Cyclic Quotient Surface Singularities: Constructing the Artin Component via the McKay–Quiver

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In memoriam Nobuo Sasakura

As a participant of the workshops in Kyoto and Tokyo in July 1997 where I gave lectures on deformation theory of rational surface singularities and the result in the present notes I was certainly looking forward to see Nobuo Sasakura again whom I met in Japan five years ago as my host and a little later as my guest in Hamburg. It was a great shock for me to learn just before my departure to Japan that there will be no discussions with him anymore.

There exists an extensive literature on the deformation theory of Kleinian singularities, i. e. quotient surface singularities of embedding dimension 3 alias *rational double points*. E. g. P. Kronheimer [K] gave an *invariant–theoretic* construction of the (monodromy trivializing finite covering of the) versal deformation and the simultaneous resolution of this family. A *quiver–theoretic* approach has been proposed by him and was carried out by others (Cassens [C]; see also Slodowy [S] and the literature cited there).

As one can easily check in simple examples this construction has no straigthforward generalization to quotient singularities $X = \mathbb{C}^2/\Gamma$, Γ a (small) finite subgroup of $\mathrm{GL}(2, \mathbb{C})$ (not in $\mathrm{SL}(2, \mathbb{C})$, i. e. the case of Kleinian singularities) and its associated McKay– or Auslander - Reiten–quiver. The purpose of the present note is to *restate* the quiver construction for the A_n–singularities in such a manner that it can be generalized to yield (up to a smooth factor) the monodromy covering of the Artin–component for all cyclic quotients. It is not clear at the moment (besides for some special cases) how to produce with this method the other components (not existing in the A_n–cases) or even the whole versal deformation (which also possesses a monodromy covering; see Riemenschneider [R2] and Brohme - Riemenschneider [BR]).

We should point out that our construction uses in a crucial manner the specific combinatorial properties of cyclic quotients and, hence, gives no hint how to find a conceptual way which could be followed in the general case of quotient surface singularities. However, it should be possible to provide our construction with an invariant–theoretic interpretation since the *special* representations used below have such a description by the work of Wunram [W] and the author (unpublished) which may lead to a general understanding of the deformation theory of quotient surface singularities completely in terms of the representations of the group Γ. Notice that e. g. the vector space of *infinitesimal* deformations of quotient surface singularities can be described in such a way by work of Pinkham [P]. There should also exist in the cyclic case a direct connection to the *toric* structure of these singularities.

1. *The* A_n*–case.*

We let the group $\mathbb{Z}_{n+1} = \mathbb{Z}/(n+1)\mathbb{Z}$, $n \geq 1$, act on $\mathbb{C}[u, v]$ by

$$(*) \qquad u \longmapsto \zeta_{n+1}\, u\,,\ v \longmapsto \zeta_{n+1}^{-1}\, v\,, \quad \zeta_{n+1} \text{ a primitive } (n+1)\text{st root of unity}\,,$$

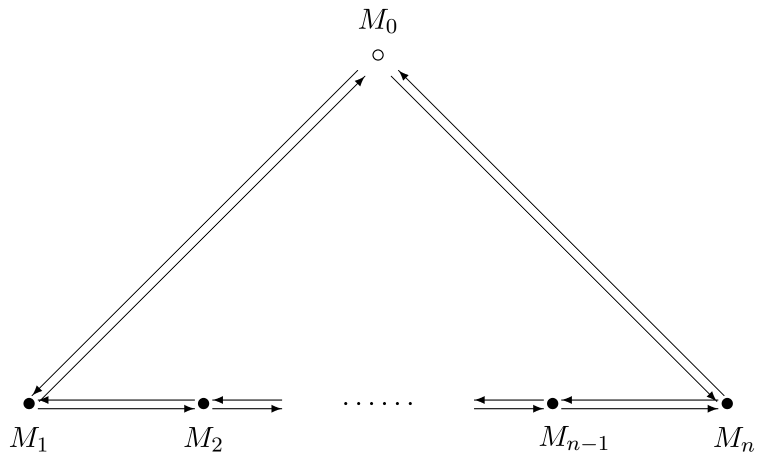
with (generating) invariants

$$x_0 \;=\; u^{n+1}\,,\; x_1 \;=\; uv\,,\; x_2 \;=\; v^{n+1}$$

and (generating) equation

$$x_0\,x_2 \;=\; (uv)^{n+1} \;=\; x_1^{n+1}\,.$$

The McKay–quiver is of the form



where the $n+1$ vertices are the one–dimensional representations

$$M_j : u \longmapsto \zeta_{n+1}^{j}\, u\,, \quad u \in \mathbb{C}\,, \quad j = 0, \ldots, n\,,$$

and tensoring M_j with the natural representation on $N = \mathbb{C}^2$ as in $(*)$ yields

$$M_j \otimes N \;=\; M_{j-1} \oplus M_{j+1}\,, \quad j \in \mathbb{Z}_{n+1}$$

which explains the two arrows from M_j to M_{j+1} and backwards. Interpreting $M_j \to M_{j+1}$ as a linear homomorphism we can identify it, after fixing a basis for each M_j, with a number u_j, and

$$U \;=\; \begin{pmatrix} 0 & 0 & \ldots & 0 & u_n \\ u_0 & 0 & \ldots & 0 & 0 \\ 0 & u_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & u_{n-1} & 0 \end{pmatrix}$$

describes the corresponding endomorphism

$$U \in \operatorname{End} M\,, \quad M \;=\; M_0 \oplus \cdots \oplus M_n\,.$$

Similarly, we have an endomorphism $V \in \operatorname{End} M$ with

$$V \;=\; \begin{pmatrix} 0 & v_1 & 0 & \ldots & 0 \\ 0 & 0 & v_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & v_n \\ v_0 & 0 & 0 & \ldots & 0 \end{pmatrix}\,.$$

The endomorphisms U^{n+1}, $U\,B$ and B^{n+1} have diagonal form:

$$U^{n+1} = u_0 \dots u_n \, E_{n+1} \ , \quad U\,V = \mathrm{diag}\,(v_0\,u_n,\, v_1\,u_0, \dots, v_n\,u_{n-1}) \ , \quad V^{n+1} = v_0 \dots v_n \, E_{n+1} \ .$$

Take the entries of these matrices, viz.

$$u_0 \dots u_n \ , \ v_0\,u_n \ , \ v_1\,u_0 \dots, v_n\,u_{n-1}, \, v_0 \dots u_n$$

as generators of a subalgebra of $\mathbb{C}\,[\,u_0, \dots, u_n,\, v_0, \dots, v_n\,]$. This, then, is presented by

$$\mathbb{C}\,[\,x_0,\, x_1^{(0)}, \dots, x_1^{(n)}\,,\, x_2\,]\,/\,(x_0\,x_2 = x_1^{(0)} \cdot \ldots \cdot x_1^{(n)})$$

with $x_0 = u_0 \dots u_n$, $x_2 = v_0 \dots v_n$, $x_1^{(j)} = v_j\,u_{j-1}$, $j \in \mathbb{Z}_{n+1}$. It is well–known that this algebra describes the total space of the (monodromy) covering of the versal deformation ([R1], [S]) of the given singularity. Notice that the corresponding deformation can be written in the form

$$[\,U,\, V\,] \, = \, U\,V \, - \, V\,U \, = \, \mathrm{diag}\,(\lambda_0, \dots, \lambda_n) \ , \quad \sum_{j=0}^{n} \lambda_j \, = \, 0 \ .$$

2. *Statement of the result in the* $\mathrm{A}_{n,q}$*–case.*

Take now $0 < q < n$ and $\gcd\,(n,\, q) = 1$. We have a natural action of $\mathbb{Z}_n$ on $\mathbb{C}\,[\,u,\, v\,]$ by

$$u \longmapsto \zeta_n\,u \ , \quad v \longmapsto \zeta_n^q\,v \ .$$

We call $\mathbb{C}^2/\mathbb{Z}_n$ the quotient singularity of type $\mathrm{A}_{n,q}$. Remark that A_n–type is the same as $\mathrm{A}_{n+1,n}$–type.

The McKay–quiver, with vertices $M_0, \dots, M_{n-1}$, has now arrows $u_j : M_j \to M_{j+1}$ and $v_j : M_j \to M_{j+q}$, $j = 0, \dots, n-1$, all indices taken modulo n, giving endomorphisms

$$U,\, V \in \mathrm{End}\, M \ , \quad M \, = \, M_0 \oplus \dots \oplus M_{n-1} \ .$$

It is well–known [R1] that the equations of an $\mathrm{A}_{n,q}$–singularity are determined by the pair $(n,\, q)$ and its continued fraction expansion

$$\frac{n}{n-q} \, = \, a_1 \, - \, 1 \,\rfloor\, a_2 \, - \dots - \, 1 \,\rfloor\, a_r \ , \quad a_\rho \geq 2 \ :$$

if we set

$$i_0 \, = \, n \, , \; i_1 \, = \, n - q \, , \; i_{\rho+1} \, = \, a_\rho\, i_\rho - i_{\rho-1} \ ; \quad j_0 \, = \, 0, \, , \; j_1 \, = \, 1 \, , \; j_{\rho+1} \, = \, a_\rho\, j_\rho - j_{\rho-1} \ ,$$

we get $r+2$ numbers

$$i_0 > i_1 > \dots > i_r = 1 > i_{r+1} \, = \, 0 \ , \quad j_0 \, = \, 0 < j_1 < \dots < j_{r+1} \, = \, n \ ,$$

and the monomials $x_\rho \, = \, u^{i_\rho}\, v^{j_\rho}$, $\rho \, = \, 0, \dots, r+1$, generate the invariant algebra. They satisfy the equations $x_0\,x_2 \, = \, x_1^{a_1}$, $x_1\,x_3 = x_2^{a_2}, \dots, x_{r-1}\,x_{r+1} = x_r^{a_r}$ plus some other equations which can compactly be written in *quasideterminantal* form, i. e. as the 2×2–*quasiminors* of the $2 \times (r+1)$–*quasimatrix*

$$\begin{pmatrix} x_0 & & x_1 & & x_2 & & \cdots & & x_r \\ & x_1^{a_1-2} & & x_2^{a_2-2} & & \cdots & & x_r^{a_r-2} & \\ x_1 & & x_2 & & x_3 & & \cdots & & x_{r+1} \end{pmatrix} .$$

Recall moreover that the exceptional divisor in the minimal resolution of such a singularity is a string of s smooth rational curves with selfintersection numbers $-b_\sigma$, $\sigma = 1, \ldots, s$, where

$$\frac{n}{q} = b_1 - \underline{1 \rfloor} \overline{b_2} - \cdots - \underline{1 \rfloor} \overline{b_s} \ , \quad b_\sigma \geq 2 \ .$$

As we mentioned at the beginning, our construction uses *special representations*. Recall that the famous McKay–*correspondence* yields for any binary polyhedral group $\Gamma \in \mathrm{SL}\,(2, \mathbb{C})$ a bijective mapping between the set of (nontrivial) irreducible complex representations of Γ and the set of irreducible components of the exceptional set in the minimal resolution of the singularity $\mathbb{C}^2/\Gamma$. For the other finite (small) subgroups $\Gamma \in \mathrm{GL}\,(2, \mathbb{C})$ this is no longer true; in fact there are in these cases always more nontrivial irreducible representations than curves. But by the work of Wunram [W] one can associate to each curve a well–defined nontrivial representation. We call these (together with the trivial one) *special* representations. They form a (proper) subset of the set of vertices in the McKay–quiver. For cyclic quotient singularities this set can be easily described [W]: The special representations are precisely the M_α with $\alpha = t_\sigma$, $\sigma = 0, \ldots, s$, where

$$t_0 := n \,,\ t_1 := q, \ldots, t_{\sigma+1} = b_\sigma\, t_\sigma - t_{\sigma-1}, \ldots, t_s = 1 \quad (t_{s+1} = 0) \ .$$

Here, of course, M_α belongs for $\alpha = t_\sigma$, $1 \leq \sigma \leq s$, to the curve with label σ. If one wants to have these numbers in their natural ordering, i. e. (modulo n)

$$\widetilde{t}_\sigma := t_{s+1-\sigma} \ , \quad \sigma = 0, \ldots, s \ ,$$

one may find them by reversing the ordering of the b_σ: writing $\beta_\sigma = b_{s+1-\sigma}$, $\sigma = 1, \ldots, s$, yields the continued fraction expansion

$$\beta_1 - \underline{1 \rfloor} \overline{\beta_2} - \cdots - \underline{1 \rfloor} \overline{\beta_s} = \frac{n}{\widetilde{q}} \ , \quad q\widetilde{q} \equiv 1 \bmod n \ ,$$

and the nonnegative integers $\widetilde{t}_\sigma$ are determined by

$$\widetilde{t}_0 = 0 \ , \quad \widetilde{t}_1 = 1 \ , \quad \widetilde{t}_{\sigma+1} = \beta_\sigma \widetilde{t}_\sigma - \widetilde{t}_{\sigma-1} \ .$$

We are now able to state our *Theorem*. It says in particular that in order to understand the deformation theory of cyclic quotient surface singularities one has to consider the underlying quiver seriously as the McKay–quiver, not just as an abstract one ! In particular the cyclic symmetry of the abstract quiver has to be broken in the general case.

Theorem *For given* (n, q), *take in* $\mathbb{C}[u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}]$ *the subalgebra generated by those elements of the diagonal matrices*

$$U^{i_\rho} V^{j_\rho}, \quad \rho = 0, \ldots, r+1,$$

which belong to special *representations. Then this algebra is canonically isomorphic to the algebra of the total space of the (monodromy covering) of the* ARTIN *component of the* $\mathrm{A}_{n,q}$*–singularity up to a smooth factor.*

Remark. Since in the A_n–case all representations are special this statement includes the former one.

3. *More explicit statement of the result.*

In order to prepare the grounds for proving the theorem we derive now an explicit formulation. We first have to compute the entries of the matrices $U^{i_\rho} V^{j_\rho}$, $\rho = 0, \ldots, r+1$. These are diagonal matrices of type

$$\mathrm{diag}\left(v_0 \cdot v_q \cdot \ldots \cdot v_{(j_\rho-1)q} \cdot u_{(j_\rho-1)q+1} \cdot u_{(j_\rho-1)q+2} \cdot \ldots \cdot u_{(j_\rho-1)q+i_\rho}, \ldots\right),$$

where the next entries are obtained from the first one by permutating the indices cyclically modulo n.

The theorem advices us to choose only the entries belonging to the indices $\widetilde{t}_\sigma$, $\sigma = 0, \ldots, s$. Since these monomials are completely determined by the v-factors we suppress the u's and write for them

$$v_{\tilde{t}_\sigma} v_{\tilde{t}_\sigma+q} \cdot \ldots \cdot v_{\tilde{t}_\sigma+(j_\rho-1)q}, \quad \rho = 0, \ldots, r+1, \quad \sigma = 0, \ldots, s.$$

It now seems reasonable to make the coordinate change

$$v_0 \longmapsto v_0, \quad v_1 \longmapsto v_q, \quad v_2 \longmapsto v_{2q}, \ldots, \text{ i.e. } v_j \longmapsto v_{jq}$$

(all indices taken modulo n). - We have the following

Lemma *If* $k_0 = 0$, $k_1 = 1, \ldots, k_{\sigma+1} = b_\sigma k_\sigma - k_{\sigma-1}$, $\sigma = 1, \ldots, s$, $(k_{s+1} = n)$, *then*

$$\widetilde{t}_{(s+1)-\sigma} = qk_\sigma \bmod n, \quad \sigma = 0, \ldots, s+1.$$

Proof. For $\sigma = 0$, we have $\widetilde{t}_{s+1} = n$ and $k_0 = 0$, for $\sigma = s-1$ it follows that $\widetilde{t}_1 = t_1 = q$, $k_1 = 1$. The rest follows by induction:

$$\begin{aligned} qk_{\sigma+1} - \widetilde{t}_{(s+1)-(\sigma+1)} &= q(b_\sigma k_\sigma - k_{\sigma-1}) - (\beta_{(s+1)-\sigma}\widetilde{t}_{(s+1)-\sigma} - \widetilde{t}_{(s+1)-(\sigma-1)}) \\ &= b_\sigma (qk_\sigma - \widetilde{t}_{(s+1)-\sigma}) - (qk_{\sigma-1} - \widetilde{t}_{(s+1)-(\sigma-1)}) \\ &\equiv 0 \bmod n. \end{aligned}$$

Combining these results we get the following description of the algebra under discussion as the subalgebra $\widetilde{A}_{n,q}$ of the polynomial ring $\mathbb{C}[v_0, \ldots, v_{n-1}]$ generated by the elements

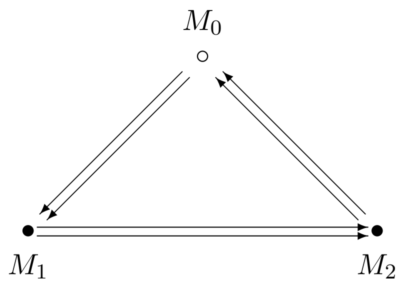
$$x_\rho^{(\sigma)} = v_{k_\sigma} v_{k_\sigma+1} \cdot \ldots \cdot v_{k_\sigma+(j_\rho-1)}, \quad \rho = 0, \ldots, r+1, \quad \sigma = 0, \ldots, s.$$

Notice that $x_0^{(\sigma)} = 1$ and $x_{r+1}^{(\sigma)} = v_0 v_1 \cdot \ldots \cdot v_{n-1}$ for *all* σ. – Hence, the statement of the theorem is equivalent to the following claim:

The algebra $\widetilde{A}_{n,q}$ *is, up to a smooth factor, the coordinate ring of the total space of the monodromy covering of the* ARTIN *component of the singularity* $X_{n,q}$.

4. *Some examples.*

Before sketching a general proof we discuss some examples. We start with the simplest non A_n–case, viz. the case $(n, q) = (3, 1)$. The MCKAY–quiver is of the form



We have to take into consideration the endomorphisms

$$U^3,\, U^2V,\, UV^2,\, V^3 .$$

The special representations are M_0 and M_1 (there is just one component in the exceptional set of the minimal resolution of the $\mathrm{A}_{3,1}$–singularity). Thus, we have to regard the invariants

$$u_0u_1u_2\,,\ v_0u_1u_2\,,\ u_0v_1u_2\,,\ v_0v_1u_2\,,\ u_0v_1v_2\,,\ v_0v_1v_2$$

which satisfy the determinantal relations

$$\begin{pmatrix} u_0u_1u_2 & u_0v_1u_2 & u_0v_1v_2 \\ v_0u_1u_2 & v_0v_1u_2 & v_0v_1v_2 \end{pmatrix}$$

and no others.

Remark. Considering *all* invariants together with the obvious 4–parameter family given by

$$[\,U^2,\, V\,] = \mathrm{diag}\,(\lambda_1,\, \lambda_2,\, \lambda_3)\,, \quad [\,U,\, V^2\,] = \mathrm{diag}\,(\lambda_4,\, \lambda_5,\, \lambda_6)$$

with $\sum_{j=1}^{3} \lambda_j = 0$, $\sum_{j=4}^{6} \lambda_j = 0$ yields the correct singularity over the origin, but no *flat* deformation of it !

This example can immediately be generalized to the *cone* $\mathrm{A}_{n,1}$ with endomorphisms $U^n,\, U^{n-1}V,\, U^{n-2}V^2, \ldots, V^n$ and special modules $M_0,\, M_1$ yielding the correct invariants and equations

$$\begin{pmatrix} u_0u_1\cdots u_{n-1} & u_0v_1u_2\cdots u_{n-1} & u_0v_1v_2\cdots u_{n-1} & \cdots & u_0v_1\cdots v_{n-1} \\ v_0u_1\cdots u_{n-1} & v_0v_1u_2\cdots u_{n-1} & v_0v_1v_2\cdots u_{n-1} & \cdots & v_0v_1\cdots v_{n-1} \end{pmatrix} .$$

We discuss another somewhat more complicated example, viz. the case $(n,\, q) = (46,\, 27)$. Here we have

$$\frac{n}{q} = 2 - \underline{1 \rceil 4} - \underline{1 \rceil 2} - \underline{1 \rceil 3} - \underline{1 \rceil 2}\ ,$$

hence $s = 5$, and

$$\frac{n}{n - q} = \frac{46}{19} = 3 - \underline{1 \rceil 2} - \underline{1 \rceil 4} - \underline{1 \rceil 3}$$

such that $e = r + 2 = 6$. The sequences (i_ρ) and (j_ρ) are given by 46, 19, 11, 3, 1, 0 and 0, 1, 3, 5, 17, 46, resp.; the sequence (t_σ) by 46, 27, 8, 5, 2, 1, 0. Therefore, we have to consider the following monomials (where we delete, as mentioned above, the variables u_j):

$$\begin{array}{llll}
x_0^{(0)} = 1 & & & \\
x_1^{(0)} = v_0 & x_2^{(0)} = v_0 v_1 v_2 & x_3^{(0)} = v_0 \cdot \ldots \cdot v_4 & x_4^{(0)} = v_0 \cdot \ldots \cdot v_{16} \\
x_1^{(1)} = v_1 & x_2^{(1)} = v_1 v_2 v_3 & x_3^{(1)} = v_1 \cdot \ldots \cdot v_5 & x_4^{(1)} = v_1 \cdot \ldots \cdot v_{17} \\
x_1^{(2)} = v_2 & x_2^{(2)} = v_2 v_3 v_4 & x_3^{(2)} = v_2 \cdot \ldots \cdot v_6 & x_4^{(2)} = v_2 \cdot \ldots \cdot v_{18} \\
x_1^{(3)} = v_7 & x_2^{(3)} = v_7 v_8 v_9 & x_3^{(3)} = v_7 \cdot \ldots \cdot v_{11} & x_4^{(3)} = v_7 \cdot \ldots \cdot v_{23} \\
x_1^{(4)} = v_{12} & x_2^{(4)} = v_{12} v_{13} v_{14} & x_3^{(4)} = v_{12} \cdot \ldots \cdot v_{16} & x_4^{(4)} = v_{12} \cdot \ldots \cdot v_{28} \\
x_1^{(5)} = v_{29} & x_2^{(5)} = v_{29} v_{30} v_{31} & x_3^{(5)} = v_{29} \cdot \ldots \cdot v_{33} & x_4^{(5)} = v_{29} \cdot \ldots \cdot v_{45} \\
 & & & x_5^{(0)} = v_0 v_1 \cdot \ldots \cdot v_{45}
\end{array}$$

A computation by hand or via [GPS] yields the following equations in *quasideterminantal* form, leaving the remaining variables untouched:

$$\begin{pmatrix}
x_0^{(0)} & & x_1^{(2)} & x_2^{(2)} & & x_3^{(4)} & & x_4^{(5)} \\
 & x_1^{(1)} & & & x_3^{(2)} x_3^{(3)} & & x_4^{(4)} & \\
x_1^{(0)} & & x_2^{(0)} & x_3^{(0)} & & x_4^{(0)} & & x_5^{(0)}
\end{pmatrix} .$$

5. *Sketch of proof*.

The main point is to show that the algebra $\widetilde{A}_{n,q}$ is - up to a smooth factor - the *generic* deformation of the given singularity in its canonical *quasideterminantal format* ([R1], [Ro]). The last example gives the right idea. Recall [R1] that

$$e - 3 = r - 1 = \sum_{\sigma=1}^{s} (b_\sigma - 2)$$

and, dually,

$$s - 1 = \sum_{\ell=1}^{r} (a_\ell - 2) .$$

Define now

$$s_\rho = \sum_{\ell=1}^{\rho} (a_\ell - 2) + 1 , \quad \rho = 1, \ldots, r ,$$

such that $s_1 = a_1 - 1 \leq s_2 \leq \cdots \leq s_r = s$, and select the following variables out of all $x_\rho^{(\sigma)}$:

$$\begin{array}{lll} x_\rho^{(0)}\,, & & \rho = 0, \ldots, r+1\,, \\ x_1^{(\sigma)}\,, & \sigma = 1, \ldots, s_1\,, & \\ x_\rho^{(\sigma)}\,, & \sigma = s_{\rho-1}, \ldots, s_\rho\,, & \rho = 2, \ldots, r\,. \end{array}$$

Notice that they form a system of $e + \sum_{\rho=1}^{r}(a_\rho - 1)$ variables. Obviously, it is sufficient to show that the *complete* set $x_\rho^{(\sigma)}$ of variables satisfies the following (generating) *quasideterminantal* relations:

$$\begin{pmatrix} x_0^{(0)} & & x_1^{(s_1)} & & x_2^{(s_2)} & \cdots & & x_r^{(s_r)} \\ & x_1^{(1)} \cdot \ldots \cdot x_1^{(s_1-1)} & & x_2^{(s_1)} \cdot \ldots \cdot x_2^{(s_2-1)} & & \cdots & x_r^{(s_{r-1})} \cdot \ldots \cdot x_r^{(s_r-1)} & \\ x_1^{(0)} & & x_2^{(0)} & & x_3^{(0)} & \cdots & & x_{r+1}^{(0)} \end{pmatrix}$$

which is just the generic deformation of the quasideterminantal format in question.

In order to do this, one can use induction with respect to the number r, the case $r = 1$ being trivial. Therefore, we start with a fixed number $r \geq 1$ and a continued fraction expansion

$$\frac{n'}{n' - q'} = a_1 - \underline{1 \,|\, a_2} - \cdots - \underline{1 \,|\, a_{r+1}}\,,$$

where

$$\frac{n}{n - q} := a_1 - \underline{1 \,|\, a_2} - \cdots - \underline{1 \,|\, a_r}\,.$$

Similarly, we provide the indices referring to the pair (n', q') by an accent; in particular, we have $r' = r + 1$, $a'_\rho = a_\rho$, $\rho = 1, \ldots, r$, $a'_{r'} = a_{r+1}$ and $j'_\rho = j_\rho$, $\rho = 1, \ldots, r+1$, $j'_{r'} = n$ and $j'_{r'+1} = n'$. Moreover, invoking the „dot“–diagram in [R1], it follows that

$$\begin{array}{l} s' = s + (a_{r+1} - 2)\,, \\ b'_\sigma = b_\sigma\,,\ \sigma = 1, \ldots, s-1\,, \quad b'_s = b_s + 1\,, \quad b'_\sigma = 2\,,\ \sigma = s+1, \ldots, s'\,. \end{array}$$

These identities imply

$$k'_\sigma = k_\sigma\,,\ \sigma = 1, \ldots, s\,, \quad k'_{s+\sigma} = k_s + \sigma n\,,\ \sigma = 1, \ldots, s' - s\,.$$

and the following

Lemma $$k_s = n - j_r\,.$$

Proof. We have by induction $k'_{s'} = (a_{r+1} - 2)n + k_s = (a_{r+1} - 1)n - j_r = a_{r+1}n - j_r - n = a_{r+1}j_{r+1} - j_r - j_{r+1} = n' - j'_{r'}$.

Moreover, we get by the same reasoning

$$x_r^{(s_r)} = v_{n-j_r} \cdot \ldots \cdot v_{n-1}\,, \quad x_{r+1}^{(s_r)} \cdot \ldots \cdot x_{r+1}^{(s_{r+1})} = v_{n-j_r} \cdot \ldots \cdot v_{n'-1}$$

and hence the relation

$$x_r^{(s_r)} x_{r+2}^{(0)} = x_{r+1}^{(0)} \cdot x_{r+1}^{(s_r)} \cdot \ldots \cdot x_{r+1}^{(s_{r+1})}\,,$$

since $x_{r+1}^{(0)} = v_0 \cdot \ldots \cdot v_{n-1}$ etc. This shows that the variables chosen satisfy the quasideterminantal equations as above.

The only problem left is to show that there are no other relations and moreover only trivial ones with respect to the remaining variables. We leave this as an exercise to the reader.

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