trivial homomorphism. In fact, the assignment $m \mapsto \chi_m$ establishes an isomorphism of M with the *character group* of T_N.

On the other hand, every $n \in N$ defines a *one parameter subgroup* $\gamma_n : \mathbb{C}^* \to T_N$ by

$$\gamma_n(\lambda)\,(m) = \lambda^{\langle m,n \rangle} , \quad \lambda \in \mathbb{C}^* , \quad m \in M .$$

Since $\gamma_{n'+n''} = \gamma_{n'}\,\gamma_{n''}$, we may and will identify N with the *group* of one–parameter subgroups of T_N.

Working with fixed coordinates on T, i.e. assuming a fixed isomorphism $T \cong \mathbb{C}^* \times \cdots \times \mathbb{C}^*$, we have $M \cong \mathbb{Z}^r$, where

$$\chi_m(t) = t^m := t_1^{m_1} \cdot \ldots \cdot t_r^{m_r} , \quad m = (m_1, \ldots , m_r) \in \mathbb{Z}^r , \quad t = (t_1, \ldots , t_r) \in T ,$$

and $N \cong \mathbb{Z}^r$, where

$$\gamma_n(s) = (s^{n_1}, \ldots , s^{n_r}) , \quad n = (n_1, \ldots n_r) \in \mathbb{Z}^r , \quad s \in \mathbb{C}^* ;$$

moreover,

$$\chi_m \circ \gamma_n(s) = s^{\langle m,n \rangle} , \quad \langle m, n \rangle = \sum_{j=1}^{r} m_j\, n_j .$$

We are now going to associate to each strongly convex rational polyhedral cone σ a certain affine variety. In algebraic terms, we form the group algebra

$$\mathbb{C}[M] = \bigoplus_{m \in M} \mathbb{C} \cdot \chi_m$$

with multiplication given by $\chi_{m'} \chi_{m''} = \chi_{m' + m''}$. Obviously, we may identify this algebra with the algebra of *Laurent polynomials*

$$\mathbb{C}[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}]$$

whose algebraic spectrum is just our torus T_N. Now, the semigroup $H_\sigma \subset M$ is finitely generated such that the group algebra

$$\mathbb{C}[H_\sigma]$$

is a finitely generated subalgebra of $\mathbb{C}[M]$. Hence, the spectrum $V_\sigma := \operatorname{spec} \mathbb{C}[H_\sigma]$ is an affine algebraic variety, admitting a canonical morphism of algebraic varieties:

$$T_N = \operatorname{spec} \mathbb{C}[M] \longrightarrow \operatorname{spec} \mathbb{C}[H_\sigma] = V_\sigma .$$

In fact, one can show much more:

***Theorem 7.27** *V_σ is an irreducible normal affine algebraic variety of dimension r, containing T_N via the canonical morphism $T_N \to V_\sigma$ as a (dense) open subset. The canonical action of T_N on itself by multiplication extends uniquely to an algebraic action of T_N on T_σ.*

Let us examine some *Examples*, especially in case $r = 2$.

1. For $\sigma = \sigma_0 = \{0\}$, we have $H_\sigma = \check{\sigma} \cap M = M$ and $\mathbb{C}[H_\sigma] = \mathbb{C}[M]$, i.e. $V_\sigma = T_N$.
2. For $\sigma = \sigma_1 = \mathbb{R}_+(1, 0) \subset \mathbb{R}^2$, it follows easily that $\check{\sigma} = \mathbb{R}_+(1, 0) + \mathbb{R}(1, 0)$ and $\mathbb{C}[H_\sigma] \cong \mathbb{C}[t_1, t_2, t_2^{-1}]$, i.e. $V_\sigma = \mathbb{C} \times \mathbb{C}^*$.
3. For $\sigma = \sigma_2 = \mathbb{R}_+(1, 0) + \mathbb{R}_+(0, 1)$, we get immediately $V_\sigma = \mathbb{C}^2$.
4. For $\sigma = \sigma_3 = \mathbb{R}_+(1, 0) + \mathbb{R}_+(1, 2)$, it is easily checked that $\check{\sigma} = \mathbb{R}_+(2, -1) + \mathbb{R}_+(0, 1)$ and $H_\sigma = \check{\sigma} \cap M$ is minimally generated by the three vectors $(1, 0)$, $(0, 1)$, and $(2, -1)$ such that

   $$\mathbb{C}[H_\sigma] \cong \mathbb{C}[x_1, x_2, x_3]/(x_1^2 - x_2 x_3) .$$

   Hence, V_σ is a normal affine variety with an A_1–singularity at the origin.

Remark that in these examples $\sigma_0 \le \sigma_1 \le \sigma_2, \sigma_3$ and V_{σ_j} is a dense open subset of V_{σ_k}, $k > j = 0, 1$. This is a general result, including the second assertion of the above mentioned Theorem:

***Theorem 7.28** *If $\tau \le \sigma$, then V_τ is a (dense) open subset of V_σ.*

This result, of course, enables us to transfer the construction of V_σ to *fans* $\Delta \subset N$. First, construct V_σ for each cone $\sigma \in \Delta$. Since $\sigma_1 \cap \sigma_2$ is in Δ for $\sigma_1, \sigma_2 \in \Delta$, $V_{\sigma_1 \cap \sigma_2}$ is open and dense in both, V_{σ_1} and V_{σ_2}. Hence, we may glue together V_{σ_1} and V_{σ_2} along $V_{\sigma_1 \cap \sigma_2}$. By that procedure, we construct an algebraic variety which we call V_Δ. The only nontrivial part to show is that V_Δ is a Hausdorff space. For this, one has to use the fact that $H_{\sigma_1 \cap \sigma_2} = H_{\sigma_1} + H_{\sigma_2}$.

***Theorem 7.29** *For a fan Δ in $N \cong \mathbb{Z}^r$, the variety V_Δ satisfies all the conditions in Theorem* [**??**] *(besides, of course, the assumption to be affine).*

Let us discuss here some more *Examples*.

1. Let $\sigma = \mathbb{R}_+ \subset \mathbb{R}$, and $\Delta = \{\sigma, -\sigma, \{0\}\}$. Then $V_\sigma \cong \mathbb{C}$ and $V_{-\sigma} \cong \mathbb{C}$ are glued along $V_{\sigma \cap (-\sigma)} = V_{\{0\}} = \mathbb{C}^*$. Hence $V_\Delta \cong \mathbb{P}_1$.

2. $\Delta = \{ \mathbb{R}_+(1, 0), \mathbb{R}_+(0, 1), \{(0, 0)\} \}$ is a fan with $V_\Delta = (\mathbb{C} \times \mathbb{C}^*) \cup (\mathbb{C}^* \times \mathbb{C}) = \mathbb{C}^2 \setminus \{(0, 0)\}$.
3. Taking $\sigma = \mathbb{R}_+(1, 0) + \mathbb{R}_+(0, 1)$, $\tau_1 = \mathbb{R}_+(-1, -1) + \mathbb{R}_+(1, 0)$, $\tau_2 = \mathbb{R}_+(-1, -1) + \mathbb{R}_+(0, 1)$ and their faces, we get $V_\Delta \cong \mathbb{P}_2$.

These examples might suggest a conjecture concerning compactness to the reader which, in fact, is correct:

***Theorem 7.30** *The toric variety V_Δ associated to the fan Δ in $N_\mathbb{R} \cong \mathbb{R}^r$ is compact if and only if Δ is finite and $|\Delta| = \mathbb{R}^r$.*

One can also easily read off the fan Δ whether the toric variety V_Δ is smooth:

***Theorem 7.31** *The toric variety V_Δ is smooth if and only if each $\sigma \in \Delta$ is nonsingular in that there exists a $\mathbb{Z}$–basis $n_1, \ldots, n_s$ of N such that $\sigma = \mathbb{R}_+ n_1 + \cdots + \mathbb{R}_+ n_{s'}$ for some $s' \le s$.*

We close this Section by an alternative description of the varieties V_σ associated to a strongly convex rational polyhedral cone σ. Write $H_\sigma = \mathbb{R}_+ m_1 + \cdots + \mathbb{R}_+ m_t$, define

$$U_\sigma = \{ u : H_\sigma \to \mathbb{C} : u(0) = 1, \ u(m' + m'') = u(m')\, u(m''), \quad m', m'' \in H_\sigma \},$$

and set $\chi_m(u) = u(m)$ for $m \in H_\sigma$, $u \in U_\sigma$. Then one can show that the image W_σ of U_σ under the injective map

$$(\chi_{m_1}, \ldots, \chi_{m_t}) : U_\sigma \longrightarrow \mathbb{C}^t$$

is a closed algebraic subset of $\mathbb{C}^t$ whose ideal is generated by finitely many polynomials of type

$$x_1^{\nu_1} \cdot \ldots \cdot x_t^{\nu_t} - x_1^{\mu_1} \cdot \ldots \cdot x_t^{\mu_t},$$

where $\nu_1 m_1 + \cdots + \nu_t m_t = \mu_1 m_1 + \cdots + \mu_t m_t$, and that W_σ is canonically isomorphic to V_σ.

7.B.3 Classification and resolution of toric varieties

We are now ready to introduce a good class of equivariant morphisms between toric varieties. We call a $\mathbb{Z}$–linear homomorphism $\varphi : N_1 \to N_2$ of free lattices N_1 and N_2 a map of the fan Δ_1 in N_1 into the fan Δ_2 in N_2, if the scalar extension $\varphi_\mathbb{R} : N_{1\mathbb{R}} \to N_{2\mathbb{R}}$ has the following property: For every $\sigma_1 \in \Delta_1$ there exists a cone $\sigma_2 \in \Delta_2$ such that $\varphi_\mathbb{R}(\sigma_1) \subset \sigma_2$. It is then an easy exercise to construct a rational map $\varphi_* : V_{\Delta_1} \to V_{\Delta_2}$ associated to φ in a canonical manner and to prove

***Theorem 7.32** *The map $\varphi_* : V_{\Delta_1} \to V_{\Delta_2}$ is equivariant with respect to the actions of T_{N_1} and T_{N_2} on these toric varieties via its restriction*

$$\varphi_{*|T_{N_1}} = \varphi \otimes \mathrm{id} : T_{N_1} = N_1 \otimes_\mathbb{Z} \mathbb{C}^* \longrightarrow N_2 \otimes_\mathbb{Z} \mathbb{C}^* = T_{N_2},$$

a homomorphism of algebraic tori.

Conversely, if $f : V_{\Delta_1} \to V_{\Delta_2}$ is such an equivariant rational (or holomorphic) map (via a homomorphism $f_{|T_{N_1}} : T_{N_1} \to T_{N_2}$ of algebraic tori), then $f = \varphi_*$ for a suitably chosen map $\varphi : N_1 \to N_2$.

The map φ_* is proper if, and only if, for each $\sigma_2 \in \Delta_2$ the set $\Delta_1' = \{ \sigma_1 \in \Delta_1 : \varphi(\sigma_1) \subset \sigma_2 \}$ is finite and

$$\varphi^{-1}(\sigma_2) = \bigcup_{\sigma \in \Delta_1'} \sigma_1 .$$

φ_* is a proper modification, if $\varphi : N_1 \to N_2$ is an isomorphism of lattices and Δ_1 is a locally finite subdivision of Δ_2 under the identification $N_{1\mathbb{R}} \cong N_{2\mathbb{R}}$.

Finally, we are in the position to formulate the converse of Theorem [**??**], the *Classification Theorem for Toric Varieties*.

***Theorem 7.33** *Let X be an irreducible normal algebraic variety on which the torus T_N acts algebraically such that X contains an open (dense) orbit isomorphic to T_N. Then there exists a (uniquely determined) fan Δ such that X and V_Δ are equivariantly isomorphic.*

The *proof* uses heavily the *complete reducibility* of algebraic tori and the following result (due to Sumihiro) to be used later again:

***Theorem 7.34** *Let the connected linear algebraic group G act algebraically on the irreducible normal algebraic variety X. Then X is the union of G–stable quasiprojective open subsets. If G is an algebraic torus, then X is the union of G–stable affine open subsets.*

In view of Theorem 22, in order to resolve the singularities of a toric variety V_Δ equivariantly, we have to find a locally finite *nonsingular* subdivision $\widetilde{\Delta}$ of Δ. This is always possible:

***Theorem 7.35** *Any toric variety V_Δ admits an equivariant resolution $V_{\widetilde{\Delta}}$ of singularities.*

From the preceding results, one can easily deduce that the 2–dimensional (normal) affine toric varieties are precisely the cyclic quotient singularities. Moreover, it is an amusing exercise to perform the construction of their resolutions by finding the nonsingular subdivisions as above. This leads directly to the Hirzebruch resolutions $\widetilde{X}(b_1, \ldots, b_r)$ with their canonical structures as toric varieties.

Notes and References

After the achievement of resolving complex analytic algebraic curves in the last century by Kronecker, Max Noether and others (see e.g.

[07 - 01] M. Noether, A. Brill: Die Entwicklung der Theorie der algebraischen Funktionen in älterer und neuerer Zeit. Jahresbericht der Deutschen Math. Vereinigung III, 107–566 (1892–93),

and, for a modern treatment, [04–03]), several approaches for resolving algebraic surfaces were proposed by the Italian school of algebraic geometers. The history of these attempts was thoroughly surveyed in Chapter I of

[07 - 02] O. Zariski: Algebraic Surfaces. Second Supplemented Edition. With Appendices by S. S. Abhyankar, J. Lipman and D. Mumford. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 61. Berlin–Heidelberg–New York: Springer–Verlag 1971;

whose first edition appeared in 1935. In a *Note added during the reading of the proofs*, Zariski says there on p. 23: "It has come to my knowledge that R.J.Walker in his Princeton dissertation in course of publication in the Annals of Mathematics gives a complete function–theoretic proof of the reduction theorem for algebraic surfaces. Having read the thesis by the courtesy of the author we believe that Walker's proof stands the most critical examination and settles the validity of the theorem beyond any doubt". Walker's thesis appeared as

[07 - 03] R. J. Walker: Reduction of the singularities of an algebraic surface. Annals of Math. 36, 336–365 (1935).

He used strongly local resolutions of what we called Jung singularities that go back to

[07 - 04] H. W. E. Jung: Darstellung der Funktionen eines algebraischen Körpers zweier unabhängigen Veränderlichen x, y in der Umgebung einer Stelle $x = a$, $y = b$. J. Reine Angew. Math. 133, 289–314 (1908).

Our treatment for the resolution of surface singularities is an adaptation of Hirzebruch's thesis which is also based on Jung's work, but completely in terms of modern analytic geometry:

[07 - 05] F. Hirzebruch: Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen. Math. Ann. 126, 1–22 (1953).

As an excellent introduction to this circle of ideas including the problem of *embedded* resolution for surfaces, we strongly recommend the article:

[07 - 06] J. Lipman: Introduction to resolution of singularities, pp. 187–230; in:

[07 - 07] R. Hartshorne (ed.): Algebraic Geometry, Arcata 1974. Proceedings of Symposia in Pure Mathematics, Vol. 29. Providence, Rhode Island: American Mathematical Society 1975.

Three other papers should be mentioned in connection with Jung singularities.

[07 - 08] K. Brauner: Zur Geometrie der Funktionen zweier komplexer Veränderlichen, III, IV. Abh. Math. Seminar Univ. Hamburg 6, 8–55 (1928);

[07 - 09] E. Kähler: Über die Verzweigung einer algebraischen Funktion zweier Veränderlichen in der Umgebung einer singulären Stelle. Math. Zeitschrift 30, 188–204 (1929).

Results about the structure of non–normal Jung singularities can be found in the next manuscript which is based on the author's thesis at Harvard 1965:

[07 - 10] J. Lipman: Quasi–ordinary singularities of surfaces in $\mathbb{C}^3$. pp. 161–172; in Part 2 of

[07 - 11] P. Orlik (ed.): Singularities, Arcata 1981. Proceedings of Symposia in Pure Mathematics, Vol. 40. Providence, Rhode Island: American Mathematical Society 1983.

This work was continued by:

[07 - 12] Y.–N. Gau: Topology of the quasi–ordinary surface singularities. Topology 25, 495–519 (1986).

Lipman himself extended the study of quasi–ordinary singularities also to higher dimensions. We mention:

[07 - 13] J. Lipman: Topological invariants of quasi–ordinary singularities. Preprint 1986.

The Hirzebruch–Jung algorithm is taken from [07–04] and [07–05]. Our version (including the determination of the equations for cyclic quotients) appeared in

[07 - 14] O. Riemenschneider: Deformationen von Quotientensingularitäten (nach zyklischen Gruppen). Math. Ann. 209, 211–248 (1974).[2]

More on the concepts of *determinantal* and *quasi-determinantal* formats of equations can be found in Chapter 13.

A conceptual approach to the Hirzebruch-Jung algorithm and some infinite generalizations is due to

[07 - 15] H. Cohn: Support polygons and the resolution of modular functional singularities. Acta Arithmetica 24, 261–278 (1973).

The procedure for resolving cyclic quotients step by step is contained in

[07 - 16] A. Fujiki: On resolutions of cyclic quotient singularities. Publ. RIMS Kyoto University 10, 293–328 (1974).

Our exposition is taken from

[2] In this paper, there is an obvious incorrect statement on the Betti numbers of cyclic surface singularities, and a not so obvious one about explicit equations for the base space of the versal deformation in a special example.

[07 - 17] H. Pinkham: Singularités de Klein – I.II. pp. 1–20, in: [04–20].

We assume that the reader is familiar with the theory of coverings shortly sketched in Section 2. If not, he or she may consult any (good) text on Algebraic Topology. There is also a concise introduction to this topic in Forster's book on Riemann surfaces. Since this shall be our main source for the Function Theory in one Variable, we may cite it here:

[07 - 18] O. Forster: Lectures on Riemann Surfaces. Graduate Texts in Mathematics 81. Berlin–Heidelberg–New York: Springer–Verlag 1981. (First published by Springer under the title *Riemannsche Flächen*).

For the examples at the end of the Chapter and in Appendix A, we follow closely the exposition (including the terminology) in Henry Laufer's book [01 - 13]. Lemma 18 is - in a slightly different version - due to Tadashi Tomaru in

[07 - 19] T. Tomaru: On Kodaira singularities defined by $z^n = f(x, y)$. Math. Z. 236, 133–149 (2001).

Another method can be found in Laufer's book.

The notions of *torus embeddings (toroidal embeddings, toric varieties)* etc. in Appendix B have been introduced in the smooth case by

[07 - 20] M. Demazure: Sous–groupes algébriques de rang maximum du groupe de Cremona. Ann. Sci. École Norm. Sup.(4) 3, 507–588 (1970);

in the general case, the foundations were laid down in

[07 - 21] G. Kempf, F. Knudsen, D. Mumford, B. Saint–Donat: Toroidal embeddings. I. Lecture Notes in Mathematics 339, Berlin–Heidelberg–New York: Springer–Verlag 1973.

From the literature concerning this theory, we select only the survey article of

[07 - 22] V. I. Danilov: The geometry of toric varieties. Russian Math. Surveys 33 : 2, 97–154 (1978),

and the book

[07 - 23] T. Oda: Convex bodies and algebraic geometry. An introduction to the theory of toric varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 15. Berlin–Heidelberg–New York–London–Paris–Tokyo: Springer–Verlag 1988,

which we followed very closely in our presentation of Appendix B. All unproven results may be found there with a proof or at least with a precise reference. Further, we mention the articles by J. L. Brylinski, M. Merle and M. Lejeune–Jalabert in

[07 - 24] M. Demazure, H. Pinkham and B. Teissier (eds.): Séminaire sur les singularités des surfaces. Lecture Notes in Mathematics 777, Berlin–Heidelberg–New York: Springer–Verlag 1980.

(Last modified: November 26, 2020)