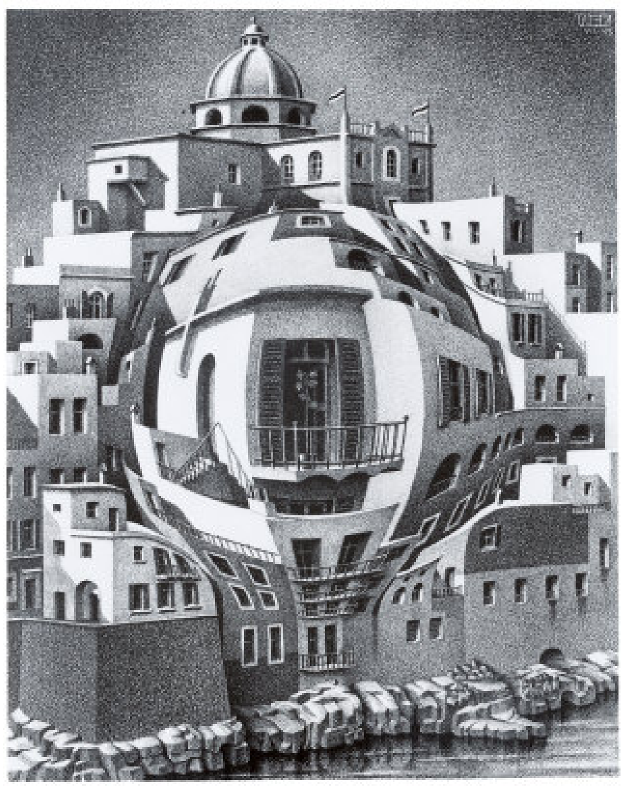
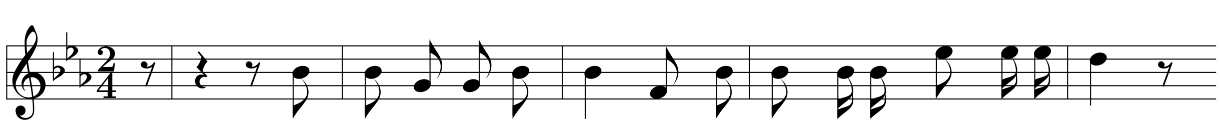
Chapter 5

Es ist eine alte Geschichte,
doch bleibt sie immer neu;
und wem sie just passieret,
dem bricht das Herz entzwei.

(Heinrich Heine,
Buch der Lieder)





Chapter 5

Blowing up, tangent cone and embedded resolution of plane curve singularities

We are going to use the σ–modification of Chapter 4.9 for "resolving" the singularities of a *plane curve* C, i.e. of an analytic hypersurface C in $\mathbb{C}^2$ (or, more generally, in a two–dimensional complex analytic manifold M). More precisely, we will construct a new manifold N together with a proper holomorphic map $\pi : N \to M$ that is (locally with respect to M) a finite iteration of σ–modifications such that the *strict transform* $\overline{C}$ of C in N is smooth. An essential constant in this construction is the *multiplicity* of such a singularity which is an invariant of its *tangent cone*. We introduce the notion of multiplicity in full generality and study it more closely in the next Chapter.

5.1 The resolution of the ordinary double point and of the cusp

The ordinary double point is isomorphic to the singularity of the cone

$$C = \{ x = (x_1, x_2) \in \mathbb{C}^2 : x_1 x_2 = 0 \}$$

at the vertex $0 = (0, 0)$. Thus, it can be resolved by the general method developed in Chapter 4. In this specific example, the base $\underline{C} \subset \mathbb{P}_1$ consists only of two distinct points, and therefore, the restriction $L_{|\underline{C}}$ of the tautological bundle L on $\mathbb{P}_1$ to $\underline{C}$ is a disjoint union of two lines. If σ denotes the canonical map $L \to \mathbb{C}^2$, then we know that, moreover,

$$\overline{C} = L_{|\underline{C}} = \overline{\sigma^{-1}(C \setminus \{ 0 \})} .$$

We want to show in the present Chapter that all plane curves C can be resolved by taking strict transforms of C under (iterated) σ–modifications as above. Before we embark into the formal proof, we fix our notations and present some more examples.

As explained in Chapter 4, the tautological bundle L on $\mathbb{P}_1$ can be constructed by identifying

$$(u_0, v_0) \sim (u_1, v_1) \Longleftrightarrow u_0 = \frac{1}{u_1} , \quad v_0 = u_1 v_1 ;$$

we write the map σ, slightly deviating from our former notations, in the form

$$\begin{cases} x_1 = v_0 \\ x_2 = u_0 v_0 \end{cases} \text{resp.} \quad \begin{array}{l} x_1 = u_1 v_1 \\ x_2 = v_1 \end{array} .$$

The zero–section of L is of course given by the vanishing of v_0 and of v_1, resp.

Let us now investigate the isolated singularity 0 of the *cusp (Neil's parabola)*

$$C = \{ (x_1, x_2) : x_1^3 - x_2^2 = 0 \}$$

which is not a cone. Under the σ–modification $\sigma : L \to \mathbb{C}^2$, the total transform $\sigma^{-1}(C)$ has the (local) equations

$$v_0^3 - (u_0 v_0)^2 = 0 , \quad (u_1 v_1)^3 - v_1^2 = 0 .$$

To get the *strict* transform $\overline{C} = \overline{\sigma^{-1}(C \setminus \{0\})}$, we must divide out v_0 and v_1, resp., as often as possible. Hence, $\overline{C}$ is described in the two coordinate charts of L by

$$v_0 - u_0^2 = 0 \text{ and } u_1^3 v_1 - 1 = 0 , \text{ resp.}$$

and therefore, it is a smooth submanifold of L which resolves the singularity of C. A (real) picture is roughly the following:

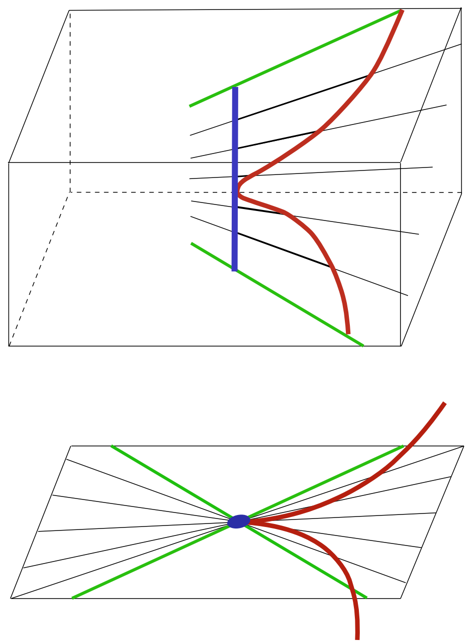


Figure 5.1

Besides the coinciding fact that both, the ordinary double point and the cusp, are resolved by a single σ–process, there are also striking differences: In the case of the ordinary double point, the resolution $\overline{C}$ consists of two *connected* components (in accordance with the reducibility of C into two *analytic* components), whereas, for the cusp, $\overline{C}$ is connected (this can easily be verified by parametrizing $\overline{C}$ via

$$\mathbb{C} \ni t \longmapsto \begin{cases} (u_0, v_0) = (t, t^2) \\ (u_1, v_1) = (t^{-1}, t^3) , \quad t \neq 0 . \end{cases}$$

Moreover, the two components of $\overline{C}$ intersect the zero–section $\mathbb{P}_1 \cong \sigma^{-1}(0)$ of L *transversely* in the first case, whereas in the second case $\overline{C}$ *touches* the zero–section (i.e. $\overline{C}$ and $\sigma^{-1}(0)$ have linearly independent resp. dependent tangent vectors at their common point).

To give also an *Example* for the fact that in general we have to perform more than one σ–modification in order to resolve plane curve singularities, we look at the isolated singular point 0 of the curves

$$C_k = \{ x_1^k - x_2^2 = 0 \} \subset \mathbb{C}^2 , \quad k \geq 2 ,$$

the case $k = 2$ being (up to isomorphism) the ordinary double point. A now straightforward calculation shows that the strict transform $\overline{C}_k$ has at most one singular point (in the chart with coordinates u_0, v_0) given by the equation of *type* C_{k-2} :

$$v_0^{k-2} - u_0^2 = 0 .$$

So, for $k \geq 4$, the game has to be continued, but it is clear that it stops after finitely many steps.

5.2 The σ - process for complex analytic manifolds

The last Example in the previous Section shows that we must be able to perform σ–modifications at any point of an arbitrary complex analytic manifold M (of dimension $m = n + 1 \geq 2$). But this is easily done, since the interesting features of the map $\sigma : L \to \mathbb{C}^m$ are concentrated around the origin $0 \in \mathbb{C}^m$; to be more precise: If V is any open neighborhood of $0 \in \mathbb{C}^m$ and $\widetilde{V} = \sigma^{-1}(V) \subset L$ is the preimage, then the restriction $\sigma_{|\widetilde{V}} \to V$ is a proper holomorphic map such that $\sigma^{-1}(0)$ is biholomorphic to $\mathbb{P}_{m-1}$ and nowhere dense in $\widetilde{V}$, and the induced map $\widetilde{V} \setminus \sigma^{-1}(0) \to V \setminus \{0\}$ is biholomorphic.

So, if M is a complex analytic manifold of dimension $m \geq 2$ together with a distinguished point $x^{(0)} \in M$, we choose any coordinate chart $\xi : U \to V \subset \mathbb{C}^m$ sending $x^{(0)}$ to 0 and form the composition $\pi = \xi^{-1} \circ (\sigma_{|\widetilde{V}}) : \widetilde{V} \to U$ which induces the analytic isomorphism

$$\widetilde{V} \setminus \pi^{-1}(x^{(0)}) \cong \widetilde{V} \setminus \sigma^{-1}(0) \xrightarrow{\sim} V \setminus \{0\} \cong U \setminus \{x^{(0)}\} .$$

Hence, patching together $\widetilde{V}$ and $M \setminus \{x^{(0)}\}$ along $\widetilde{V} \setminus \pi^{-1}(x^{(0)})$ resp. $U \setminus \{x^{(0)}\}$ via this isomorphism yields a new manifold $\widetilde{M}$ together with a proper holomorphic map

$$\pi : \widetilde{M} \longrightarrow M$$

such that $\pi^{-1}(x^{(0)}) \cong \mathbb{P}_{m-1}$ is nowhere dense in $\widetilde{M}$, inducing an isomorphism

$$\widetilde{M} \setminus \pi^{-1}(x^{(0)}) \xrightarrow{\sim} M \setminus \{x^{(0)}\} .$$

We call this map the σ–process of M at $x^{(0)}$. $\widetilde{M}$ is very often also referred to as the *blow–up* or *blowing up* of $x^{(0)}$ in M or the *Hopf modification* at $x^{(0)}$.

Of course, this construction depends on the chosen local coordinates. However, the following is true:

***Theorem 5.1** *Let M be a $m = n + 1$–dimensional complex manifold, and denote by $\pi_j : \widetilde{M}_j \to M$, $j = 1, 2$, two blow–ups of a point $x^{(0)} \in M$ (with respect to two possibly different holomorphic charts of M around $x^{(0)}$). Then there exists a unique biholomorphic map making the following diagram commutative*:

$$\begin{array}{ccc} \widetilde{M}_1 & \xrightarrow{\sim} & \widetilde{M}_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M & \xrightarrow{\text{id}} & M \end{array}$$

We will postpone the *proof* in the surface case ($m = 2$) until Chapter 09, Section 26.

5.3 Blowing up and strict transforms of subvarieties

In contrast to the situation in the case of plane curves, higher dimensional objects can generally not be resolved just by taking strict transforms under σ–processes. However, for surface singularities, there is the classical result due to Zariski that one can obtain a resolution by successively blowing up and forming *normalizations*. More about this approach will be said in Chapter 15. As a preparation, we study here some examples and develop the general theory of blowing up analytic subsets, coherent analytic ideal sheaves and arbitrary coherent analytic sheaves. This will be - among others - used for introducing the concept of the multiplicity of a singularity in the next Chapter.

For the sake of computations, it is first of all useful to have an explicit description of the σ–process $\sigma : \widetilde{M} \to M$ at the point $x^{(0)}$ in terms of local coordinates $x_1, \ldots, x_n$ of the manifold M vanishing at $x^{(0)}$. According to the local description of $\widetilde{M}$ as total space of the tautological bundle $L \to \mathbb{P}_{n-1}$, we may cover $\widetilde{M}$ near $\sigma^{-1}(x^{(0)})$ by n coordinate neighborhoods $V_1, \ldots, V_n$

with coordinates $(u_1^{(j)}, \ldots, u_n^{(j)})$, $j = 1, \ldots, n$, $u_j^{(j)}$ small enough, such that $\sigma_{|V_j} : V_j \to M$ is given by

$$\sigma(u_1^{(j)}, \ldots, u_n^{(j)}) = (u_1^{(j)} u_j^{(j)}, \ldots, u_{j-1}^{(j)} u_j^{(j)}, u_j^{(j)}, u_{j+1}^{(j)} u_j^{(j)}, \ldots, u_n^{(j)} u_j^{(j)}) .$$

Recall that, for $n = 2$, we used instead coordinates (v_1, u_1) and (u_2, v_2) with $\sigma(v_1, u_1) = (v_1, v_1 u_1)$, $\sigma(u_2, v_2) = (v_2 u_2, v_2)$. For $n = 3$, we will always calculate in coordinates

$$(v_1, \xi_1, \xi_2), \quad (\eta_1, v_2, \eta_2), \quad (\zeta_1, \zeta_2, v_3)$$

such that

$$\begin{aligned} \sigma(v_1, \xi_1, \xi_2) &= (v_1, v_1\xi_1, v_1\xi_2) \\ \sigma(\eta_1, v_2, \eta_2) &= (v_2\eta_1, v_2, v_2\eta_2) \\ \sigma(\zeta_1, \zeta_2, v_3) &= (v_3\zeta_1, v_3\zeta_2, v_3) . \end{aligned}$$

The reason for the *uniqueness* of the σ–process (Theorem 5.1) is simply the following: Although $\mathbb{P}_{n-1}$ was introduced as the projective space with homogeneous coordinates $[x_1 : \cdots : x_n]$ it may be invariantly considered to be the projective space $\mathbb{P}(T_{M,x^{(0)}}) = T^*_{M,x^{(0)}}/\mathbb{C}^*$ associated to the *tangent vector space* $T_{M,x^{(0)}}$ of M at $x^{(0)}$ which is independent of local coordinates. The tautological line bundle L is also easily described by $T_{M,x^{(0)}}$ and $\mathbb{P}(T_{M,x^{(0)}})$ without referring to concrete variables. Finally, M being a complex analytic manifold, there exists a canonical local biholomorphic isomorphism $(M, x^{(0)}) \xrightarrow{\sim} (T_{M,x^{(0)}}, 0)$. With respect to concrete calculations, this remark implies that the isomorphism $\widetilde{M}_1 \xrightarrow{\sim} \widetilde{M}_2$ between two σ–processes σ_1 and σ_2 induces an isomorphism $\mathbb{P}_{n-1} \cong \sigma_1^{-1}(x^{(0)}) \xrightarrow{\sim} \sigma_2^{-1}(x^{(0)}) \cong \mathbb{P}_{n-1}$ which is given by the projective linear automorphism associated to the Jacobi matrix of the coordinate transformation at $x^{(0)}$.

If $Z \subset M$ denotes any (not necessarily reduced) positive dimensional complex analytic subvariety, we call the closure

$$\overline{Z} = \overline{\sigma^{-1}(Z \setminus \{x^{(0)}\})} \text{ in } \widetilde{M}$$

the *blow–up* of Z at $x^{(0)}$ or the *strict transform* of Z in $\widetilde{M}$ (as opposed to the full preimage or *total transform* $\sigma^{-1}(Z)$ which contains $\overline{Z}$). Notice that this process depends on the embedding of Z in M. But in the case that Z is a smooth submanifold at $x^{(0)}$ the fact that the restriction of the tautological bundle on $\mathbb{P}_{n-1}$ to a linear subspace is again (isomorphic to) the tautological bundle implies that the restriction $\overline{Z} \to Z$ is the σ–modification of Z at $x^{(0)}$ if $\dim Z \geq 2$ and an isomorphism for $\dim Z = 1$. That $\overline{Z}$ is in fact a complex analytic subvariety of $\widetilde{M}$ is easily seen by the explicit description given above. If f is any function vanishing on Z, then

$$\widetilde{f}_j(u_1^{(j)}, \ldots, u_n^{(j)}) = f_j(u_1^{(j)} u_j^{(j)}, \ldots, u_j^{(j)}, \ldots, u_n^{(j)} u_j^{(j)})$$

vanishes on $\sigma^{-1}(Z) \cap V_j$. There is a unique decomposition

$$\widetilde{f}_j = \left(u_j^{(j)}\right)^{r_j} \cdot g_j ,$$

such that g_j does not vanish identically for $u_j^{(j)} = 0$. It is clear that the closure of $\sigma^{-1}(Z \setminus \{x^{(0)}\})$ in V_j is precisely the set of points where all the functions g_j constructed in this manner vanish. The same reasoning shows that

$$\sigma^{-1}(Z) = \overline{Z} \cup \sigma^{-1}(x^{(0)})$$

and that $\sigma^{-1}(x^{(0)}) \cong \mathbb{P}_{n-1}$ is an irreducible component of $\sigma^{-1}(Z)$.

We should remark here that the concept of strict transforms was already used in Chapter 4 for resolving cones $C \subset \mathbb{C}^{n+1}$ over smooth projective varieties $\underline{C} \subset \mathbb{P}_n$: The construction of the submanifold $\widetilde{C}$ inside the tautological bundle L on $\mathbb{P}_n$ is obviously nothing else but taking the closure of $\sigma^{-1}(C \setminus \{0\})$ in L with respect to the σ–process $\sigma : L \to \mathbb{C}^{n+1}$, i.e. $\widetilde{C}$ coincides with the strict transform $\overline{C}$ of C under σ. In other words, we can phrase parts of the results obtained there by saying that *isolated singular vertices of (algebraic) cones* $C \subset \mathbb{C}^n$ *can be resolved by their strict transforms* $\overline{C}$ *after one blow–up*. Recall that under the map $\sigma : \overline{C} \to C$ the exceptional set $\sigma^{-1}(0) \subset \mathbb{P}_{n-1}$ is isomorphic to the base $\underline{C}$ of C.

5.4 Absolutely isolated singularities

An isolated singular point $x^{(0)} \in X \subset U \subset \mathbb{C}^n$ is called *infinitesimally isolated* (up to first order), if its strict transform $\overline{X}$ with respect to the σ–process at $x^{(0)}$ has only (necessarily finitely many) isolated singular points. In such a situation one can blow up the new singular points and hope that the iterated process will stop after finitely many steps. If this actually happens, X is called *absolutely isolated*. Examples are again the cones over projective algebraic manifolds, but also all plane curve singularities as we will prove later in this Chapter. In the present Section we wish to give some counterexamples in dimension larger than one.

It is easily seen that the strict transform of an isolated hypersurface singularity given by an equation

$$F(x_1, \ldots, x_n) = g(x_1, \ldots, x_k) + \sum_{j=k+1}^{n} x_j^2 = 0$$

has no singularities in the coordinate charts $V_{k+1}, \ldots, V_n$. Therefore, investigating e.g. the function

$$f(x, y, z) = z^2 - (x^4 + y^4),$$

we have only to consider the first coordinate system (using the symmetry between x and y) along $\sigma^{-1}(0)$. Thus, we substitute $(x, y, z) = (v_1, \xi_1 v_1, \xi_2 v_1)$, and get - after dividing v_1^2 out of $f(v_1, \xi_1 v_1, \xi_2 v_1)$ - the defining function

$$g_1(v_1, \xi_1, \xi_2) = \xi_2^2 - v_1^2 (1 + \xi_1^2).$$

Near $\xi_1 = 0$, the zero set of g_1 is the product of a complex line with the one–dimensional ordinary double point. Therefore, the original singularity X is not infinitesimally isolated and there is no way to resolve it by blowing up isolated points only. However, it is clear that the normalization of the strict transform $\overline{X}$ is already smooth such that X can be resolved by *normalized* blow–ups.

This being true for all *two–dimensional* singularities, a slight variation gives a stronger counterexample in dimension 3. The same calculation as above for

$$X = \{ f(x, y, z, w) = (z^2 + w^2) - (x^4 + y^4) = 0 \}$$

yields

$$g_1(v_1, \xi_1, \xi_2, \xi_3) = \xi_2^2 + \xi_3^2 - v_1^2 (1 + \xi_1^2),$$

such that the zero set $\{ g_1 = 0 \}$ is (near $\xi_1 = 0$) the product of the complex line with the two–dimensional ordinary double point, hence normal with a one–dimensional singular locus which has to be blown up simultaneously in order to resolve the original isolated singularity.

In the first example, $\overline{X}$ was not locally irreducible, but could be resolved just by separating the irreducible components. That this behaviour is not generally adopted can be concluded from the function $f(x, y, z) = z^3 + x^5 + y^5$ which leads to a product of a line with the cusp $\{ v_1^2 + \xi_2^3 = 0 \}$.

Finally, let us consider the two–dimensional singularity defined by the polynomial $f(x, y, z) = x^4 + y^4 + z^6$. Here, we have to do the computations in each of the three coordinate patches:

$$\begin{aligned} f(v_1, \xi_1 v_1, \xi_2 v_1) &= v_1^4 (1 + \xi_1^4 + \xi_2^6 v_1^2) \\ f(\eta_1 v_2, v_2, \eta_2 v_2) &= v_2^4 (\eta_1^4 + 1 + \eta_2^6 v_2^2) \\ f(\zeta_1 v_3, \zeta_2 v_3, v_3) &= v_3^4 (\zeta_1^4 + \zeta_2^4 + v_3^2) \quad . \end{aligned}$$

There is only one singularity in the strict transform, of type $z^2 = x^4 + y^4$. Thus, the original singularity is infinitesimally isolated up to second, but not to first order.

5.5 Surface singularities of type A_k, D_k and E_k

The problem to determine all absolutely isolated hypersurface singularities in $\mathbb{C}^3$ of type

$$(*) \qquad z^2 = g(x, y)$$

leads to the same list of function germs g as does the classification of all simple plane curve singularities (see the Appendix to the present Chapter for this list). The corresponding equations $(*)$ in three dimensions define singular objects, the *rational double points*, that occur in completely different branches of mathematics. The whole Chapter 16 will be devoted to such characterizations that are discernible as equivalent without relating on these equations too much. There we will also present a conceptual proof for the equivalence of rational double points and function germs of multiplicity 2 in three variables defining absolutely isolated singularities.

In the present Section we are going to show by brute force that all these singularities are indeed absolutely isolated. Let us start with the singularity of type E_8 and equation $z^2 = x^3 + y^5$. Straightforward calculations imply equations

$$\begin{cases} \xi_2^2 = v_1 + \xi_1^5 v_1^3 & \text{on} \quad V_1 \\ \eta_2^2 = \eta_1^3 v_2 + v_2^3 & \text{on} \quad V_2 \ . \end{cases}$$

There is no singularity in V_1 and precisely one in V_2, of type $E_7 : z^2 = x^3 + xy^3$.

In the case E_7, we get

$$\begin{cases} \xi_2^2 = v_1 + \xi_1^3 v_1^2 & \text{on} \quad V_1 \\ \eta_2^2 = \eta_1^3 v_2 + \eta_1 v_2^2 & \text{on} \quad V_2 \ . \end{cases}$$

Again, the only singularity lies at the origin of the second coordinate system. Substituting $z = \eta_2$, $x = v_2 + \eta_1^2/2$, $y = \eta_1$ leads to an equation of the form $z^2 = x^2 y - y^5/4$ that is of type D_6.

Type E_6 is given by $z^2 = x^3 + y^4$. Almost the same calculation as in the case E_8 leads to the singularity $z^2 = x^3 y + y^2$ of type A_5 (substitute $y - x^3/2$ for y).

For the series $D_k : z^2 = x^2 y + y^{k-1}$, $k \geq 5$, one obtains the singularities $z^2 = x^2 y + y^{k-3}$ in the second coordinate system, i.e. D_{k-2} for $k \geq 6$ and A_3 for $k = 5$ (substitute $y - x^2/2$ for y). For D_4, the strict transform is smooth in the second coordinate chart. But in the first one, we easily get an equation $z^2 = xy + xy^3 = xy(1 + y^2)$ which has three singularities of type A_1 (at $x = 0$, $y = 0, i, -i$), whereas for D_k, $k \geq 5$, there lies precisely one A_1–singularity on that chart.

Since the A_k, given by $z^2 = x^2 + y^{k+1}$, $k \geq 1$, generate an A_{k-2} for $k \geq 3$ and no singularity for $k = 1, 2$ (which we simply denote by A_0), we can finally indicate the full system of dependence relations in the following somewhat mysterious diagram (which will be explained conceptually in the Appendix to Chapter 12):

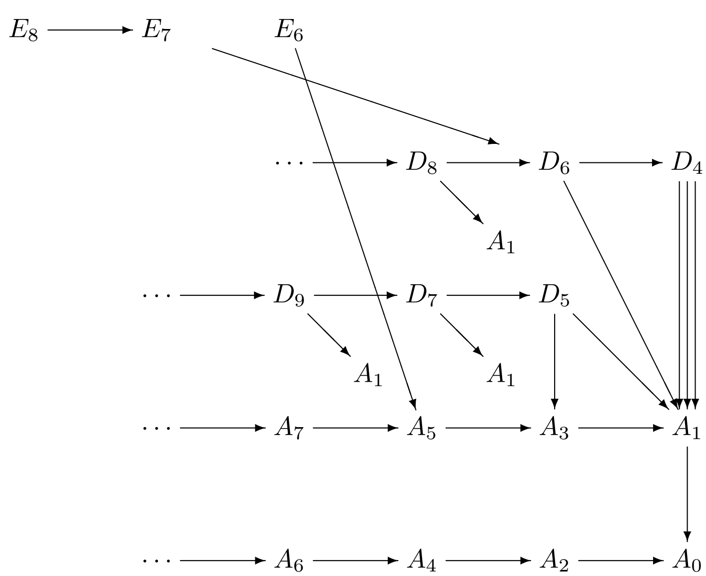


Figure 5.2

One could use the same pattern of reasoning more accurately to determine the precise structure of the resolutions obtained by successively blowing up. We will apply invariant theory instead (see Chapter 11). As it turns out there is always a simple relation between resolutions of surface singularities of type A_k, D_k and E_k and the corresponding Coxeter–Dynkin–Witt (CDW) diagram.

Let us close this Section with an *Example* of a germ of multiplicity two which is infinitesimally isolated up to order 1, but not to order 2: the germ $z^2 = x^3 + y^6$ gives after one blow–up the isolated singularity $z^2 = x^3 y + y^4$, but the last one leads to $z^2 = x^3 y^2 + y^2$ which is singular along $y = z = 0$, x arbitrary.

5.6 Projective and affine tangential cones

We now develop the necessary tools for developing the general theory of *multiplicities* for isolated singularities $X \subset U \subset \mathbb{C}^n$, the distinguished point $x^{(0)}$ being always the origin (for more details, see the next Chapter). Denote by $\sigma : \widetilde{U} \to U$ the Hopf blow–up at the origin. Then $\overline{X}$, the *blow–up* of the distinguished point in X, is the analytic closure of $\sigma^{-1}(X \setminus \{0\})$ in $\widetilde{U}$, and the restriction of σ to $\overline{X}$ will be denoted by π:

$$\begin{array}{ccc} \overline{\sigma^{-1}(X \setminus \{0\})} = \overline{X} & \longrightarrow & \widetilde{U} \\ \Big\downarrow \pi & & \Big\downarrow \sigma \\ X & \longrightarrow & U \end{array}$$

Under this assumption, the fiber $\pi^{-1}(0) = \overline{X} \cap \sigma^{-1}(0)$ is called the *projective tangential cone* of X (or more precisely: of the pair $(X, \mathbb{C}^n)$ since it depends on the embedding of X in $\mathbb{C}^n$). We denote it always by $T^c_{X,0}$.

As we have seen, $T^c_{X,0}$ is an analytic subvariety of $\sigma^{-1}(0) \cong \mathbb{P}_{n-1}$. Hence, it is projective algebraic. The cone over $T^c_{X,0}$ in $\mathbb{C}^n$ is called the *affine tangential cone* of X, denoted by $T^a_{X,0}$. By construction, $T^c_{X,0}$ consists of all points in $\mathbb{P}_{n-1}$ (interpreted as lines in $\mathbb{C}^n$ through the origin) which are limits of secants going through 0 and another point $x \in X \setminus \{0\}$. Therefore, $T^a_{X,0}$ is just the union of all these limit lines - but counted carefully: The (algebraic and) complex analytic structures on $T^c_{X,0}$ and $T^a_{X,0}$ have to be taken as the *full* structures; that is, one has to compute locally the functions g_j mentioned in the proof that $\overline{X}$ is complex analytic (following the definition of the blow–up $\overline{X}$), and then one has to plug in the coordinates of the origin to get the locally defining equations for $T^c_{X,0}$ in $\mathbb{P}_{n-1}$. In general, this structure is not reduced, as we will see in a moment.

Regard as a first *Example* the equation $x^3 + xy + y^3 = 0$ in $\mathbb{C}^2$. This is the "folium cartesium" (for the parameter $a = 1/3$ and reflected on the line $y = -x$; see Figure 1.6). So, we expect $T^a_{X,0} = \{xy = 0\}$. In fact, the total transform of X under σ has the equations

$$\begin{cases} v_1^3 + v_1^2 u_1 + v_1^3 u_1^3 = 0 & \text{on} \quad V_1 \\ v_2^3 u_2^3 + v_2^2 u_2^2 + v_2^3 = 0 & \text{on} \quad V_2 \,. \end{cases}$$

The equations of the strict transform are again derived by dividing out the highest possible power of v_j. Therefore, we have the following description of $\overline{X}$:

$$\begin{cases} v_1 + u_1 + v_1 u_1^3 = 0 & \text{on} \quad V_1 \\ v_2 u_2^3 + u_2 + v_2 = 0 & \text{on} \quad V_2 \,, \end{cases}$$

and from these identities we get the equations for $T^c_{X,0}$ by putting $v_1 = 0$, $v_2 = 0$. Thus, we obtain the union of the two lines $\{u_1 = 0\}$ and $\{u_2 = 0\}$ which correspond to $\{y = 0\}$ and $\{x = 0\}$, resp.

Remark. It is clear that we get the same result for all polynomials $xy + P(x, y)$ where P is of total degree ≥ 3 (or $P = 0$).

As a next *Example* we consider once more the *cusp* $X = \{x^2 = y^3\} \subset \mathbb{C}^2$.

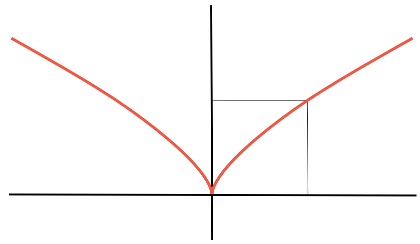


Figure 5.3

The usual calculations imply

$$\begin{cases} v_1^2\,(1 + v_1 u_1^3) = 0 \\ v_2^2\,(u_2^2 + v_2) = 0\ . \end{cases}$$

Consequently, $T^c_{X,0}$ has only points in the second coordinate neighborhood, where it is described by the non–reduced equation $u_2^2 = 0$. - The same phenomenon occurs in our examples of type $z^2 = g\,(x,\,y)$, where g is of degree ≥ 3. Here, the $(x,\,y)$–plane must be counted twice.

We leave it as an *Exercise* to the reader to determine $T^c_{X,0}$ for the other curves in Chapter 1 and the two clover-leaves $(x^2 + y^2)^2 + 3x^2y - y^3 = 0$ resp. $(x^2 + y^2)^3 - 4x^2y^2 = 0$.

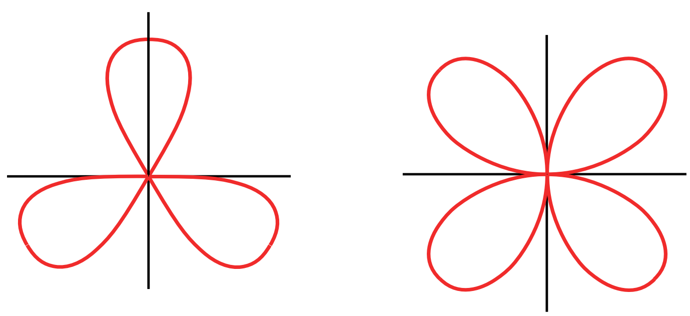


Figure 5.4

The examples suggest that the tangential cone does only depend on the *leading terms* of the defining functions for X. Recall that for a nonvanishing convergent power series $f \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$ there is a unique expansion

$$f = \sum_{d \geq m} f_d\ , \quad f_m \neq 0\ ,$$

with homogeneous polynomials $f_d \in \mathbb{C}\,[\,x_1, \ldots, x_n\,]$ of degree d. f_m is called the leading or *inital* term of f, in symbols

$$f_m := \mathrm{in}\,(f)\ ,$$

and m is the multiplicity of the function germ f. By a careful analysis of the examples above, the reader should be able to convince himself that for a hypersurface singularity $X = \{\,f = 0\,\}$ the tangential cone $T^c_{X,0}$ is given by $\mathrm{in}\,(f_0) = 0$. However, for arbitrary singularities $X = N\,(f_1, \ldots, f_r)$, the ideal defining $T^c_{X,0}$ is in general not just generated by the initial forms of $f_{1,0}, \ldots, f_{r,0}$. Instead, one has to take *all* initial forms of elements in the ideal generated by $f_1, \ldots, f_r$:

***Theorem 5.2**

a) *If X is the variety associated to a coherent ideal I, then the ideal defining $T^c_{X,0}$ is generated by the initial forms of the germs $f_0 \in I_0$.*

b) *If the ideal of $T^c_{X,0}$ is generated by $\mathrm{in}\,(f_{1,0}), \ldots, \mathrm{in}\,(f_{t,0})$ with elements $f_{1,0}, \ldots, f_{t,0} \in I_0$, then I_0 is generated by the germs $f_{1,0}, \ldots, f_{t,0}$.*

5.7 Statement of the main Theorem and strategy of proof

The main objective of the present Chapter is the proof of the following:

Theorem 5.3 *Let C be a curve in a two–dimensional complex manifold M (i.e. a complex analytic hyperplane in M). Then there exists a manifold N and a proper holomorphic map $\pi : N \to M$ which is locally (with respect to M) a finite iteration of σ–processes such that the strict transform $\overline{C}$ of C in N is smooth. In particular, $\pi_{|\overline{C}} : \overline{C} \to C$ has only finite fibers and induces a biholomorphic map after removing discrete sets from C and $\overline{C}$, resp.*

Remarks. 1. We do not assume in the statement that our curve C has a *reduced* structure, that is: (local) defining equations $f = 0$ for C may have *multiple* prime factors (for this and the following, see Chapter 1.4). Therefore, local equations $\overline{f}$ for $\overline{C}$ we construct during the proof may also have this deficiency such that *smoothness* of $\overline{C}$ means that $N(\overline{f})$ is smooth in its reduced structure, i. e. $\overline{f} = \overline{x}_2^m$ for some $m \geq 1$ and some local coordinates $\overline{x}_1, \overline{x}_2$ on N.

2. It is, however, not difficult to show in the process described below that starting with reduced equations leads to such after each blowing up.

3. A *consequence* of our statement is the fact that a reduced curve C has only isolated singularities. (The discrete sets mentioned at the end of Theorem 3 are, of course, the set of singular points of the reduction and its preimage under $\pi_{|\overline{C}}$) . However, for the proof to work, we must know a priori that this is true. We will give the argument later in Section [**??**] in a more general situation and will *assume* for the moment that C has only isolated singularities. But then the statement of the theorem is of purely a local nature.

Proof. So, we may assume that $C = \{ f = 0 \} \subset U \subset \mathbb{C}^2$ and that the origin 0 is the only singular point of C. We have to show that, after blowing up 0 in U, the strict transform $\overline{C}$ has only finitely many singularities which - in a certain sense - are less terrifying than the original one.

The crucial measure for the singular behaviour of C will be the *multiplicity* m of f at 0; i.e.

$$m = \mathrm{mult}_0(f) = \max_{\mu \in \mathbb{N}} \{ f_0 \in \mathfrak{m}_2^{\mu} \} .$$

If this number would drop at least by one for each singular point of $\overline{C}$, we could finish the game in at most $m - 1$ steps (since $\mathrm{mult}_0(f) = 1$ implies $df(0) \neq 0$). However, the behaviour of m is a little more complicated, as can be seen from the sequence of singularities $\{ f_k(x_1, x_2) = x_1^2 + x_2^k = 0 \}$ which all have multiplicity 2 for $k \geq 2$. Therefore, we have to introduce one more invariant. By the Weierstraß Preparation Theorem, we can write f (after a linear change of coordinates) in the form

$$f = e\,(x^m + a_1(y)x^{m-1} + \cdots + a_m(y))$$

with $e\,(0, 0) \neq 0$, $a_1(0) = \cdots = a_m(0) = 0$ and $m = \mathrm{mult}_0(f)$ - we only have to arrange matters such that

$$f_m(x, 0) \neq 0 , \text{ where } f_m = j^m f - j^{m+1} f .$$

After using the Tschirnhaus transformation

$$x \longmapsto x - \frac{a_1(y)}{m} ,$$

if necessary, we always can assume that $a_1(y) \equiv 0$. Since $\mathrm{mult}_0 f = m$, the a_j must vanish of order $\mathrm{mult}_0 a_j \geq j$. We then define

$$\nu = \nu_0(f) = \min_{2 \leq j \leq m} \left(\frac{\mathrm{mult}_0 a_j}{j} \right) ,$$

where we put $\mathrm{mult}_0 a_j = \infty$ for $a_j = 0$. Hence, $\nu_0(f) \geq 1$ and

$$\nu_0(f) \in \frac{1}{m!} \mathbb{N} \cup \{ \infty \} ,$$

where $\nu_0(f) = \infty$ belongs to $f = e \cdot x^m$ in which case $N(f) = N(x)$ is smooth.

We now perform a σ–modification $\sigma : \widetilde{U} \to U$ at 0. Writing f as a convergent series

$$f = \sum_{\mu=m}^{\infty} f_\mu$$

with homogeneous polynomials f_μ of degree μ, we immediately check that

$$f_\mu(u_0 v_0, v_0) = v_0^\mu f_\mu(u_0, 1)$$

such that $\overline{C} \cap \sigma^{-1}(0)$ is given by

$$\{ f_m(u_0, 1) = 0 \} \cup \{ f_m(1, u_1) = 0 \} \subset \mathbb{P}_1 = \sigma^{-1}(0) ,$$

in other words, if (ξ_0, ξ_1) denote homogeneous coordinates on $\sigma^{-1}(0) = \mathbb{P}_1$ with $u_0 = \xi_0/\xi_1$, $u_1 = \xi_1/\xi_0$, then

$$\overline{C} \cap \sigma^{-1}(0) = \{ [\xi_0 : \xi_1] \in \mathbb{P}_1 : f_m(\xi_0, \xi_1) = 0 \} .$$

As an analytic subset of a one–dimensional compact complex manifold, this set consists of finitely many points only, say $y_1, \ldots, y_\ell$. Denote by $m_1, \ldots, m_\ell$ and $\nu_1, \ldots, \nu_\ell$ the multiplicities resp. ν–invariants of the canonical defining equations of $\overline{C}$ at $y_1, \ldots, y_\ell$. Then Theorem 3 can obviously be deduced from the following two facts which we will prove in the next Section:

1. If $\nu = \nu_0(f) = 1$, then $\ell > 1$ and $m_\lambda < m = \mathrm{mult}_0(f)$ for all $\lambda = 1, \ldots, \ell$;
2. If $\nu = \nu_0(f) > 1$, $\nu < \infty$, then $\ell = 1$ and either $m_1 < m$ or $m_1 = m$ but $\nu_1 = \nu - 1$.

5.8 Proof of the two crucial facts

We first investigate the conditions under which we get $\ell = 1$. Supposing without loss of generality that $f_m(1, 0) \neq 0$ and assuming that $\ell = 1$, we have for some $\alpha \in \mathbb{C}$, $c_0 \in \mathbb{C}^*$:

$$f_m(u_0, 1) = c_0(u_0 - \alpha)^m .$$

By homogeneity of f_m, this implies

$$f_m(x, y) = c_0 y^m f_m \left(\frac{x}{y} - \alpha \right)^m = c_0(x - \alpha y)^m .$$

On the other hand, denoting the homogeneous part of $a_j(y)$ of degree j by $a_{jj}(y)$, we find that

$$f_m(x, y) = e\,(0, 0)\,(x^m + a_{22}(y) x^{m-2} + \cdots + a_{mm}(y)) .$$

These two expressions can only coexist, when

$$\alpha = 0 \text{ and } a_{22}(y) = \cdots = a_{mm}(y) = 0 ,$$

and then we necessarily have

$$\nu_0(f) > 1 .$$

By reversing the argument we see also that $\nu_0(f) > 1$ leads to $\ell = 1$. Thus, the first statements in 1. and 2. are shown to be correct.

To finish the proof for the second parts, we have to study the defining equation for $\overline{C}$:

$$g\,(u_0, v_0) = \sum_{\mu=m}^{\infty} v_0^{\mu-m} f_\mu(u_0, 1) = e\,(u_0 v_0, v_0) \left(u_0^m + \frac{a_2(v_0)}{v_0^2}\, u_0^{m-2} + \cdots + \frac{a_m(v_0)}{v_0^m} \right)$$

near the points $(\overline{u}_0, \overline{v}_0)$ with $f_m(\overline{u}_0, 1) = 0$ and $\overline{v}_0 = 0$. In case 2, $\overline{u}_0 = 0$ such that we always have $m_1 \le m$, and from $m_1 = m$ we can easily deduce that

$$\nu_1 = \min_{2 \le j \le m} \frac{\mathrm{mult}_0(a_j / v_0^j)}{j} = \min_{2 \le j \le m} \frac{\mathrm{mult}_0\, a_j - j}{j} = \nu - 1\,.$$

In case 1, there exists a number $m' < m$ such that

$$f_m(u_0, 1) = (u_0 - \overline{u}_0)^{m'} A(u_0)$$

with a holomorphic function $A(u_0)$ defined near $\overline{u}_0$ and not vanishing there. Thus, we see that the function

$$g(u_0, v_0) = (u_0 - \overline{u}_0)^{m'} A(u_0) + v_0 B(u_0, v_0)$$

has multiplicity $\le m' < m$ at $(\overline{u}_0, 0)$. □

5.9 Divisors with normal crossings

We now return to the map $\pi : N \to M$ in Theorem 3. Denoting by S the *singular locus* of the curve $C \subset M$:

$$S = \mathrm{sing}\, C = \{ x^{(0)} \in C : x^{(0)} \text{ is a singular point of } C \}\,,$$

we are going to clarify the structure of the preimage $\pi^{-1}(S)$ and its relation to the strict transform $\overline{C}$ of C.

Blowing up a point $x^{(0)} \in M$ creates a curve $E_1 \cong \mathbb{P}_1 \subset \widetilde{M}_1$. Blowing up again a point $x_1 \in E_1$ creates a new rational curve $E_2 \subset \widetilde{M}_2$, and the total transform of E_1 under the second σ–process $\sigma_2 : \widetilde{M}_2 \to \widetilde{M}_1$ equals

$$E_2 \cup \overline{E}_1\,,$$

$\overline{E}_1$ denoting the strict transform of E_1. Using the explicit description for the two–dimensional σ–process in Section 1, it is easily checked that σ_2 induces a biholomorphic map $\overline{E}_1 \to E_1$ (this is true, of course, for an arbitrary smooth curve in a manifold and its strict transform under a σ–process). So, $\overline{E}_1$ is a rational curve which obviously meets E_2 transversely in just one point. We visualize the situation by the following picture:

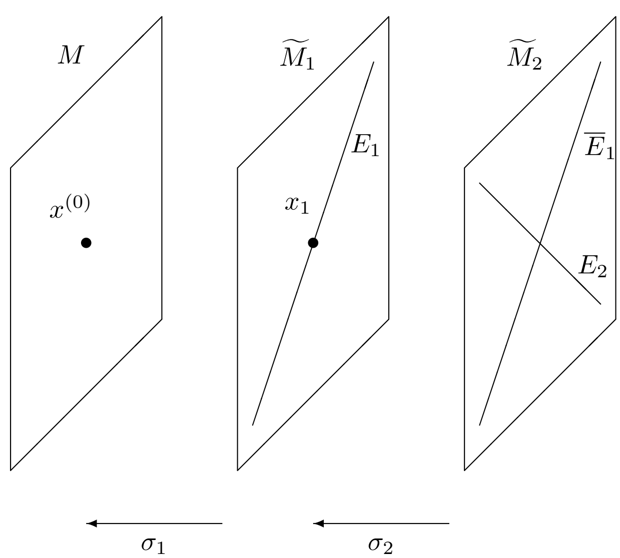


Figure 5.5

For the next step, there are two possibilities: either x_2, the point we are going to blow up, is a smooth point of $E_2 \cup \overline{E}_1$, or it is the singular point. After blowing up, the picture is schematically the following - only the position of the newly created curve depends on the position of the point blown up:

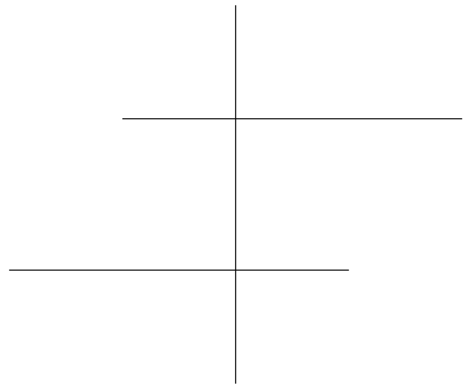


Figure 5.6

Proceeding by induction, this consideration yields the following:

Theorem 5.4 *Under the assumptions of* Theorem 3, *the preimage* $\pi^{-1}(S)$ *of* $S = \operatorname{sing} C$ *decomposes into the connected components*

$$\pi^{-1}(x)\ , \quad x \in S\ ,$$

and each connected component is a finite union

$$E_1 \cup E_2 \cup \ldots \cup E_r$$

of smooth rational curves E_ρ *which intersect each other (transversely) in at most one point.*

Moreover, the proof of Theorem 4 shows that the sets $E_1 \cup \ldots \cup E_r$ form *trees*:

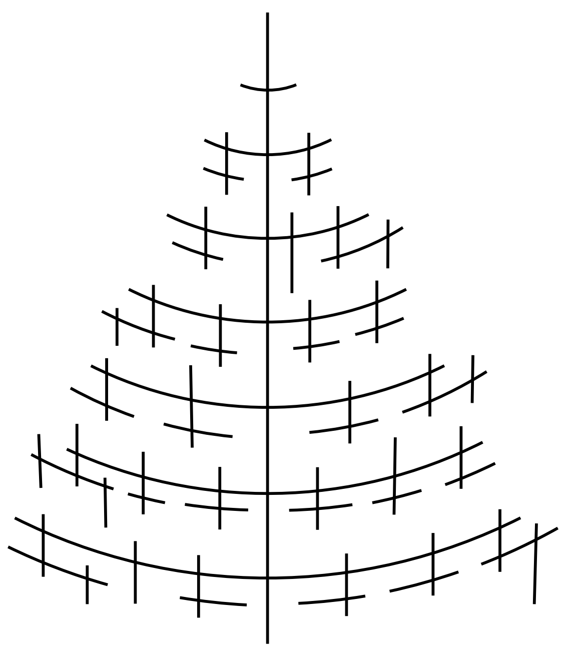


Figure 5.7

i.e., they do not contain *cyclic configurations* like

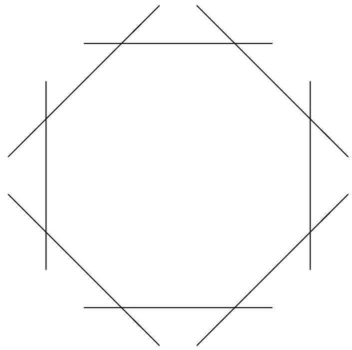


Figure 5.8

In particular, the preimage $\pi^{-1}(S)$ is a set of the type which we will call a *divisor with normal crossings*, namely a curve in a two–dimensional complex analytic manifold which has no singularities except ordinary double points:

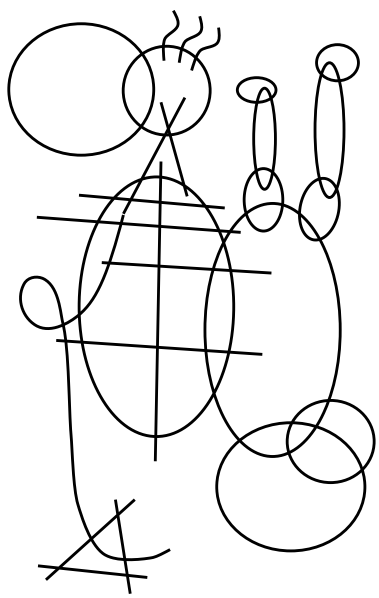


Figure 5.9

It is plain that the preimage $\pi^{-1}(C)$ of the curve $C \subset M$ consists of the union of the smooth strict transform $\overline{C}$ and the set $\pi^{-1}(S)$ which in general do not fit together to form a normal crossing divisor. However, this nice situation can be achieved after blowing up some more points as we are going to explain on the remaining pages of the present Section.

We have to deal with the following two kinds of singular points of $\pi^{-1}(C)$:

1. $\overline{C}$ meets $\pi^{-1}(S)$ in an ordinary double point, intersecting both components transversely;

2. $\overline{C}$ touches one of the irreducible components (at a smooth or a singular point of $\pi^{-1}(S)$).

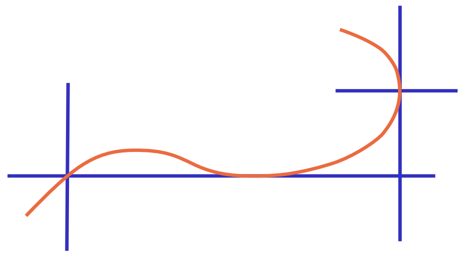


Figure 5.10

In case 1, if $\pi^{-1}(S)$ is locally given by $x_1 x_2 = 0$ in appropriately chosen coordinates, then a local equation for $\overline{C}$ is necessarily of the form

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2 + \text{ higher order terms}$$

with $a_1 \neq 0$, $a_2 \neq 0$. After blowing up the origin, the strict transform of $\overline{C}$ has an equation

$$g(u_0, v_0) = a_1 + a_2 u_0 + v_0\, h(u_0, v_0)$$

and intersects $\mathbb{P}_1 \cap \{(u_0, v_0) \in \mathbb{C}^2\} = \{(u_0, v_0) \in \mathbb{C}^2 : v_0 = 0\}$ transversely at the point $(\overline{u}_0, 0)$, $\overline{u}_0 = -a_1/a_2 \neq 0$.

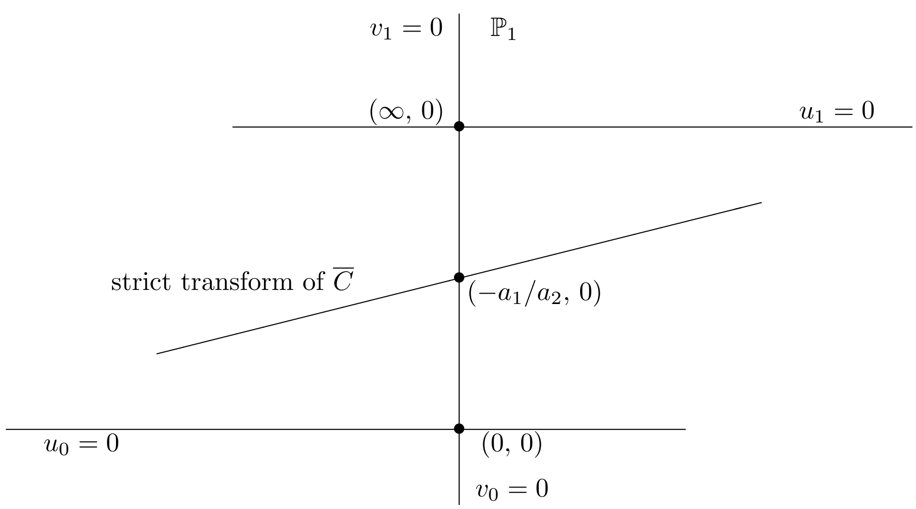


Figure 5.11

In the second case, we may assume that the (local) component E of $\pi^{-1}(S)$, which is touched by $\overline{C}$, is given by $x_2 = 0$ near the origin, and that the local equation for $\overline{C}$ is of the form

$$f(x_1, x_2) = x_2 + \sum_{j+k \geq 2} a_{jk}\, x_1^j\, x_2^k .$$

There must be a $j \geq 2$ with $a_{j0} \neq 0$, because otherwise $f = x_2 \cdot g$, $g(0, 0) \neq 0$, such that $\overline{C}$ would be locally contained in E which is impossible for a strict transform. We call the number

$$c = \min_{j \geq 2} \{ a_{j0} \neq 0 \} - 1 \geq 1$$

the *order of contact* between $\overline{C}$ and E. By dividing out the unit

$$1 + \sum_{\substack{2 \le j+k \le c \\ 1 \le k}} a_{jk}\, x_1^j\, x_2^{k-1} \,,$$

we may assume that f is given in the form

$$f(x_1, x_2) = x_2 + a\, x_1^{c+1} + \text{ other terms of degree } \mu \ge c+1 \,, \quad a \ne 0 \,.$$

After performing a σ–process, the strict transform of $\overline{C}$ has an equation

$$g(u_0, v_0) = u_0 + a\, v_0^c + \text{ other terms of degree } \ge c \,.$$

Hence, the contact order decreases by one for $c \ge 2$, and for $c = 1$ the σ–process reduces the case 2 to case 1 (such that we will refer to a transverse intersection as being a contact point of order 0).

Summarizing, we may say that the following holds true:

Theorem 5.5 *Let C be a curve in a two–dimensional complex manifold M. Then there exists a manifold $\widetilde{N}$ and a proper holomorphic map $\widetilde{\pi} : \widetilde{N} \to M$ which is locally (with respect to M) a finite iteration of σ–processes such that the preimage $\widetilde{\pi}^{-1}(C)$ is a divisor in $\widetilde{N}$ with normal crossings.*

This result will serve us for two different purposes: First, we realize an arbitrary surface singularity X as a branched analytic covering $X \to U \subset \mathbb{C}^2$ and transform the branch locus to a normal crossing divisor. By analyzing (normal) surface singularities with such a special branch locus and by explicit resolution of those singularities, we are able to construct resolutions for all singularities. This is the method that goes back to Jung. Once we have constructed a resolution $\pi : \widetilde{X} \to X$, the so–called *exceptional set* $E = \pi^{-1}(\text{sing } X)$ is a curve in the two–dimensional manifold $\widetilde{X}$. Using Theorem 5, we are led to the existence of the more specific *good* resolutions for which E has (among others) the property of being a normal crossing divisor.

5.10 Divisors and line bundles

In fact, the considerations of the preceding Sections apply more generally to so called *divisors* which we have to discuss anyway.

A *divisor* D on M is a formal (locally finite) sum $D = \sum n_\tau C_\tau$ with $n_\tau \in \mathbb{Z}$ and C_τ are irreducible curves in M (possibly with singularities, but reduced). The set $\text{Div}\, M$ of all divisors has an obvious abelian group structure. More generally, we can introduce this notion for an arbitrary complex manifold M of dimension n when the curves C_τ are replaced by (irreducible) *hypersurfaces*.

If h is a meromorphic function on M, then there exists a unique divisor

$$\text{div } h = \sum n_\tau C_\tau$$

such that

$$h_{|M \setminus \bigcup_\tau C_\tau}$$

is holomorphic and nowhere vanishing and h has multiplicity n_τ along

$$C'_\tau = \text{reg } C_\tau \setminus \bigcup_{\sigma \ne \tau} C_\sigma$$

(which means that it has zeros of order n_τ along C'_τ if $n_\tau > 0$ and poles of order $|n_\tau|$ if $n_\tau < 0$).

Now, locally, every divisor D is the divisor of a meromorphic function, and two such defining functions have a well–defined nowhere vanishing holomorphic ratio. Hence, if

$$D_{|U_j} = \text{div } h_j \,, \quad M = \bigcup_j U_j \,,$$

then

$$f_{kj} := \frac{h_k}{h_j} \in \mathcal{O}^*_M(U_{jk})$$

defines a holomorphic line bundle on M which is determined by D up to analytic isomorphism. This bundle (more precisely: its class in $H^1(M, \mathcal{O}^*)$) is always denoted by $[\,D\,]$. Note that $[\,D\,]$ has a global meromorphic section $\{\,h_j\,\}$ such that

$$D_{|U_j} = \operatorname{div} h_j \ .$$

We abbreviate the symbol $\mathcal{O}([\,D\,])$ in the following by $\mathcal{O}(D)$.

Notice that we have $[\,D_1 + D_2\,] \cong [\,D_1\,] \otimes [\,D_2\,]$ for all divisors D_1, D_2. In particular $[\,D\,] \otimes [\,-D\,]$ is the trivial line bundle such that $[\,-D\,]$ is canonically isomorphic to the dual bundle of $[\,D\,]$:

$$[\,-D\,] \cong [\,D\,]^* \ .$$

On the other hand, if a holomorphic line bundle L on the manifold M has a global nontrivial meromorphic section $\{h_j\}$, then

$$\operatorname{div} h_j = \operatorname{div} f_{jk} h_k = \operatorname{div} f_{jk} + \operatorname{div} h_k = \operatorname{div} h_k \ ,$$

such that

$$D_{|U_j} = \operatorname{div} h_j$$

defines a divisor D on M with $L \cong [\,D\,]$.

This correspondence between holomorphic line bundles which carry a nontrivial meromorphic section and line bundles associated to divisors is valid for complex analytic manifolds of arbitrary dimension. We will use the fact that line bundles on compact Riemann surfaces have a nontrivial meromorphic section and hence are given by divisors extensively in later Chapters. More generally, this statement is true for any projective algebraic manifold and even varieties. For us, it is more important that the same principle holds for resolutions of surface singularities (cf. Chapter 9 for this and also for more details about our claims at the beginning of the present Section).

As an *Example*, we look at the hyperplane $H = \{\,x_0 = 0\,\}$ in $\mathbb{P}_n$. Considered as the divisor $1 \cdot H$, H has local defining equations

$$h_0 = \frac{x_0}{x_0} \ , \ h_1 = \frac{x_0}{x_1} \ , \ldots , \ h_n = \frac{x_0}{x_n} \ ,$$

such that the transition functions f_{kj} of $[1 \cdot H]$ are given by

$$f_{kj} = \left(\frac{x_0}{x_k}\right) \cdot \left(\frac{x_0}{x_j}\right)^{-1} = \frac{x_j}{x_k} \ ,$$

i.e. by the transition functions of the hyperplane bundle on $\mathbb{P}_n$. Therefore, our notation $\mathcal{O}(1)$ is only a short version of the correct symbol $\mathcal{O}_{\mathbb{P}_n}(1 \cdot H)$, and, more generally, $\mathcal{O}(\ell)$ stands for $\mathcal{O}_{\mathbb{P}_n}(\ell \cdot H)$, $\ell \in \mathbb{Z}$.

5.11 Local intersection multiplicities

Let two curves C, D on M be given with local equations $f = 0$ resp. $g = 0$ near the point $x \in M$, and suppose that the intersection $C \cap D$ is isolated at x. Then

$$(C \cdot D)_x := \dim_{\mathbb{C}} \mathcal{O}_{M,x} / (f_x, g_x)$$

is finite and independent of the choices of f and g. It is called the *intersection multiplicity* of C and D at x. Clearly,

$$(C \cdot D)_x = 1$$

if and only if f_x and g_x form coordinates of M at x (and $(C \cdot D)_x = 0$ if and only if $x \notin C \cap D$).

Assume now that C is reduced and denote by $\nu : \overline{C} \to C$ the *embedded resolution* of C. We write $\nu^{-1}(x) = \{\,x^{(1)}, \ldots, x^{(r)}\,\}$ and $\overline{g} = (g_{|C}) \circ \nu$. We are going to prove:

Lemma 5.6 *One has the equality*

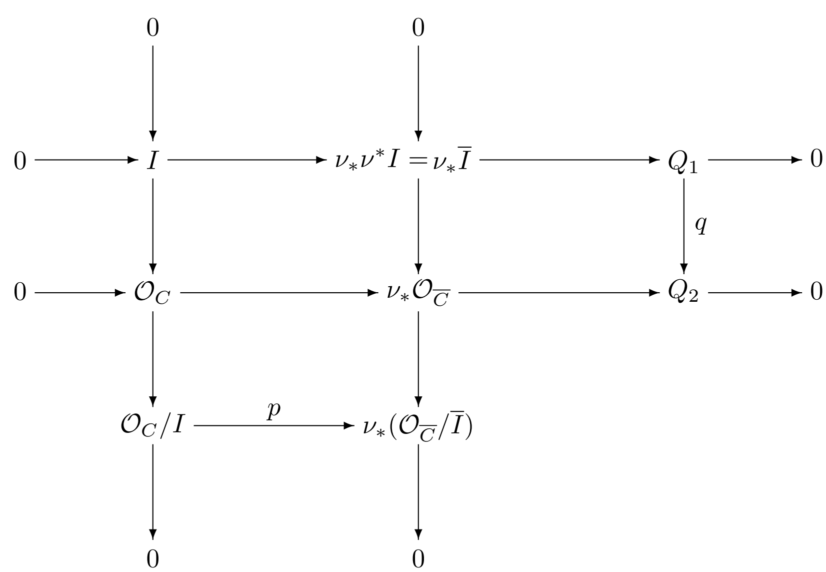
$$(C \cdot D)_x = \sum_{k=1}^{r} \operatorname{ord}_{x^{(k)}} \overline{g} \,.$$

Remark. The following proof uses sheaf theory, in particular the exactness of the direct image functor for finite holomorphic maps. The uninitiated reader may skip the proof for the moment and should come back to it later. However, the same proof yields the following result which can be proven by elementary calculations. The preceding Lemma 6 is derived by applying this finitely many times.

***Lemma 5.7** *Let $C \subset M$ be a reduced curve and denote by $\nu : \overline{C} \to C$ the restriction of a σ–process $\sigma : \overline{M} \to M$ to the strict transform $\overline{C}$ of C. We write $\nu^{-1}(x) = \{ x^{(1)}, \ldots, x^{(r)} \}$ and denote by $\sigma^{-1}(D)$ the total preimage of D. Then,*

$$(C \cdot D)_x = \sum_{k=1}^{r} (\overline{C} \cdot \sigma^{-1}(D))_{x^{(k)}} \,.$$

Proof of Lemma 6. Let I be the ideal on C generated by $g_{|C}$, and let $\overline{I} = I\,\mathcal{O}_{\overline{C}}$ be the image of $\nu^* I = I \otimes_{\mathcal{O}_C} \mathcal{O}_{\overline{C}} \to \mathcal{O}_{\overline{C}}$ which is generated by $\overline{g}$ (since I is locally free, $\nu^* I \cong \overline{I}$). We get a commutative diagram with exact rows and columns (ν_* being an exact functor (c.f. the Supplement, Section [**??**])):



Clearly, the coherent sheaves Q_1 and Q_2 are concentrated at x (if C is chosen so small that x is the only singular point), as are the sheaves on the last row. It suffices to show that

$$(+) \qquad \dim \ker p_x = \dim \operatorname{coker} p_x$$

since then

$$\dim_x \mathcal{O}_C / I = \dim_x \nu_* (\mathcal{O}_{\overline{C}} / \overline{I}) \,,$$

which gives the claim because of $\dim_x \mathcal{O}_C / I = \dim_x \mathcal{O}_M / (f_x, g_x) = (C \cdot D)_x$ and

$$\dim_x \nu_* (\mathcal{O}_{\overline{C}} / \overline{I}) = \dim_x \bigoplus_{k=1}^{r} \mathcal{O}_{\overline{C}, x^{(k)}} / \overline{g}_{x^{(k)}} = \sum_{k=1}^{r} \operatorname{ord}_{x^{(k)}} \overline{g} \,.$$

Now, the snake lemma implies $\ker p_x \cong \ker q_x$ and $\operatorname{coker} p_x \cong \operatorname{coker} q_x$. Thus the equation (+) is equivalent to

$$\dim Q_{1,x} = \dim Q_{2,x} \,.$$

But this follows from the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_C & \longrightarrow & \nu_* \mathcal{O}_{\overline{C}} \\ {\scriptstyle \cdot g} \big\downarrow & & \big\downarrow {\scriptstyle \cdot \overline{g}} \\ \mathcal{O}_C & \longrightarrow & \nu_* \mathcal{O}_{\overline{C}} \end{array}$$

which induces compatible isomorphisms of $\mathcal{O}_C$ with I and $\nu_*\mathcal{O}_{\overline{C}}$ and $\nu_* \overline{I}$ and therefore an isomorphism $Q_{2,x} \cong Q_{1,x}$. □

If C is compact and D arbitrary (or vice versa) with $C \cap D$ discrete, we define

$$(C \cdot D) := \sum_{x \in C \cap D} (C \cdot D)_x = \sum_{x \in C \cap D} \sum_{y \in \nu^{-1}(x)} \operatorname{ord}_y \overline{g} \,.$$

This is called the *intersection number* of C and D, sometimes also denoted by (C, D) or $C \cdot D$. The definition is symmetric in C and D and bilinear because of $\operatorname{ord}_y(\overline{gh}) = \operatorname{ord}_y \overline{g} + \operatorname{ord}_y \overline{h}$. Therefore, it may be linearly extended to *divisors* C and D with appropriate conditions.

We are next rephrasing this definition in terms of numbers attached to line bundles. If D is (locally near C) given by the vanishing of the holomorphic functions g_j, the line bundle $[\,D\,]$ has a holomorphic section given locally by g_j. After finitely many σ–processes on M we get a diagram

$$\begin{array}{ccc} \overline{C} & \xrightarrow{\ \nu\ } & C \\ \big\downarrow & & \big\downarrow \\ \widetilde{M} & \xrightarrow[\ \sigma\ ]{} & M \end{array}$$

in which the *strict transform* $\overline{C}$ of C is smooth (and hence the *normalization of* C). Clearly, $\sigma^*([\,D\,])$ has a holomorphic section given by the functions $g_j \circ \sigma$. Since

$$\sigma^*([\,D\,])_{|\overline{C}} = \nu^*([\,D\,]_{|C}) \,,$$

the line bundle on the right hand side has a holomorphic section consisting of the functions $\overline{g}_j = g_j \circ \nu$. Thus,

$$(C \cdot D) = \deg \nu^*([\,D\,]_{|C})$$

where $\deg L$ denotes the *degree* of a holomorphic line bundle on a compact Riemann surface, i.e. the total order of zeros and poles of a nontrivial meromorphic section (see Chapter 9.6).

This identity allows us to generalize our definition to *holomorphic line bundles* L on C by

$$(C, L) := \deg \nu^* L \,, \quad \nu : \overline{C} \longrightarrow C \text{ the normalization} \,.$$

In particular, if C is smooth,

$$(C, L) = \deg L \,.$$

Moreover, we can introduce *self–intersection numbers* by

$$(C \cdot C) := (C, [C]_{|C}) \,.$$

5.12 Intersection numbers and blowing up

It will be important for later applications to control the behaviour of the intersection numbers under blowing up. Recall Lemma 6 which, in turn, is a consequence of Lemma 7. We can prove even more.

Lemma 5.8 *If* $\sigma : \overline{M} \to M$ *is the* σ*–process at* $x \in M$*, then*

$$(C \cdot D) = (\overline{C} \cdot \sigma^{-1}(D)) , \quad (C, L) = (\overline{C}, \sigma^* L)$$

where $\overline{C}$ *denotes the strict transform of* C *in* $\overline{M}$ *and* $\sigma^{-1}(D)$ *the total transform.*

Proof. Either x is a smooth point of C in which case the induced mapping $\overline{C} \to C$ is biholomorphic, or σ is one step in the process of normalizing C such that the normalization factors through $\overline{C}$. $\square$

We apply this Lemma to study the intersection matrix $((E_j \cdot E_k))$ for compact irreducible curves E_j with $E_j \cap E_k$ discrete, $j \neq k$, under blowing up at a point x. Notice that this matrix is *symmetric* (see the previous Section).

Denote by m_j the *multiplicity* of (a defining equation of) E_j near x. Then,

$$\sigma^{-1}(E_j) = m_j \overline{E}_0 + \overline{E}_j ,$$

where $\overline{E}_0 = \sigma^{-1}(x) \cong \mathbb{P}_1$ as one can easily check by a concrete calculation. Since $[E_j]$ is trivial near x, we have

$$0 = (\overline{E}_0 \cdot \sigma^{-1}(E_j)) = (\overline{E}_0 \cdot m_j \overline{E}_0) + (\overline{E}_0 \cdot \overline{E}_j) ,$$

hence $(\overline{E}_0 \cdot \overline{E}_j) = m_j$ because of $(\overline{E}_0 \cdot \overline{E}_0) = -1$. Further, for $j, k \geq 1$,

$$(\overline{E}_j \cdot \overline{E}_k) = (\overline{E}_j \cdot (\sigma^{-1}(E_k) - m_k \overline{E}_0)) = (E_j \cdot E_k) - m_j m_k .$$

So, the new intersection matrix looks as follows:

	$\overline{E}_0$	$\overline{E}_1$	$\overline{E}_2$	$\cdots$
$\overline{E}_0$	-1	m_1	m_2	$\cdots$
$\overline{E}_1$	m_1	$(E_1 \cdot E_1) - m_1^2$	$(E_1 \cdot E_2) - m_1 m_2$	$\cdots$
$\overline{E}_2$	m_2	$(E_2 \cdot E_1) - m_2 m_1$	$(E_2 \cdot E_2) - m_2^2$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Adding m_1 times the first column to the second one and then m_1 times the first row to the second and proceeding with the other columns and rows it becomes evident that the quadratic form associated to the new matrix is equivalent to the matrix

$$\left(\begin{array}{c|c} -1 & 0 \ \cdots \ 0 \\ \hline 0 & \\ \vdots & (E_j \cdot E_k) \\ 0 & \end{array}\right) .$$

This proves the following theorem which will play a central role in connection with the Grauert–Mumford criterion.

Theorem 5.9 *The intersection matrix* $((E_j \cdot E_k))_{j,k=1,\ldots,r}$ *is negative definite if and only if the corresponding matrix* $((\overline{E}_j \cdot \overline{E}_k))_{j,k=0,\ldots,r}$ *has this property.*

5.A Appendix: The classification of simple plane curve singularities

In the present Appendix, we introduce the fundamental concept of an *unfolding* of a holomorphic function germ $f \in \mathcal{O}_0^{(n)}$ in the sense of a family of *small* variations of the original function. f is called *simple*, if f deforms locally only into finitely many different function germs.

5.A.1 Simple germs of holomorphic functions

An *unfolding* of a germ $f_0 \in \mathcal{O}_0^{(n)}$ is the germ F_0 of a function

$$F \in H^0(U \times V, \mathcal{O}_{\mathbb{C}^{n+k}}) ,$$

$0 \in U \subset \mathbb{C}^n$, $0 \in V \subset \mathbb{C}^k$, with

$$f(x_1, \ldots, x_n) := F(x_1, \ldots, x_n, 0, \ldots, 0)$$

inducing f_0. We think of F as being a deformation of f with parameters $t = (t_1, \ldots, t_k) \in V \subset \mathbb{C}^k$. A germ $f_0 \in \mathfrak{m}_n$ is called *simple* (or, more precisely, *right–simple*), if there exist finitely many germs $g_1, \ldots, g_r \in \mathcal{O}_0^{(n)}$ such that for any unfolding F_0 of f_0 and all zeros $(x^{(0)}, t_0)$ of a representative F in a suitable small neighborhood $U \times V$ of $0 \in \mathbb{C}^{n+k}$, the germ of the function

$$f^{t_0}(x) := F(x, t_0)$$

at $x^{(0)}$ is (right) equivalent to one of the germs $g_1, \ldots, g_r$.

The purpose of this Appendix is the beginning of the classification of such simple germs: We will completely solve the problem for $n = 2$. The general problem will be reduced to this case in Appendix A to Chapter 14.

Before we present the somewhat computational details, we study as an *Example* the function

$$f(x_1, x_2) = x_1 x_2 (x_1 + x_2)(x_1 - x_2) ,$$

whose zero set consists of four different lines intersecting at the origin. It is sufficient to look at the unfolding

$$F(x_1, x_2, t) := x_1 x_2 (x_1 - t x_2)(x_1 - x_2)$$

for t close to -1 and $x^{(0)} = 0$. If f^t and $f^{t'}$ were equivalent (for small $t + 1$ and $t' + 1$) at $x^{(0)} = 0$, we would find a linear automorphism $\varphi \in \mathrm{GL}(2, \mathbb{C})$ with

$$f_0^{t'} = f_0^t \circ \varphi$$

(since f^t and $f^{t'}$ are homogeneous functions in x of the same degree). The corresponding projective linear automorphism $\overline{\varphi}$ of $\mathbb{P}_1$ maps the points $[t' : 1]$, $[1 : 0]$, $[0 : 1]$, $[1 : 1]$ to the points $[t : 1]$, $[1 : 0]$, $[0 : 1]$, $[1 : 1]$. Since $\overline{\varphi}$ preserves the cross ratio

$$\frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4}$$

for any quadruple of points $z_1, z_2, z_3, z_4 \in \mathbb{P}_1$, where we identify $\mathbb{P}_1$ with $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ via $[\xi_0 : \xi_1] \mapsto \xi_1/\xi_0$, t' must be equal to one of the values

$$t , \quad \frac{1}{t} , \quad 1 - t , \quad \frac{1}{1 - t} , \quad \frac{t - 1}{t} , \quad \frac{t}{t - 1} ,$$

which is possible (if and) only if $t' = t$ or $t' = 1/t$. Therefore, the unfolding F contains in any neighborhood of -1 infinitely (in fact uncountably) many inequivalent function germs: the germ of $f(x_1, x_2)$ is *not* simple.

5.A.2 The generalized Morse Lemma and the main result

Regular germs $f_0 \in \mathfrak{m}_n$ are obviously simple. We thus may restrict our considerations to critical germs $f_0 \in \mathfrak{m}_n^2$. For a nondegenerate critical germ f - we suppress from now on for the rest of this Chapter the suffix 0 - all nearby germs $f_{x(0)}^t$ of an arbitrary unfolding F of f are either regular or critical, but nondegenerate, as a simple calculation using the Taylor series of F shows. So, the germ

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j^2$$

is simple. For the case, that the rank r of the two–jet $j^2 f$ of f at 0, viz.

$$r = \operatorname{rank}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right),$$

is smaller than n, we need some substitute for the Morse Lemma. This is provided by

Theorem 5.10 *If $f \in \mathfrak{m}_n^2$ has a two–jet of rank r, $0 \le r \le n$, then f is right equivalent to the germ of a function of type*

$$x_1^2 + \cdots + x_r^2 + g(x_{r+1}, \ldots, x_n), \quad g \in \mathfrak{m}_{n-r}^3.$$

The *proof* will be given in Appendix A.8. In Appendix A to Chapter 14 we will show that the simple germs $g \in \mathfrak{m}_n^2$, $n \ge 2$, are exactly those of type

$$g(x_1, \ldots, x_n) = f(x_1, x_2) + x_3^2 + \cdots + x_n^2$$

with a simple germ $f \in \mathfrak{m}_2^2$ (up to right equivalence). So, the following list - which is due to V. I. Arnol'd - contains already all information about the simple holomorphic function germs with isolated critical points (we write (x, y) instead of the coordinates (x_1, x_2)):

Theorem 5.11 *Any simple germ $f \in \mathfrak{m}_2^2$ with isolated critical point is right equivalent to one and only one of the following list*:

(A_k)	$x^{k+1} + y^2$,	$k \ge 1$,
(D_k)	$x^2 y + y^{k-1}$,	$k \ge 4$,
(E_6)	$x^3 + y^4$,	
(E_7)	$x^3 + xy^3$,	
(E_8)	$x^3 + y^5$.	

Remark. Of course, for $n = 1$ there is only the A_k–series consisting of the germs x^{k+1}, $k \ge 1$.

5.A.3 Normal forms for two - and three - jets

Since $r = \operatorname{rank} j^2 f = 2$ leads to the singularity of type A_1, we assume that $r < 2$. The case $r = 1$ is easily discussed. By Theorem 10, we have

$$f(x, y) = x^2 + g(y), \quad g \in \mathfrak{m}_1^3.$$

For $g = 0$, we get a germ that is not simple, since for (fixed) $t \neq 0$ the germs of the functions

$$x^2 + t\, y^k, \quad k \in \mathbb{N},$$

are pairwise nonequivalent. Therefore, $g(y) = y^\ell \cdot \text{unit}$, $\ell \ge 3$, and f is equivalent to a function of type

(A_k) $$x^2 + y^{k+1}, \quad k \geq 2 .$$

It remains to investigate the much more complex situation, where $j^2 f = 0$. Here, we first have to find normal forms for the homogeneous three–jet $j^3 f$. Since the number of zeros of a nontrivial cubic form on $\mathbb{P}_1$ is 3 (counted with multiplicity) and any 3 points on $\mathbb{P}_1$ can be moved to $0, 1, \infty$ by a projective linear automorphism, we have

Lemma 5.12 *Each cubic form $a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3$ is reduced by $\mathbb{C}$–linear transformations to one (and only one) of the forms*

$$x^2y + y^3 , \quad x^2y , \quad x^3 , \quad 0 .$$

In the next Section, we will show that the case $x^2y + y^3$ leads to D_4, whereas x^2y leads to the D_k, $k \geq 5$. In Section A.5, we shall see that only the germs of type E_k can be simple with $j^3f = x^3$. Finally, Section A.6 is devoted to the fact that simple germs must have nontrivial three–jets. In Section A.7 we give some hints how to prove that all the germs in Theorem 7 are indeed simple.

5.A.4 $j^3f = x^2y + y^3, x^2y$

By Theorem 2.30, the germ $x^2y + y^3$ is 3–determined; hence, $f \sim x^2y + y^3$ is of type D_4. The germ x^2y is not simple, since

$$x^2y + t\,y^k , \quad k \in \mathbb{N} ,$$

are pairwise nonequivalent germs (for fixed t). Therefore, we must find an ℓ–jet, $\ell \geq 4$, of type

$$j^\ell f = x^2y + ay^\ell + 2bxy^{\ell-1} + x^2g(x, y) , \quad g \in \mathfrak{m}_2^{\ell-2} .$$

By a linear substitution $x_1 = x + by^{\ell-2}$, $y_1 = y + g(x, y)$, the ℓ–jet is reduced to the form

$$j^\ell f = x_1^2y_1 + ay_1^\ell .$$

If $a = 0$, we can repeat this process, and we come to the conclusion that (since f is finitely determined) either $f \sim x^2y$ - which is impossible - or that there exists an $\ell \geq 4$ and a complex number $a \neq 0$ with

$$j^\ell f = x^2y + ay^\ell .$$

The last germ being ℓ–determined by Theorem 2.30, we have

$$f \sim x^2y + y^{k-1} , \quad k \geq 5 .$$

5.A.5 $j^3f = x^3$

In this case, we write the four–jet of f in the form

$$j^4f = x^3 + ay^4 + bxy^3 + 3x^2g(x, y) , \quad g \in \mathfrak{m}_2^2 .$$

Using the substitution $x_1 = x + g(x, y)$, $y_1 = y$ yields the form

$$j^4f = x^3 + ay^4 + bxy^3$$

(with a possibly different constant b).

In the case $a \neq 0$, we substitute $y \mapsto cy + dx$ for suitable c, d, and derive a form

$$j^4f = x^3 + y^4 + 3x^2h(x, y) , \quad h \in \mathfrak{m}_2^2 .$$

After the transformation $x_1 = x + h(x, y)$, $y_1 = y$, we are led to the four–jet

$$x_1^3 + y_1^4$$

which is 4–determined. Hence, $f \sim x^3 + y^4$, i.e. f is a germ of type (E_6).

If $a = 0$, but $b \neq 0$, we may assume $b = 1$. We are then going to prove that the germ

$$x^3 + xy^3$$

is 4–determined such that in this case f must be equivalent to the germ of type (E_7). Unfortunately, Theorem 2.30 yields only the 5–determinacy of this germ. Therefore, we are forced to show that each germ

$$x^3 + xy^3 + \alpha x^5 + \beta x^4 y + \gamma x^3 y^2 + \delta y^5 + xy^3 g(x, y), \quad g \in \mathfrak{m}_2,$$

is equivalent to $x^3 + xy^3$ modulo $\mathfrak{m}_2^6$. Because of the term xy^3, we are reduced at once to the case $g = 0$ by the substitution $y_1 = y(1 + g(x, y))^{1/3}$. Repeating the same trick for the term x^3 leads to $\alpha = \beta = \gamma = 0$, and we may assume that $\delta = 1$. The nontrivial transformation

$$\begin{cases} x = x_1 - x_1^2/3 - x_1 y_1 - y_1^2 \\ y = y_1 + x_1 \end{cases}$$

gives finally a 5–jet of type

$$x_1^3 + x_1 y_1^3 + x_1 h(x_1, y_1), \quad h \in \mathfrak{m}_2^4,$$

which is seen to being equivalent to the jet $x_1^3 + x_1 y_1^3$ by following our arguments once more.

We are left with the case $a = b = 0$. Writing the 5–jet in the form

$$x^3 + cy^5 + dxy^4 + 3x^2 g(x, y), \quad g \in \mathfrak{m}_2^3,$$

we can easily transform this to $x^3 + y^5$ modulo $\mathfrak{m}_2^6$, if $c \neq 0$. Since the germ of type (E_8) is 5–determined, we are done.

To complete the classification in the given case, we have to show that an arbitrary germ f with 5–jet of type

$$x^3 + dxy^4 + 3x^2 g(x, y), \quad g \in \mathfrak{m}_2^3,$$

is not simple. Substituting $x_1 = x + g(x, y)$, $y_1 = y$ reduces our considerations to $g = 0$. Following the same strategy as in the case $a = 0$, $b \neq 0$, we easily simplify the 6–jet of such a germ in case $d \neq 0$ to one of the form

$$x^3 + xy^4 + ey^6.$$

We now introduce the unfolding

$$F(x, y, \lambda) = f(x, y) + \lambda y^6$$

whose 6–jet equals

$$x^3 + xy^4 + t y^6, \quad t = e + \lambda,$$

which is 6–determined for $t \neq 0$ (see Chapter 2.13). In case $d = 0$, we study the two–parameter family

$$F(x, y, \lambda, \mu) = f(x, y) + \lambda y^6 + \mu xy^4$$

near $\mu = 0$. That both families violate the condition for f to be simple is an easy consequence of

Lemma 5.13 *The germs $f^t \in \mathcal{O}_0^{(2)}$ of the functions $f^t(x, y) = x^3 + xy^4 + t y^6$, $t \in \mathbb{C}$, are generically not equivalent.*

Proof. Let t_1, t_2, t_3 be the roots of the equation $\lambda^3 + \lambda + t = 0$. Then

$$f^t(x, y) = (x - t_1 y^2)(x - t_2 y^2)(x - t_3 y^2),$$

and the analytic hypersurface $\{ f^t(x, y) = 0 \}$ consists for generic t of three different parabolae touching each other at the origin:

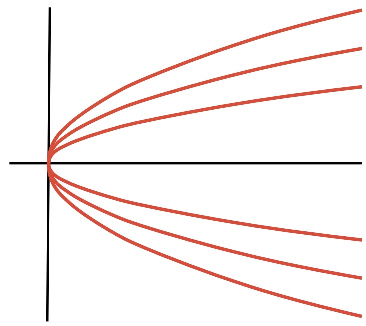


Figure 5.12

Blowing up the origin yields the equations

$$\ell_j = \{ u = t_j v \} , \quad j = 1, 2, 3 ,$$

for the strict transforms ℓ_j of these parabolae (in the coordinate system $(u, v) = (u_1, v_1)$ at "infinity").

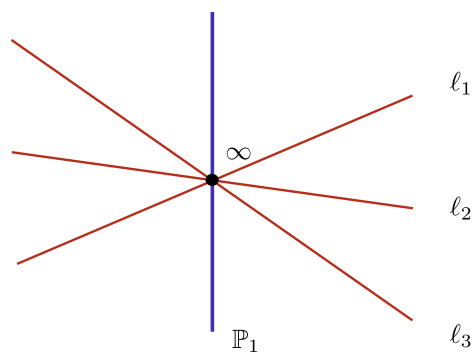


Figure 5.13

For another complex number t', we denote by t'_1, t'_2, t'_3 the corresponding roots of the equation $\lambda^3 + \lambda + t' = 0$. It is easily checked that an arbitrary biholomorphic automorphism germ φ_0 mapping f^t to $f^{t'}$ induces a linear automorphism of the tangent plane of the blow–up of $\mathbb{C}^2$ at the point $\infty \in \sigma^{-1}(0) \cong \mathbb{P}_1 \cong \overline{\mathbb{C}}$ which maps the three directions t_1, t_2, t_3 to the directions t'_1, t'_2, t'_3, keeping the fourth direction ∞ fixed. Hence, by projectivizing the tangent plane, we see that the cross ratios

$$\tau = \frac{t_3 - t_1}{t_3 - t_2} \text{ and } \tau' = \frac{t'_3 - t'_1}{t'_3 - t'_2}$$

coincide up to the natural action of the symmetric group $\mathfrak{S}_3$ on $\mathbb{P}_1$ (which associates to the transposition $(1\,2)$ the rational map $\tau \mapsto \tau^{-1}$ and to the cycle $(1\,2\,3)$ the rational map $\tau \mapsto (\tau - 1)\,\tau^{-1}$).

Obviously, it is enough (for finishing the *proof* of Lemma 13) to show that for all $t \in \mathbb{C}$ outside the discriminant locus of the equation $\lambda^3 + \lambda + t = 0$ (which consists of the two points $\pm t_0$ satisfying $4 + 27\,t_0^2 = 0$) there is no infinite sequence $t' \to t$ with $\varphi^{t'} \sim \varphi^t$ for all t'. So, assume by reductio ad absurdum that such a converging sequence exists. It is clear that we can locally define holomorphic maps

$$\mathbb{C} \setminus \{ \pm t_0 \} \ni t' \stackrel{\tau}{\longmapsto} \tau\,(t') = \frac{t'_3 - t'_1}{t'_2 - t'_1} \in \mathbb{C} \setminus \{ 0 \} \subset \mathbb{P}_1$$

which induce a globally defined continuous map

$$\mathbb{C} \setminus \{ \pm t_0 \} \ni t' \stackrel{\overline{\tau}}{\longmapsto} \overline{\tau}\,(t') = \tau\,(t') \bmod \mathfrak{S}_3$$

from $\mathbb{C} \setminus \{\pm t_0\}$ to the topological quotient $\mathbb{P}_1/\mathfrak{S}_3$. By our assumption and the Identity Theorem for holomorphic functions in one variable, the maps τ and $\overline{\tau}$ would be constant near t. As it is well–known (and will be discussed in more detail in Chapter 8) the quotient $\mathbb{P}_1/\mathfrak{S}_3$ carries a natural complex analytic manifold structure making the projection $\mathbb{P}_1 \to \mathbb{P}_1/\mathfrak{S}_3$ - and hence the map $\overline{\tau}$ - holomorphic. A straightforward generalization of the Identity Theorem then implies that $\overline{\tau} : \mathbb{C} \setminus \{\pm t_0\} \to \mathbb{P}_1/\mathfrak{S}_3$ must be globally constant. But this is impossible since for $t \neq \pm t_0$ converging to $\pm t_0$, the image of t in $\mathbb{P}_1/\mathfrak{S}_3$ consists of an $\mathfrak{S}_3$–orbit

$$\left\{\tau\,,\quad \frac{1}{\tau}\,,\quad 1-\tau\,,\quad \frac{1}{1-\tau}\,,\quad \frac{\tau-1}{\tau}\,,\quad \frac{\tau}{\tau-1}\right\}$$

with $\tau \neq 0, 1, \infty$ which converges to the exceptional orbit

$$\{0, \infty, 1\}$$

(since there are no triple roots of $\lambda^3 + \lambda + \pm t_0 = 0$). □

5.A.6 $j^3 f = 0$

If g is a second germ with $j^3 g = 0$, and if $f \sim g$, then necessarily the homogeneous parts of degree 4 of f and g resp. are equivalent. So, if f were simple, there would not exist an unfolding F of f with $j^3 F^t = 0$ for all t and having only finitely many equivalence classes among the $j^4 F^t$. But such a family is easily written down:

$$F(x, y, t_0, t_1, t_2, t_3, t_4) = t_0 x^4 + t_1 x^3 y + t_2 x^2 y^2 + t_3 x y^3 + t_4 y^4 + f(x, y)\,.$$

The family of 4–jets being the variety of all homogeneous polynomials of degree 4, it contains the open and dense subset corresponding to 4–forms having only simple zeros on $\mathbb{P}_1$. But those germs are equivalent to the nonsimple germs

$$x\,y\,(x - y)\,(x - ty)\,,\quad t \neq 0, 1$$

(see Section A.1).

Let us formalize the last argument a little further: In fact, we regard the 5–dimensional vector space S_4 of all homogeneous 4–forms in two variables together with the action of the general linear group $\mathrm{GL}(2, \mathbb{C})$ which is induced from the action of $\mathrm{Aut}\,\mathcal{O}_{\mathbb{C}^2,0} \supset \mathrm{GL}(2, \mathbb{C})$ on $\mathcal{O}_{\mathbb{C}^2,0} \supset S_4$. But $\mathrm{GL}(2, \mathbb{C})$ has dimension 4 such that orbits of this action can only have dimensions at most 4. So, every neighborhood of an element in S_4 ought to meet uncountably many such orbits.

5.A.7 The versal unfolding of holomorphic function germs

An unfolding $F(x_1, \ldots, x_n, t_1, \ldots, t_r)$ of the function $f(x_1, \ldots, x_n)$ is called *versal*, if each unfolding $G(x_1, \ldots, x_n, s_1, \ldots, s_m)$ of f can be derived from F up to right equivalence by a holomorphic substitution $t = \varphi(s)$. More precisely, that means that there exist holomorphic maps

$$\varphi : \mathbb{C}^m \longrightarrow \mathbb{C}^r \text{ and } \Phi : \mathbb{C}^n \times \mathbb{C}^m \longrightarrow \mathbb{C}^n$$

with $\Phi(x, 0) = x$ such that

$$G(x, s) = F(\Phi(x, s), \varphi(s))\,.$$

It is clear, how one has to modify this definition for germs of functions.

It follows from the condition $\Phi(x, 0) = x$ that $\Phi^t(x) := \Phi(x, t)$ gives rise to a family of biholomorphic map germs for t close to the origin. In particular, the knowledge of such a versal unfolding F and the equivalence classes of the germs $F^t(x)$, t near 0, is sufficient for knowing *all* germs $G^s(x)$ for *all* unfoldings G of f, s small.

Fortunately, there exists a versal unfolding in the cases we are interested in which, moreover, is easily computable. It is not difficult to show that the existence of such an unfolding for a germ $f \in \mathfrak{m}_n$ implies that

$$\dim_{\mathbb{C}} \mathcal{O}_0^{(n)} / J_f < \infty , \quad J_f = \left(\frac{\partial f}{\partial x_1} , \ldots , \frac{\partial f}{\partial x_n} \right) \mathcal{O}_0^{(n)} .$$

The converse is also true:

***Theorem 5.14** *Let $h_1, \ldots, h_r$ be representatives of a finite $\mathbb{C}$–basis of the vector space $\mathcal{O}_0^{(n)} / J_f$. Then f possesses a versal unfolding F which is given by*

$$F(x, t) = f(x) + t_1 h_1(x) + \cdots + t_r h_r(x) .$$

By an easy computation we are able to write down the versal unfolding for the presumably simple function germs.

Theorem 5.15 *The versal unfoldings of the germs of type (A_k), (D_k) and (E_k) are the following:*

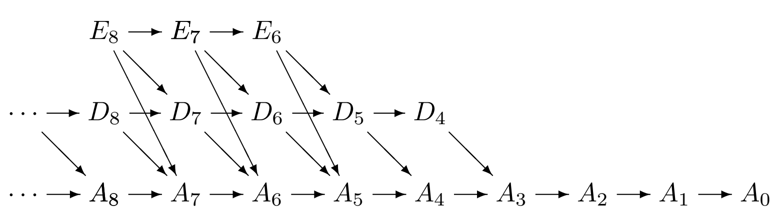
(A_k)	$x^{k+1} + y^2 + t_1 + t_2 x + \cdots + t_k x^{k-1}$
(D_k)	$x^2 y + y^{k-1} + t_1 + t_2 y + \cdots + t_{k-1} y^{k-2} + t_k x$
(E_6)	$x^3 + y^4 + t_1 + t_2 y + t_3 y^2 + x(t_4 + t_5 y + t_6 y^2)$
(E_7)	$x^3 + xy^3 + t_1 + t_2 y + t_3 y^2 + t_4 y^3 + t_5 y^4 + x(t_6 + t_7 y)$
(E_8)	$x^3 + y^5 + t_1 + t_2 y + t_3 y^2 + t_4 y^3 + x(t_5 + t_6 y + t_7 y^2 + t_8 y^3)$.

For fixed t, we can now subject these germs to the algorithm developed in the Sections A.3, A.4 and A.5. The result is usually documented in form of an *adjacency diagram* in which a consecutive sequence of arrows

$$C \longleftarrow \cdots \longleftarrow D$$

indicates that the germ of type C arises at the germ of a fiber F^t of the versal unfolding F of D for arbitrarily small t. The germ C is then called *adjacent* to D. An adjacency diagram is always assumed to be complete in that it should comprise all possible adjacencies.

Theorem 5.16 *The following is the adjacency diagram for the germs of type (A_k), (D_k) and (E_k):*



As a Corollary, we finally may conclude that these germs are actually simple, thereby finishing the *proof* of Theorem 11 up to the demonstration of the generalized Morse Lemma which we supply subsequently.

5.A.8 Proof of the generalized Morse Lemma

As in the statement of Theorem 10, let us assume that the critical germ $f \in \mathfrak{m}_n^2$ has a two–jet $j^2 f$ of rank r. Then we may write f in the form

$$f = q + g ,$$

where $q = q(x_1, \ldots, x_r) = x_1^2 + \cdots + x_r^2$ and $g \in \mathfrak{m}_n^3$. We expand g as

$$g = g_0 + x_{r+1} g_{r+1} + \cdots + x_n g_n ,$$

where $g_0 = g_0(x_1, \ldots, x_r) \in \mathfrak{m}_r^3$ and $g_{r+1}, \ldots, g_n \in \mathfrak{m}_n^2$. Using the Morse Lemma for q we can drop g_0. Hence,

$$f(x_1, \ldots, x_r, x_{r+1}, \ldots, x_n) = q(x_1, \ldots, x_r) + \sum_{\rho=r+1}^{n} x_\rho g_\rho(x_1, \ldots, x_n)$$

can be viewed as an unfolding of q with parameters $x_{r+1}, \ldots, x_n$. By Theorem 14, the universal unfolding of q is given by

$$q(x_1, \ldots, x_r) + t \, .$$

Hence, there exist functions

$$\begin{cases} \varphi = \varphi(x_{r+1}, \ldots, x_n) \\ \Phi_k = \Phi_k(x_1, \ldots, x_n) = x_k + \displaystyle\sum_{\rho=r+1}^{n} x_\rho \, h_{\rho k}(x_1, \ldots, x_n) \, , \quad k = 1, \ldots, r \, , \end{cases}$$

such that

$$f(x_1, \ldots, x_n) = q(\Phi_1(x), \ldots, \Phi_k(x)) + \varphi(x_{r+1}, \ldots, x_n) \, .$$

The claim then follows by (the inverse of) the substitution $x'_k = \Phi_k(x)$, $k = 1, \ldots, r$, $x'_j = x_j$, $j = r + 1, \ldots, n$.

Notes and References

The proof of Theorem 3 is a slight variation of the version given in [04 - 04], paragraph 8B. According to Mumford (loc.cit., p. 161), it is due to Hironaka and is extracted from his proof of resolution in the general case published in the famous paper

[05 - 01] H. Hironaka: Resolution of singularities of an algebraic variety over a field of characteristic zero. Annals of Math. 79, 109–326 (1964).

There is a nice interpretation of the invariant ν in “geometric” terms related to the *Newton diagram*

$$\mathrm{Newt}(f) = \{ (j, k) \in \mathbb{N}^2 : a_{jk} \neq 0 \}$$

for a given power series $f = \sum a_{jk} x^j y^k$ in two variables. Due to our assumptions,

$$(m, 0) \in \mathrm{Newt}(f) \subset \{ (j, k) : k \geq m - j \}.$$

Regard now *all* lines through $(m, 0)$ of the form

$$k = \omega(m - j) \text{ with } \omega \geq 1 \, .$$

Then,

$$\nu(f) = \sup_{\omega} \{ \mathrm{Newt}(f) \subset \{ (j, k) : k \geq \omega(m - j) \} \} \, ,$$

when $\nu(f) < \infty$.

Concerning intersection theory of curves on surfaces we consulted mainly [01 - 18], pp. 65 ff, in particular with respect to the sheaf theoretic proof of Lemma 6.

The classification of the simple holomorphic function germs is due to V. I. Arnol’d. The full picture including the hierarchy of simple germs appeared in

[05 - 02] V. I. Arnol’d: Normal forms for functions near degenerate critical points, the Weyl groups A_k, D_k, E_k and Lagrangian singularities. Functional Analysis and its Applications 6, 254–272 (1972).

This work of Arnol'd together with its generalization to the classification of 1–modular and 2–modular germs (the simple germs being 0–modular in this context) is an integral part of the book [01 - 20]. It also contains a complete list of his papers. In Chapter 8 of the present manuscript, the reader may find all the results on the versal unfolding we used in the Appendix. Our arrangement of the material is strongly influenced by [01 - 15] (see also [01 - 18], pp. 61 ff).

Of course, there is much more to say about the beautiful domain of (algebraic) plane curves. From our point of view, the best suited accompanying text on this subject is

[05 - 03] E. Brieskorn, H. Knörrer: Plane Algebraic Curves. Basel–Boston–Stuttgart: Birkhäuser 1986. (First published 1981 by Birkhäuser under the title *Ebene algebraische Kurven*.)