Two elementary analytic evaluations of $\zeta(2)$ found by Euler and another one he missed

Oswald Riemenschneider

Mathematisches Seminar
Universität Hamburg

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Ovid, *Metamorphoses*

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where $\zeta(2)$ is the value of the *Riemann Zeta - function*

$$\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \text{etc.}$$

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It is easy to see that $\zeta(2)$ is a convergent series. Hence, by WEIERSTRASS's M-test, $\zeta(s)$, $\zeta_{\rm odd}(s)$ and $\zeta_{\rm even}(s) := \zeta(s) - \zeta_{\rm odd}(s)$ are absolutely convergent for real $s \geq 2$ and also for complex s with $\mathrm{Re}\, s \geq 2$. For seeing that this is even true in the region $\mathrm{Re}\, s > 1$, one has to use the more complicated RAABE *convergence test* or the even deeper *comparison test*.

Now, obviously

$$\frac{1}{2^s}\,\zeta(s) = \zeta_{\rm even}(s)\,, \quad \left(1 - \frac{1}{2^s}\right)\zeta(s) = \zeta_{\rm odd}(s)$$

such that there is no essential difference between $\zeta(s)$, $\zeta_{\rm even}(s)$ and $\zeta_{\rm odd}(s)$.

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Plan of lecture

1. The Basel problem

The Basel Problem

In 1650, the Italian mathematician Pietro Mengoli showed in his book *Novae Quadraturae Arithmetica* that the sum of the reciprocals of the *triangular numbers*

$$1\,,\ 1+2=3\,,\ 1+2+3=6\,,\ 1+2+3+4=10\,,\ \ldots$$

is 2. In other words:

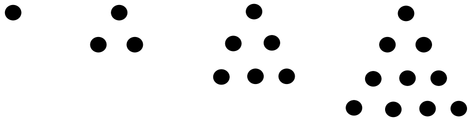
$$\sum_{n=1}^{\infty} \frac{2}{n\,(n+1)} = 2\,.$$

This result is - for modern standards - almost trivial since we have a *telescope* series (which, by the way, dominates $\zeta\,(2)$!):

$$\frac{1}{n\,(n+1)} = \frac{1}{n} - \frac{1}{n+1}\,.$$

The Basel Problem

He asked the obvious question: After having treated



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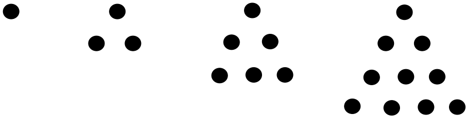
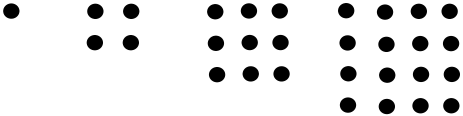


What about the reciprocals of the *quadrangular* numbers, i.e. the *squares*?



The task to express the limit of the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.},$$

by known terms has also been mentioned by John Wallis in the 17th century and became one of the most urgent challenges at the beginning of the 18th century. Despite the efforts of the most prestigious mathematicians like Leibniz, Stirling, de Moivre and all the Bernoullis it remained unsolved for a long time. Because of the involvement of Jakob and Johann Bernoulli the problem entered the history of mathematics as the "Basel problem".

A frustrated Jakob Bernoulli, at the time one of the most experienced mathematicians in manipulating infinite series, formulated the following urgent request:

"*... sollte jemand das, was unseren Anstrengungen bis jetzt entgangen ist, finden und uns mitteilen, so werden wir ihm sehr dankbar sein*".

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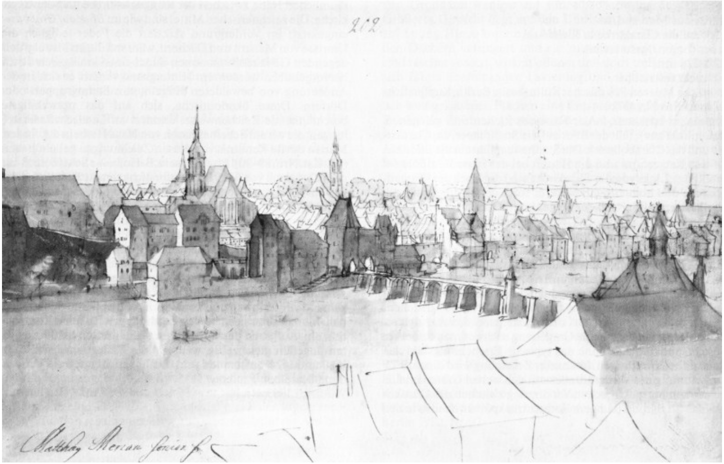
" ... should somebody find and communicate to us what escaped our endeavors we would be very grateful to him."



Matthäus Merian: Sketch of Basel (ca. 1615)

Basel today

2. Short survey on Euler's contributions

Jakob Emanuel Handmann: Leonhard Euler 1753

- In a paper he read at the Petersburg academy on December 5, 1735, Leonhard Euler (1707 - 1783) presented his ingenious method to calculate the values of the Riemann Zeta function at the even integers that after some extra work was later brought into the form

$$\zeta(2k) := \frac{(2\pi)^{2k}}{2\,(2k)!}\, B_{2k}\ .$$

In particular, he solved the Basel problem:

$$(+) \qquad \zeta(2) = \pi^2/6\ .$$

- This paper has been published only in 1740 under the title *De Summis Serierum Reciprocarum* in the proceedings of the academy (Commentarii academiae scientiarum Petropolitanae 7, pp. 123–134 [E 41]. An English translation from the Latin by Jordan Bell can be found under the title *On the sums of series of reciprocals* in arXiv:math/05064152v2). Here, E stands for *Eneström - Index*.

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Das theorema summationis seriei

$$1 + \frac{1}{4} + \frac{1}{9}\,\text{etc.} = \frac{pp}{6} \quad \textit{und} \quad 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4}\,\text{etc.} = \frac{p^4}{90}$$

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Indeed, Euler had already done parts of his ingenious contributions to the summation of (slowly) converging series. In *De summatione innuberabilium progressionem* [E 20] (submitted 1731, published 1738) he computed the value up to 6 decimals. And in *Inventio summae cujusque seriei ex dato termini generali* [E 47] (read at the St. Petersburg academy on October 13, 1735, published in 1741) he had already 20 decimals. It is believed that this approximation let him to first "guess" the result $\pi^2/6$.

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Indeed, Euler had already done parts of his ingenious contributions to the summation of (slowly) converging series. In *De summatione innuberabilium progressionem* [E 20] (submitted 1731, published 1738) he computed the value up to 6 decimals. And in *Inventio summae cujusque seriei ex dato termini generali* [E 47] (read at the St. Petersburg academy on October 13, 1735, published in 1741) he had already 20 decimals. It is believed that this approximation let him to first "guess" the result $\pi^2/6$.

Vasilij Sokolov: Leonhard Euler 1737

In his St. Petersburg paper and later on, Euler used the product formula for the Sin - function:

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{(n\pi)^2}\right)$$

just in analogy to the situation for polynomials. This has been criticized, among others, by Johann Bernoulli in a letter of April 2, 1737 to Euler.

He made some attempts to justify the formula, but did not really succeed. Some people speculate that he gave up the whole subject because he was not satisfied with this approach himself.

One should have in mind that a rigorous proof of the product formula was given only 100 years later by Weierstraß. In my opinion, evaluations of $\zeta(2k)$ or $\zeta(2)$ with formulas like this are really wonderful and should be taught in Complex Analysis courses, but they can of course not be considered to be *elementary*.

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$$(++) \qquad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots .$$

It is the purpose of this lecture to present a simultaneous proof of $(+)$ and $(++)$ with Euler's second method using "[une] équation *différentio–différientelle*" which he published only in 1743 under the title *Démonstration de la somme de cette Suite* $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}$ *etc.* in a rather obscure journal - and to compare this approach with an idea of

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3. Wallis' integrals and $\zeta(2)$

Wallis' integrals

For the Wallis' integrals

$$\mathcal{I}_m := \int_0^1 \frac{x^m \, \mathrm{d}x}{\sqrt{1 - x^2}} = \int_0^{\pi/2} \sin^m \phi \, \mathrm{d}\phi$$

one gets by *partial integration* - replacing the factor $\sin^2 \phi$ by $1 - \cos^2 \phi$ or writing the integrand in the form

$$\frac{x^m}{\sqrt{1 - x^2}} = u(x)\, v'(x) \quad \text{with} \quad u(x) = x^{m-1}\,, \; v(x) = -\sqrt{1 - x^2}$$

- the well known recursion formulas

$$(m + 1)\mathcal{I}_{m+1} = m\mathcal{I}_{m-1}$$

with the initial values $\mathcal{I}_0 = \frac{\pi}{2}\,, \quad \mathcal{I}_1 = 1\,.$

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Wallis' integrals and $\zeta(2)$

For the sake of simplicity we define the sequence(s) (a_n) for the rest of this talk via

$$(\circ) \qquad a_0 = a_1 = 1\,, \quad (n+2)\,a_{n+2} = (n+1)\,a_n\,.$$

Obviously, we then have

$$\mathcal{I}_{2k} = \frac{\pi}{2}\,a_{2k}\,, \quad \mathcal{I}_{2k+1} = a_{2k+1}\,.$$

By multiplication of $(\circ)$ with a_{n+1} we can conclude that the sequence

$$n\,a_n\,a_{n-1} \quad \text{is constant equal to} \quad 1\,a_1\,a_0 = 1\,, \quad \text{hence}$$

$$a_n\,a_{n-1} = \frac{1}{n}\,, \quad n \geq 1\,.$$

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Therefore, it follows purely formally and at first sight without any profit

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{a_n \, a_{n-1}}{n} \, ,$$

or, if n runs first only over the *even* integers:

$$\zeta_{\mathrm{even}}(2) = \frac{1}{4}\,\zeta(2) = \sum_{k=1}^{\infty} \frac{a_{2k} \, a_{2k-1}}{2k}$$

and then through the *odd* integers:

$$\zeta_{\mathrm{odd}}(2) = \frac{3}{4}\,\zeta(2) = \sum_{k=0}^{\infty} \frac{a_{2k+1} \, a_{2k}}{2k+1} \, .$$

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4. Euler's first elementary proof

By induction, it is easily seen that the coefficients a_{2k} are explicitly given by

$$a_{2k} = \frac{1^2 \cdot 3^2 \cdot \ldots \cdot (2k-1)^2}{(2k)!} = \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{2 \cdot 4 \cdot \ldots \cdot (2k)} = \frac{(2k)!}{2^{2k}\, k!^2} .$$

Or, to put it differently,

$$a_{2k} = \frac{1}{2^{2k}} \binom{2k}{k} = (-1)^k \binom{-1/2}{k} .$$

Thus,

$$\frac{1}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} a_{2k} x^{2k} .$$

(We will give another proof in the next section without using the generalized binomial coefficients.)

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By integration, it follows that

$$\arcsin x = \sum_{k=0}^{\infty} \frac{a_{2k}}{2k+1}\, x^{2k+1} .$$

Interchanging summation and integration (which is allowed; see the Appendix) this yields

$$\begin{aligned}\frac{3}{4}\,\zeta(2) &= \sum_{k=0}^{\infty} \frac{a_{2k+1}\, a_{2k}}{2k+1} = \sum_{k=0}^{\infty} \frac{a_{2k}}{2k+1}\, \mathcal{I}_{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{a_{2k}}{2k+1} \int_0^1 \frac{x^{2k+1}\, dx}{\sqrt{1-x^2}} = \int_0^1 \sum_{k=0}^{\infty} \frac{a_{2k}}{2k+1} \frac{x^{2k+1}\, dx}{\sqrt{1-x^2}} \\ &= \int_0^1 \frac{\arcsin x\, dx}{\sqrt{1-x^2}} = \frac{1}{2} \arcsin^2 x \Big|_0^1 = \frac{\pi^2}{8} .\end{aligned}$$

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\frac{3}{4}\,\zeta\,(2) &= \sum_{k=0}^{\infty} \frac{a_{2k+1}\,a_{2k}}{2k+1} = \sum_{k=0}^{\infty} \frac{a_{2k}}{2k+1}\,\mathcal{I}_{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{a_{2k}}{2k+1} \int_0^1 \frac{x^{2k+1}\,\mathrm{d}x}{\sqrt{1\,-\,x^2}} = \int_0^1 \sum_{k=0}^{\infty} \frac{a_{2k}}{2k+1}\,\frac{x^{2k+1}\,\mathrm{d}x}{\sqrt{1\,-\,x^2}} \\
&= \int_0^1 \frac{\arcsin\,x\,\mathrm{d}x}{\sqrt{1\,-\,x^2}} = \frac{1}{2}\,\arcsin^2 x\,\Big|_0^1 = \frac{\pi^2}{8}\;.
\end{aligned}$$

5. Euler's differential equation and his second elementary proof

Euler's differential equation

Let the power series $\sum_{n=0}^{\infty} \alpha_n x^n$ be convergent in a neighbourhood of the origin 0, where the sequence of the coefficients α_n is not fixed for the moment. We denote the function determined by the series by f. The connection with the Wallis' integrals will be illuminated by the following

Lemma

The following are equivalent :

i) *f satisfies in a neighborhood of 0 the linear differential equation :*

$$(\times) \qquad (1 - x^2)\, f'(x) = x\, f(x) + c\,,$$

where necessarily $c = f'(0) = \alpha_1$.

ii) *The sequence α_n is given by the recursion formulas*

$$(n+2)\,\alpha_{n+2} = (n+1)\,\alpha_n$$

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for all $n \geq 0$.

In particular, the given power series is convergent in the interval I of the real numbers x with $|x| < 1$, and the differential equation in i) is satisfied on I.

Proof. By (formal) differentiation of the power series and subsequent simple algebraic manipulations it follows that the expression

$$(1 - x^2)\, f'(x) - x\, f(x)$$

will be represented by the power series

$$\alpha_1 + (2\alpha_2 - \alpha_0)\, x + (3\alpha_3 - 2\alpha_1)\, x^2 + (4\alpha_4 - 3\alpha_2)\, x^3 + \cdots .$$

Comparison of coefficients on both sides of i) yields immediately the equivalence of i) and ii).

Since the given power series is convergent on I under the condition ii) by the quotient criterion and consequently the power series of its derivative, too, the last claim is automatically justified by the identity theorem. □

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Remark

Euler's differential equation shows up in his 1743 paper in the form

$$\mathrm{dd}y\,(1 - xx) - x\,\mathrm{d}x\,\mathrm{d}y = \mathrm{d}x^2 \,.$$

He then adds:

Mais en divisant l'équation différentio–différentielle par $\mathrm{d}x^2$ *, nous avons*

$$\frac{\mathrm{dd}y}{\mathrm{d}x^2} - \frac{xx\mathrm{dd}y}{\mathrm{d}x^2} - \frac{x\,\mathrm{d}y}{\mathrm{d}x} - 1 = 0 \,.$$

The paper has been published in the long time forgotten *Journal littéraire d'Allemagne, de Suisse et du Nord, 2:1. pp. 115–127 (1743)*. Paul Stäckel reproduced and discussed it in his article *Eine vergessene Abhandlung Leonhard Eulers über die Summe der reziproken Quadrate der natürlichen Zahlen*. Bibl. Math. (3) 8, pp. 37–54 (1907–1908).

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In order to solve the equation $(\times)$ we start with the *homogeneous* case $c = 0$. Because of linearity, we may assume that $f(0) = \alpha_0 = 1$.

Lemma

The homogeneous linear differential equation

$$(1 - x^2)\, f'(x) = x\, f(x)$$

has, with the initial condition $q(0) = 1$, the (uniquely determined) solution

$$q(x) = \frac{1}{\sqrt{1 - x^2}}\,.$$

In particular,

$$\frac{1}{\sqrt{1 - x^2}} = \sum_{k=0}^{\infty} a_{2k}\, x^{2k}\,,$$

where $a_0 = 1$ and $(2k + 2)\, a_{2k+2} = (2k + 1)\, a_{2k}$.

In order to solve the equation $(\times)$ we start with the *homogeneous* case $c = 0$. Because of linearity, we may assume that $f(0) = \alpha_0 = 1$.

Lemma

The homogeneous linear differential equation

$$(1 - x^2)\, f'(x) = x\, f(x)$$

has, with the initial condition $q(0) = 1$*, the (uniquely determined) solution*

$$q(x) = \frac{1}{\sqrt{1 - x^2}}\,.$$

In particular,

$$\frac{1}{\sqrt{1 - x^2}} = \sum_{k=0}^{\infty} a_{2k}\, x^{2k}\,,$$

where $a_0 = 1$ *and* $(2k + 2)\, a_{2k+2} = (2k + 1)\, a_{2k}$.

Proof. Because of the initial condition any solution f must be positive in a neighbourhood of the origin. Hence, there exists the logarithm $\ln f(x)$ near 0, and we have

$$2\,(\ln f(x))' = 2\,\frac{f'(x)}{f(x)} = \frac{2x}{1-x^2} = -(\ln(1-x^2))'\,.$$

Integrating and "exponentiating" results in

$$f^2(x) = \frac{C}{1-x^2}\,,$$

whence $f = q$ when $f(0) = 1$. □

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$$f(x) = c(x)\,q(x)\,.$$

f satisfies the inhomogeneous equation if and only if

$$(1 - x^2)\,c'(x)\,q(x) = c\,,$$

i.e.

$$c'(x) = c\,q(x)\,.$$

Therefore,

$$c(x) = c \arcsin x + C\,,$$

and the uniquely determined solution of the inhomogeneous equation f with $f(0) = 0$ is given by

$$f(x) = \frac{c \arcsin x}{\sqrt{1 - x^2}}\,.$$

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Corollary

On the interval I *, one has the power series expansions*

$$\frac{\arcsin x}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} a_{2k+1}\, x^{2k+1}\,, \quad \arcsin^2 x = \sum_{k=0}^{\infty} \frac{a_{2k+1}}{k+1}\, x^{2k+2}\,,$$

where $a_1 = 1$ *and* $(2k+3)\, a_{2k+3} = (2k+2)\, a_{2k+1}\,.$

Remark

Interestingly enough, this expansion for $\arcsin^2$ *has been known earlier in Japan as a tool to approximately calculate the length of the chord of a given arc of a circle. Some historians ascribe this result to the famous Japanese mathematician* Takaharu Seki *(1642 - 1708); more recent articles refer to his pupil* Katahiro Takebe *(1664 - 1739).*

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Euler's second elementary proof

After our preparations, also Euler's second elementary proof is mere child's play:

$$\frac{1}{4}\,\zeta(2) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{a_{2k-1}\mathcal{I}_{2k}}{2k} = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{a_{2k-1}}{2k}\int_0^1 \frac{x^{2k}\,\mathrm{d}x}{\sqrt{1-x^2}}\;.$$

Further, as already deduced,

$$\sum_{k=1}^{\infty}\frac{a_{2k-1}}{k}x^{2k} = \arcsin^2 x\;.$$

Hence,

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A warning of Euler

Ces deux méthodes toutes faciles qu'elles sont, mériteroient une plus grande attention, si elles se pouvoient employer également pour trouver les sommes des plus hautes puissances paires, qui sont toutes comprises dans mon autre méthode générale tirée de la considération des racines d'une équation infinie. Mais malgré toute la peine que je me suis donnée pour trouver seulement la somme des biquarrés [...] je n'ai pas encore pu réussir dans cette recherche, quoique la somme par l'autre méthode me soit connue [...]. Par faciliter la peine, que d'autres peut–être se donneront, dans cette affaire, j'y joindrai les sommes de toutes les puissances paires, que j'ai trouvées par l'autre méthode [...].

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6. Paul Levrie's approach

Both power series expansions for arcsin und $\arcsin^2$ are in fact *uniformly convergent* on the *closed* interval $\overline{I} = [0, 1]$ as one can see, e.g., by Raabe's criterion. This does not help very much in Euler's methods because of the multiplication with the unbounded function q. However, substituting $x = \sin\phi$ and replacing the coefficients a_{2k+1} by the corresponding Wallis integrals, one obtains the identity

$$(*) \qquad \phi = \sum_{k=0}^{\infty} \mathcal{I}_{2k+1} \sin^{2k+1}\phi \cos\phi$$

uniformly on $\overline{J} = \frac{\pi}{2}\overline{I} = [0, \pi/2]$.

It is Levrie's merit to having presented in loc.cit. a simple direct proof of this result *à la mode d'Euler*. We give another proof using the Wallis integrals in their "improper" guise that makes the role of the geometric series even more transparent.

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Levries proof

Integration of $(*)$ yields

$$(**) \qquad \phi^2 = \sum_{k=0}^{\infty} \frac{\mathcal{I}_{2k+1}}{k+1} \sin^{2k+2}\phi \, ,$$

and integrating once more and substituting $\phi = \pi/2$ gives again the desired result:

$$\frac{1}{3}\left(\frac{\pi}{2}\right)^3 = \sum_{k=0}^{\infty} \frac{\mathcal{I}_{2k+1}}{k+1} \mathcal{I}_{2k+2} = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{a_{2k+1}}{k+1} a_{2k+2} = 2\,\frac{\pi}{2}\,\frac{\zeta(2)}{4} \, .$$

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It remains to present a (new) proof - following Levries idea - of the identity $(*)$ in the form

$$\frac{\arcsin x}{\sqrt{1 - x^2}} = \sum_{k=0}^{\infty} \mathcal{I}_{2k+1}\, x^{2k+1} \,,$$

which we already know. For that, we start with the righthand side

$$\begin{aligned}
\sum_{k=0}^{\infty} \mathcal{I}_{2k+1}\, x^{2k+1} &= \sum_{k=0}^{\infty} \left(\int_0^1 \frac{t^{2k+1} \mathrm{d}t}{\sqrt{1 - t^2}} \right) x^{2k+1} \\
&= \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(x\, t)^{2k+1}}{\sqrt{1 - t^2}} \right) \mathrm{d}t \\
&= x \int_0^1 \frac{t\, \mathrm{d}t}{(1 - (tx)^2)\sqrt{1 - t^2}} \,.
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\end{aligned}$$

By substituting $\tau = \sqrt{1 - t^2}$, the last integral will become

$$\int_0^1 \frac{x\,\mathrm{d}\tau}{(1 - x^2) + x^2\tau^2} = \int_0^x \frac{\mathrm{d}\,T}{\sqrt{1 - x^2}^2 + T^2}\,,$$

and this one is, as everybody knows, equal to

$$\frac{1}{\sqrt{1 - x^2}}\ \arctan \frac{x}{\sqrt{1 - x^2}} = \frac{\arcsin\,x}{\sqrt{1 - x^2}}\,.$$

Remark

The same trick works also with the function q, thus yielding a fourth approach.

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The same trick works also with the function q, *thus yielding a fourth approach.*

Appendix: Interchanging integration and summation

A theorem of Beppo Levi type

Euler used implicitly the following

Lemma

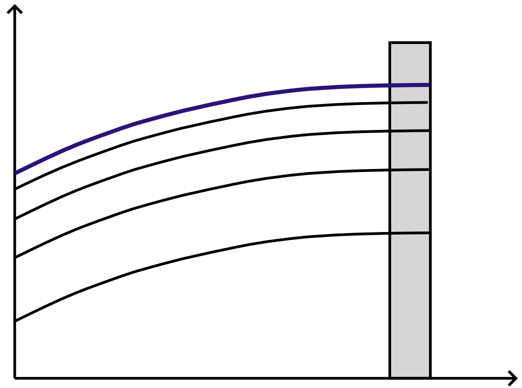
Let f_n, f be functions on the interval $\mathrm{I} = [0, 1[$ which are improper integrable such that the sequence $(f_n)_{n\in\mathbb{N}}$ converges monotone increasing to f uniformly on every closed subinterval $[0, a]$ of I. Then

$$\int_0^{1-} f(x)\,\mathrm{d}x = \int_0^{1-} \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_0^{1-} f_n(x)\,\mathrm{d}x\,.$$

Remark. This is a weak version of Beppo Levi's theorem of monotone convergent sequences of Lebesgue integrable functions.

Of course, application of *Lebesgue theory* is not what we mean by an *elementary* method.

The main idea of the *first proof* is contained in the following picture:



More precisely: We define $g_n := f - f_n$ such that $g_n \searrow 0$. Since $g_0 \geq 0$ is improper integrable, there is by the Cauchy criterion for each $\varepsilon > 0$ an $a \in \mathrm{I}$ such that for all $n \in \mathbb{N}$

$$0 \leq \int_a^{1-} g_n(x)\,\mathrm{d}x \leq \int_a^{1-} g_0(x)\,\mathrm{d}x \leq \frac{\varepsilon}{2}\ .$$

On the interval $[0, a]$ the sequence $(g_n)_{n\in\mathbb{N}}$ converges uniformly to 0. Therefore, there is an $N = N(\varepsilon)$ such that for all $n \geq N$

$$0 \leq \int_0^{a} g_n(x)\,\mathrm{d}x \leq \frac{\varepsilon}{2}\ .$$

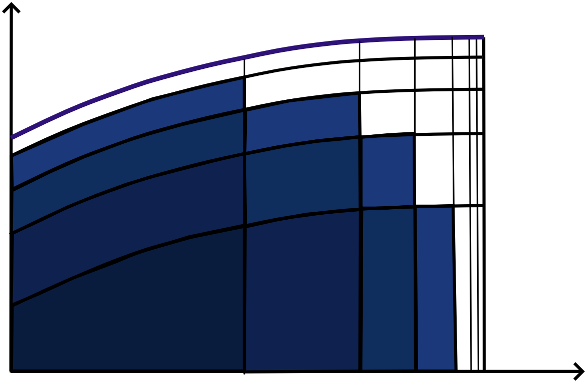
This implies

$$\lim_{n\to\infty} \int_0^{1-} g_n(x)\,\mathrm{d}x = 0 = \int_0^{1-} \lim_{n\to\infty} g_n(x)\,\mathrm{d}x\ . \qquad \square$$

The *second proof* has been communicated to me by Robin Chapman in the case

$$f = \sum_{n=0}^{\infty} f_n \, , \quad f_n \geq 0 \, .$$

It uses *Cauchy's Double Series Theorem*.



In exact symbols: Choose a sequence $0 = a_0 < a_1 < a_2 < \cdots < 1$ converging to 1 and define $\mathrm{I}_m := [\, a_m \,, a_{m+1} \,] \subset \mathrm{I}$ and

$$J_{mn} := \int_{\mathrm{I}_m} f_n(x)\,\mathrm{d}x \,.$$

Then

$$\sum_{m=0} \Big(\sum_{n=0} J_{mn} \Big) = \sum_{m=0} \int_{\mathrm{I}_m} f\,(x)\,\mathrm{d}x = \int_0^{1-} f\,(x)\,\mathrm{d}x$$

and

$$\sum_{n=0} \Big(\sum_{m=0} J_{mn} \Big) = \sum_{n=0} \int_0^{1-} f_n(x)\,\mathrm{d}x \,.$$

An excellent overview of Euler's work on $\zeta(2)$ including historical details can be found in :
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Anybody interested in other elementary calculations of $\zeta(2)$ should consult the article

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