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Special surface singularities

A survey on the geometry and combinatorics of their deformations

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Introduction

The past few years have witnessed a significant progress in understanding deformations of normal surface singularities. In general, the base space of the versal deformation of such a singularity (simply called its *versal base* in the sequel) is highly singular, having many components (including nonreduced and even embedded ones). However, due to the work of KOLLÁR and SHEPHERD–BARRON [KSB] we know that for quotient surface singularities each (reduced) component is smooth after normalization. The set of components is in one to one correspondence to a set of certain partial resolutions of the given singularity. Moreover, the families over these (normalized) components are equipped with monodromy which arises from certain other partial resolutions, called M–resolutions by K. BEHNKE and J. CHRISTOPHERSEN [BC3]; the monodromy groups may be realized as Galois groups of distinguished coverings, acting as reflection groups. The classical example of course is provided by the ARTIN component (which is always smooth), the Galois covering mentioned above being the base change needed to resolve the family simultaneously [Art2,Lip,W3].

T. DE JONG and D. VAN STRATEN developed a new method to calculate the versal base space by projecting the surface singularity generically to a (weakly normal) hypersurface singularity in $\mathbf{C}^3$ and comparing the deformation theory of the singularity and its projection [JS1,JS3]. They applied their own method to the case of rational quadruple points [JS2].

Since the work of Behnke and Christophersen is easily accessible and already described in detail in a survey paper [BR], I give in these notes only a short summary and refer the reader to the literature. However, I would like to add some new remarks on the case of cyclic quotient singularities (CQS). Here it is known [St1,Chr] that the reduced components are smooth without normalization. Therefore one might hope to patch all Galois coverings together to form one special covering of the (reduced) base space with a group acting on this covering and inducing on the various components the monodromy coverings of the components of the versal base. This is indeed the case; a somewhat fuller announcement will be given in section 5.

Also, the papers of de Jong and van Straten are published and nicely surveyed e. g. in the article of G.–M. GREUEL [Gr] in the *Festband* edited at the occasion of the centennial birthday of the DMV (German Mathematical Society). Therefore, I only include some remarks about the general method and describe an application due to S. BROHME concerning minimally elliptic singularities of embedding dimension 6 [Bro].

At the symposion arranged by Professor Shihoko Ishii at RIMS in March 1992 I gave in my talk under the title *Deformations of rational singularities and quivers* a survey on the work of Behnke and Christophersen; I also reported on work of H. CASSENS about a quiver construction concerning the (resolved) versal deformation of rational double points which might be useful to construct the families over the Galois coverings in all quotient cases [Cas]. Since the work of H. Cassens, generalizing results of P. KRONHEIMER [K], is still in progress, I present here only a short outline in a special case in order to give him priority for publishing his results himself first in full strength. I also include a short description of the thesis of A. RÖHR [Roe] which has only be mentioned in my talk at RIMS but was considered more thoroughly at other occasions during my stay in Japan in spring of 1992.

In a sense, these notes reflect the content of all my lectures given during that period. It is therefore my great pleasure to thank all colleagues in Japan who made my stay possible and so pleasant. In particular, I am very much indebted to Professors SHIHOKO ISHII and NOBUO SASAKURA, and also to KIMIO WATANABE, KEI–ICHI WATANABE and HIDEAKI KAZAMA (the last two especially in commemorating their kind hospitality and unforgettable musical events).

1 The general context

In these notes we mainly deal with normal surface singularities (X, x_0), mostly written X for short. For such a singularity, $\pi : \widetilde{X} \to X$ always denotes a *normal modification* of the singularity with *exceptional set* $E = \cup E_i$ (most of the time assumed to be the *(minimal) resolution*). It is well known that the number

$$p_g(X, x_0) = \dim_{\mathbb{C}} (R^1\pi_*\mathcal{O}_{\widetilde{X}})_{x_0}$$

is independent of the choice of a resolution; it is called the *geometric genus* of the singularity X.

We are mainly concerned with the case of *rational singularities* which are characterized by the property $p_g(X) = 0$ [Art1]. This class of singularities includes as a proper subclass the *quotient singularities* $X = \mathbb{C}^2/G$, G a finite subgroup of $\mathrm{GL}(2, \mathbb{C})$, and in particular the *rational double points* (RDP) or *Klein singularities* for which G is a subgroup of $\mathrm{SL}(2, \mathbb{C})$. If the group G is *cyclic*, we speak of *cyclic quotient surface singularities* which up to isomorphism are of type $A_{n,q}$ or shortly of type (n, q) with relatively prime integers $0 < q < n$: these are by definition the quotients of the form $\mathbb{C}^2/C_{n,q}$ where the cyclic group $C_{n,q}$ is generated by the matrix

$$\begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^q \end{pmatrix}, \quad \zeta_n = \exp(2\pi i/n).$$

Notice that the intersection of the classes of cyclic quotients and RDP's consists of the A_n–singularities with equation

$$z^2 + x^2 + y^{n+1} = 0.$$

We are also considering *minimally elliptic singularities* which are defined by $p_g = 1$ and the assumption to be *Gorenstein* [L2]. This class contains e. g. the *Hilbert modular cusps* and *simple elliptic singularities* (which have as minimal resolution the total space of a negative line bundle over an elliptic curve).

2 The Artin component for rational singularities

If X is any normal surface singularity, one would like to understand all possible *deformations* of X. The complete information is contained in the *versal deformation* of X [G] denoted by $\mathcal{X} \to S = \mathrm{Def}(X)$ in the sequel. One of the standard questions is that about *adjacencies*, i. e.: *which singularities occur in nearby fibres*? Here is a sample of some general facts:

- *rational singularities deform to rational singularities* [El];
- *quotient surface singularities deform to quotient singularities* [EV,I];
- *cyclic quotients deform to cyclic quotients* [KSB].

However, it goes without saying that in specific examples one would like to have a much more precise classification of the adjacencies.

Also, the minimal resolution $\tilde{X}$ admits a versal deformation (relatively to the exceptional set E), see e. g. [BK]:

$$\tilde{\mathcal{X}} \longrightarrow T = \mathrm{Def}(\tilde{X})$$

with a smooth base space T. Just by definition of rationality [Rie1], this deformation can for a rational singularity be blown down simultaneously to a deformation

$$\mathcal{X}_T \longrightarrow T$$

of X itself, and by definition of versality, there is a Cartesian diagram

$$\begin{array}{ccc} \mathcal{X}_T & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ T & \longrightarrow & S. \end{array}$$

It is interesting to study the mapping

$$\mathrm{Def}(\tilde{X}) = T \longrightarrow S = \mathrm{Def}(X).$$

If X is an RDP, that is a singularity of type ADE, it was already known due to work of TYURINA, BRIESKORN, GROTHENDIECK and SLODOWY [Tju,Br3,S] which relates deformations of singularities of type ADE to the theory of simple complex Lie groups and their Weyl groups of corresponding type that the base space $\mathrm{Def}(X)$ is smooth and the mapping $T \to S$ is a Galois covering with Galois group the corresponding Weyl group of type ADE acting as a reflection group on T.

In general, the following holds true:

Theorem 2.1 (Artin, Lipman, Wahl) *Let X be a rational surface singularity. Then* $\mathrm{Def}(\tilde{X})$ *maps finitely to one to a* component *of* $\mathrm{Def}(X)$, *called the* Artin component $(\mathrm{Def}(X))_{\mathrm{art}}$. *More precisely:*

$$(\mathrm{Def}(X))_{\mathrm{art}} \cong \mathrm{Def}(\tilde{X}) / \prod W_j ,$$

where the W_j are finite Weyl groups belonging to the maximal connected (-2)-configurations, i. e. the rational double point configurations supported by the exceptional set of $\tilde{X}$.

Clearly, this implies in particular that $(\mathrm{Def}(X))_{\mathrm{art}}$ is always smooth. For RDP's, this component coincides with S. The same is true for rational singularities of multiplicity 3 (*triple ponts*) due to a general result of SCHAPS and HILBERT–BURCH on Cohen–Macaulay singularities of codimension 2. The first interesting example for higher codimension was given by PINKHAM [P]: If X denotes the cone over the rational normal curve of degree 4 given by the equations

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} < 2\,,$$

the versal base space has two components, namely the Artin component

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1+s_1 & x_2+s_2 & x_3+s_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} < 2$$

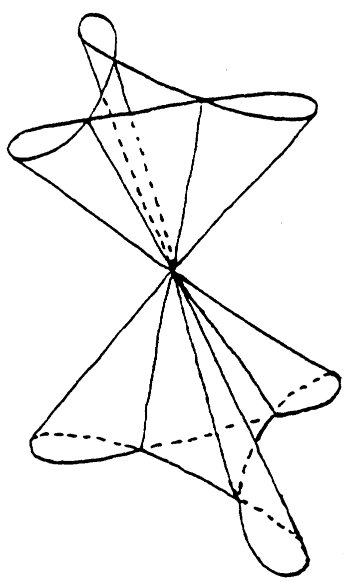
and another component described by

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2+s & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} < 2\,.$$

In [Riel] it has been proven that for a general cyclic quotient singularity of embedding dimension 5 the versal base space is a product of this space with a smooth factor. Pinkham also determined the versal base spaces for the cones of higher multiplicity and found the surprising fact that they consist of the Artin component together with an embedded component concentrated at the distinguished point. For other cyclic quotients, cf. section 5.

3 The method of de Jong - van Straten

This method provides some tools to determine $\mathrm{Def}(X)$ at least up to smooth factors. The authors project a normal surface singularity $X \subset \mathbb{C}^e$ generically to a hypersurface $X' \subset \mathbb{C}^3$ and study the interplay of deformations of X and X'. As an *example* we reproduce the generic projection of the affine cone over the rational normal curve of degree 4 from [Gr].



For the pleasant fact that X' is a hypersurface one has to pay a price: X' is in general not normal anymore (in the reduced structure). More precisely, the singular locus Σ (always assumed to be equipped with the reduced structure) is one–dimensional unless X' is normal itself, i. e. when $X = X'$ was already a hypersurface in $\mathbf{C}^3$. The authors make heavily use of the fact that the pair (X', Σ) is nevertheless *weakly normal*, and therefore generically of transverse A_1–type. In particular, X is isomorphic to the normalization of X'. *Admissible deformations* of the pair (X', Σ) are of course those of X' which induce a flat deformation of the relative singular locus over the base space. The authors prove the existence of a semiuniversal admissible deformation in this context and study the forgetful functors

$$\underline{\mathrm{Def}}_{(X \to X')} \longrightarrow \underline{\mathrm{Def}}_X \quad \text{and} \quad \underline{\mathrm{Def}}_{(X \to X')} \longrightarrow \underline{\mathrm{Def}}_{(X', \Sigma)} ,$$

for which they show that the second one is an isomorphism and the first one is at least smooth. Consequently, the base spaces of the versal deformations of X and (X', Σ) are equal up to smooth factors which we denote by the symbol $\sim$.

Theorem 3.1 (de Jong - van Straten)

1. *There exists a versal admissible deformation of* (X', Σ);
2. $\mathrm{Def}\,(X) \sim \mathrm{Def}\,(X', \Sigma)$.

For applications, the following result is very helpful:

Theorem 3.2 *Let* (X_1', Σ_1) *and* (X_2', Σ_2) *be weakly normal hypersurface singularities in* $\mathbf{C}^3$ *defined by the functions* f_1 *and* $f_2 \in \mathcal{O}_{\mathbf{C}^3,0}$. *Then*

$$\mathrm{Def}\,(X_1', \Sigma_1) \sim \mathrm{Def}\,(X_2', \Sigma_2)$$

if one of the following holds:

a) *the normalizations* X_1 *and* X_2 *are isomorphic* ; *or*

b) $\Sigma_1 = \Sigma_2$ *and* $f_1 - f_2$ *is contained in the square of the ideal defining* Σ_1.

With these principles, de Jong and van Straten are able to handle rational quadruple points by reduction to certain taut singularities depending only on one extra parameter n and finally to so–called n–stars. For the given quadruple point X the integer n is just the number of blowings up needed for dropping the multiplicity of X (and is also the number of blowings up needed for resolving the n–star). The final result is that the versal base space depends only on n up to smooth factors. Call this *universal* base $B(n)$. They show that it is of embedding dimension $5n - 1$; it can be described as follows: let

$$a_0(t)\,,\ a_1(t)\,,\ a_2(t)\,,\ a_3(t) \quad \text{be polynomials of degree} \quad n-1$$

and

$$b(t) \quad \text{be a polynomial of degree} \quad n-2\,.$$

They can be considered to form a manifold parametrized by the $5n - 1$ coefficients. Then

$$B(n) = \{t^n + b(t) \quad \text{divides} \quad a_0(t)\,a_i(t) \quad \text{for} \quad i = 1, 2, 3\}\,.$$

Moreover, they prove that this base space

- is reduced,
- has $n+1$ components Y_j, $j = 0, \ldots, n$, of dimension $\dim Y_j = 2n - 1 + 2j$ and multiplicity
$$\operatorname{mult} Y_j = \binom{n}{j},$$
respectively,
- Y_0 and Y_n are smooth, all other Y_j have smooth normalization.

Clearly, for $n = 1$, this result reproduces Pinkham's example in a more general context.

In order to apply this method effectively it is useful to know the general hyperplane sections of singularities. This was provided in work of Behnke and Christophersen for rational singularities for the sake of computing the space T^2 of obstructions [BC1]. In a subsequent paper, they achieved the same task for minimally elliptic singularities [BC2].

Stephan Brohme applied their result to minimally elliptic singularities (with reduced tangent cone) of multiplicity (= embedding dimension) 6. In fact, for multiplicity $m \leq 4$, the base space is smooth by the result of Hilbert–Burch–Schaps, and for $m = 5$ we have the well known structure theorem saying that Gorenstein singularities of codimension 3 are described by Pfaffians of some symmetric matrix which also determines the deformations such that the base space is again smooth. Brohme proves the following

Theorem 3.3 *The versal base space of a minimally elliptic singularity of multiplicity 6 (with reduced tangent cone) is (up to smooth factors) isomorphic to the affine cone over the Segre embedding* $\mathbf{P}_1 \times \mathbf{P}_2 \hookrightarrow \mathbf{P}_5$.

In order to achieve this result he has to determine the singular locus (a double curve) of the projection of a minimally elliptic singularity to $\mathbb{C}^3$ in terms of *elliptic partition curves.* The main result then states that - if I denotes the ideal of this double curve - the generic projection of such a singularity can be deformed modulo I^2 to a projection of a simple elliptic singularity. Clearly, by the theory of de Jong and van Straten, this reduces the problem to the determination of the versal base space of such a simple elliptic singularity, where the result is known [P]. For the last result, he applies classical results of Segre on pencils amd linear systems of plane projective curves to a certain pencil of plane sextics with nine double points.

With a completely different method using Banach analytic considerations for nonisolated singularities, J. Stevens proved this result without restriction on the tangent cone [St4]. He found more striking relations between the deformation theory of Cohen–Macaulay singularities and that of Gorenstein singularities of one dimension or codimension higher.

4 The components for quotient surface singularities

As an application of a special case of three–dimensional Mori theory, Kollár and Shepherd–Barron were able to explain where the components of the reduced versal base space S_{red} come from in the case of quotient surface singularities.

Theorem 4.1 *Let* X *be a quotient surface singularity and* $\mathcal{X} \to T$ *a one-parameter smoothing. Then there exists a modification* $\mathcal{Y} \xrightarrow{f} \mathcal{X}$ *with the following properties:*

i) *$\mathcal{Y}$ has only terminal singularities* ;

ii) *for each curve C which is contracted by f we have $K_{\mathcal{Y}} \cdot C > 0$, i. e. $K_{\mathcal{Y}}$ is f-ample* ;

iii) *$Y := \mathcal{Y}_0$ has only quotient surface singularities.*

The singularities of Y can be determined: They have special deformations, namely so called $\mathbb{Q}$–Gorenstein smoothings.

Definition. Let (X, x_0) be a reduced surface singularity such that $X \setminus \{x_0\}$ is Gorenstein. A one–parameter smoothing $\mathcal{X} \to T$ is called *$\mathbb{Q}$–Gorenstein* or qG, if some multiple of the canonical class of $\mathcal{X}$ is a Cartier divisor. A smoothing component of the versal deformation of X is called a *qG-component*, if every smoothing in it is qG.

This allows the authors to identify the singularities of Y to being either *rational double points* (RDP) or of *type* T. The last ones are cyclic quotients with very special resolution graphs. They can be described more precisely as follows: Let $\mathbb{Z}_r$ act on $\mathbb{C}^3$ by the diagonal matrix

$$\begin{pmatrix} \zeta_r & 0 & 0 \\ 0 & \zeta_r^{-1} & 0 \\ 0 & 0 & \zeta_r^d \end{pmatrix}, \quad 0 < d < r, \quad \gcd(r, d) = 1 .$$

Then $\mathbb{C}^3/\mathbb{Z}_r$, for which we also write $\mathbb{C}^3/\mathbb{Z}_r(1, -1, d)$ for the sake of resolving the ambiguity in the defintion, is a three–dimensional quotient singularity. Let A_{rs-1} be given in $\mathbb{C}^3$ by $xy = z^{rs}$. Then $\mathbb{Z}_r$ acts on A_{rs-1}, too, and the non RDP type T–singularities are precisely the quotients

$$A_{rs-1}/\mathbb{Z}_r(1, -1, d) \cong \text{cyclic quotient of type } (r^2 s,\ drs - 1) =: A_{(r,s,d)} .$$

Definition. A *P-resolution* (partial resolution) is a modification $\pi : \overline{X} \to X$ such that

a) $K_{\overline{X}}$ is relatively ample with respect to π,

b) $\overline{X}$ has only singularities of type T.

Theorem 4.2 (Kollár, Shepherd - Barron) *For quotient surface singularities there is a $1 : 1$-correspondence between the set of P–resolutions and the set of components of S_{red}.*

The construction works as follows: Take first a component of the base space. Since X is rational, it is automatically a smoothing component. Then choose a generic curve T in the component, restrict the deformation to T: $\mathcal{X} \to T$, and call upon $\mathcal{Y} \to \mathcal{X}$ as in the first theorem. Then $\overline{X} = \mathcal{Y}_0$ is the corresponding P–resolution.

Take on the other hand a P–resolution $\overline{X}$ with singular locus $\operatorname{sing} \overline{X} = \{\overline{x}_i\}$. Then there are forgetful (smooth) functors

$$\begin{tikzcd} \underline{\mathrm{Def}}_{\overline{X}} \arrow[r] & \prod \underline{\mathrm{Def}}_{(\overline{X}, \overline{x}_i)} \\ \underline{\mathrm{Def}}^0_{\overline{X}} \arrow[r] \arrow[u] & \prod \underline{\mathrm{Def}}^0_{(\overline{X}, \overline{x}_i)} \arrow[u] \end{tikzcd}$$

where the superscript 0 on the symbol at the southeast corner denotes deformations of $(\overline{X}, \overline{x}_i)$ as a subspace of $\mathbf{C}^3/\mathbf{Z}_{r_i}$. This defines the functor $\underline{\mathrm{Def}}^0_{\overline{X}}$ on the southwest corner, and by a), deformations representing this functor can be blown down to deformations of X. The authors then prove that $\mathrm{Def}^0(\overline{X})$ is unobstructed and that the corresponding morphism of base spaces

$$\mathrm{Def}^0(\overline{X}) \longrightarrow \mathrm{Def}(X)$$

maps the smooth space $\mathrm{Def}^0(\overline{X})$ generically $1:1$ to a component of $\mathrm{Def}(X)$. □

Consequently all components have smooth normalization. (For cyclic quotient singularities, the components are even smooth).

In the general rational case, one has to be more careful due to examples by Jan Stevens and others for some rational quintupel points.

Definition. Let (X, x_0) be a rational surface singularity. A modification $Y \to X$ is called a *P-modification* or a *P-resolution*, if every singularity (Y, y_i) of Y has a qG–component and $K_Y \cdot C > 0$ for all exceptional curves C in Y.

Definition. A component of the versal deformation of X is called a *P-component*, if it is the image of a component of the deformation space of a P–resolution Y, for which the induced deformations of the singular points (Y, y_i) are all on qG–components.

With these definitions in mind, one has the following

Conjecture (Kollár) *Every component of the versal base space of a rational surface singularity is a P-component.*

Up to now, the conjecture is proven to be true for:

- quotient surface singularities (due to Kollár, Shepherd–Barron);
- rational quadruple points (due to Jan Stevens).

Jan Stevens calculated all P–resolutions for

- cyclic quotients (giving a nice interpretation in terms of continued fractions and Catalan numbers and completing work of J. Arndt and J. Christophersen - see next section);
- all quotient surface singularities (reducing this case to the former one);
- all rational quadrupel points (since the base space is known by the work of de Jong and van Straten, see section before, this confirms the conjecture).

However, he failed in the quintupel case.

K. Behnke and J. Christophersen completed the work of Kollár and Shepherd–Barron in another direction: qG–deformations of type T–singularities give smoothings with Milnor fibre of Milnor number

$$\mu + 1 = \frac{(rs - 1) + 1}{r} = s ,$$

hence $\mu = s - 1$. Therefore, in case $s = 1$ these smoothings are *homologically trivial*, i. e. without vanishing cycles. So, call the special T–singularities with $s = 1$ of *type* T_0. Jan Christophersen introduced the notion of an *M-resolution* $\widetilde{X} \to X$ which has by definition only T_0–singularities and $K_{\widetilde{X}} \cdot C \geq 0$ for all contractible curves C on $\widetilde{X}$.

Theorem 4.3 (Behnke - Christophersen)

1. *For every P-resolution $\overline{X} \to X$ there exists a unique M-resolution $\widetilde{X} \to X$ dominating $\overline{X}$:*

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f"] \arrow[dr] & & \overline{X} \arrow[dl] \\ & X & \end{tikzcd}$$

   and satisfying $K_{\widetilde{X}} \cdot C = 0$ for all f-contractible curves C (also called the crepant *M-resolution of the P-resolution, since $f^* K_{\overline{X}} = K_{\widetilde{X}}$).*

2. *The f-exceptional curves C create* flops *(like (-2)-curves in minimal resolutions) that generate a finite Weyl group W. The base change map*

$$\mathrm{Def}^0(\widetilde{X}) \longrightarrow \mathrm{Def}^0(\overline{X}) \,(= \textit{normalization of a component of } \mathrm{Def}(X))$$

   is Galois with Galois group W.

3. *W is the monodromy group of the (normalized) component.*

5 Cyclic quotient surface singularities

In the cyclic case the Galois groups W of the components are products of symmetric groups. The Galois coverings are easy to describe; they even have a toric structure. This was first realized by Christophersen. We want to present here shortly his and J. Stevens' results and a new construction for the versal deformation as indicated in the introduction by means of a Galois covering of the whole family.

Recall that the quotient surface singularity of type (n, q) is of embedding dimension e, where

$$\frac{n}{q} = a_2 - 1 \rfloor a_3 - \cdots - 1 \rfloor a_{e-1}$$

is the Hirzebruch–Jung *continued fraction expansion*, and can be described by

$$\binom{e-1}{2}$$

equations of type

$$x_\delta x_\varepsilon = P_{\delta\varepsilon}\,, \quad 2 \leq \delta + 1 \leq \varepsilon - 1 \leq e - 1\,.$$

By *the first* or *initial equations*, we always understand the equations of this type with $\gamma = \varepsilon - 1 = \delta + 1$. They are more precisely of the form

$$x_{\gamma-1} x_{\gamma+1} = x_\gamma^{a_\gamma}\,.$$

They determine inductively all other equations by the following scheme ($\delta \leq \varepsilon - 3$):

$$x_\delta x_\varepsilon = \frac{x_\delta x_{\varepsilon-1} x_{\delta+1} x_\varepsilon}{x_{\delta+1} x_{\varepsilon-1}} = \frac{P_{\delta,\varepsilon-1} P_{\delta+1,\varepsilon}}{P_{\delta+1,\varepsilon-1}} =: P_{\delta\varepsilon} .$$

If one introduces CHRISTOPHERSENS $(e-2)$*-chains representing zero* [Chr]:

$$k_2 - \underline{1} \overline{\big| k_3} - \cdots - \underline{1} \overline{\big| k_{e-1}} = 0 ,$$

one can formulate the result on the component structure of the versal base space for cyclic quotient singularities the following way:

Theorem 5.1 *The P–resolutions (and thus the components of the reduced versal base space) of a cyclic quotient surface singularity of type* $(n,\, q)$ *(and embedding dimension* e *) are in* $1:1$*–correspondence to the set of all* $(e-2)$*–chains* $(k_2, \ldots, k_{e-1})$ *satisfying the conditions*

$$k_\varepsilon \leq a_\varepsilon , \quad \varepsilon = 2, \ldots, e-1 .$$

In particular, if the exponents a_ε are large enough, there are no conditions to satisfy and hence there are exactly as many components as zero–representing chains exist, namely exactly

$$K_r = \frac{1}{r} \binom{2(r-1)}{r-1}$$

many for $r = e - 2$, where K_r is the famous r–th *Catalan number*.

Following the method of my paper [Rie2] and the dissertation of J. ARNDT [Ard], one can indeed construct a canonical candidate for the *full* Weyl group: In the case of a cyclic quotient of type $(n,\, q)$, this group is just the product of symmetric groups

$$\mathfrak{S}_{a_2-1} \times \mathfrak{S}_{a_3-1} \times \cdots \times \mathfrak{S}_{a_{e-1}-1} .$$

The corresponding Galois covering is distinguished by the property that for the lifted deformation family all polynomials in one variable occuring in the describing equations are completely factored into linear forms. This group then induces in fact the monodromy groups

$$\mathfrak{S}_{a_2-k_2} \times \mathfrak{S}_{a_3-k_3} \times \cdots \times \mathfrak{S}_{a_{e-1}-k_{e-1}}$$

on the components where again the tuples $(k_2,\, k_3, \ldots, k_{e-1})$ are the zero–representing chains attached to the various components.

Therefore, this group really deserves the name of *the* monodromy group of the given singularity. However, for the proof of this statement, I will restrict myself to the case of small codimension, that means to the representative and nontrivial cases of embedding dimension less or equal to 6 “in order not to be lost in a morass” of notational subtleties. Concerning the above mentioned papers, there will be a change in the numbering and notation of the deformation parameters.

So, let us first concentrate at the case of embedding dimension $e = 5$. According to [Rie2] and [Ard], we make an *Ansatz* for the versal deformation of the first equations of the following type:

$$\begin{aligned} x_1(x_3+t_3) &= x_2(x_2^{a_2-1} + s_2^{(1)} x_2^{a_2-2} + \cdots + s_2^{(a_2-1)}) \\ x_2 x_4 &= (x_3+t_3)(x_3^{a_3-1} + s_3^{(1)} x_3^{a_3-2} + \cdots + s_3^{(a_3-1)}) \\ x_3 x_5 &= x_4(x_4^{a_4-1} + s_4^{(1)} x_4^{a_4-2} + \cdots + s_4^{(a_4-1)}) \end{aligned}$$

Then there is no obstruction to form the next equation:

$$x_1 x_4 = (x_2^{a_2-1} + s_2^{(1)} x_2^{a_2-2} + \cdots + s_2^{(a_2-1)})(x_3^{a_3-1} + s_3^{(1)} x_3^{a_3-2} + \cdots + s_3^{(a_3-1)}) .$$

However, to form the equation with index label (2, 5), one has to make the polynomial

$$(x_3+t_3)(x_3^{a_3-1} + s_3^{(1)} x_3^{a_3-2} + \cdots + s_3^{(a_3-1)})$$

divisible by x_3. In other words: one has to impose the relation

$$t_3 s_3^{(a_3-1)} = 0 .$$

With this proviso in mind, we immediately get

$$\begin{aligned} x_2 x_5 = {} & (x_3^{a_3-1} + (s_3^{(1)} + t_3) x_3^{a_3-2} + \cdots + (s_3^{(a_3-1)} + t_3 s_3^{(a_3-2)}))\cdot \\ & \cdot(x_4^{a_4-1} + s_4^{(1)} x_4^{a_4-2} + \cdots + s_4^{(a_4-1)}) . \end{aligned}$$

For the last equation, one has to introduce two more relations, namely ([Rie2])

$$s_3^{(a_3-1)} s_2^{(a_2-1)} = s_3^{(a_3-1)} s_4^{(a_4-1)} = 0 .$$

The last equation then reads

$$\begin{aligned} x_1 x_5 = {} & (x_2^{a_2-1} + s_2^{(1)} x_2^{a_2-2} + \cdots + s_2^{(a_2-1)})\cdot \\ & \cdot(x_3^{a_3-2} + s_3^{(1)} x_3^{a_3-3} + \cdots + s_3^{(a_3-2)})\cdot \\ & \cdot(x_4^{a_4-1} + s_3^{(1)} x_3^{a_4-2} + \cdots + s_4^{(a_4-1)}) \\ & + s_3^{(a_3-1)}(x_2^{a_2-2} + s_2^{(1)} x_2^{a_2-3} + \cdots + s_2^{(a_2-2)})\cdot \\ & \cdot(x_3^{a_3-1} + s_3^{(1)} x_3^{a_3-2} + \cdots + s_3^{(a_3-1)})\cdot \\ & \cdot(x_4^{a_4-2} + s_3^{(1)} x_3^{a_4-3} + \cdots + s_4^{(a_4-2)}) . \end{aligned}$$

Obviously, the base space of this family has two components, given by the equations

$$s_3^{(a_3-1)} = 0 \quad \text{resp.} \quad t_3 = s_2^{(a_2-1)} = s_4^{(a_4-1)} = 0 .$$

The equations for the corresponding two families are easily written down [Rie2].

We now form a Galois covering of the base space with respect to the group

$$\mathfrak{S}_{a_2-1} \times \mathfrak{S}_{a_3-1} \times \mathfrak{S}_{a_4-1}$$

where the group action involves only the s-parameters, not the t-variables. In other words: Besides the variable t_3, we introduce new variables

$$t_2^{(1)}, \ldots, t_2^{(a_2-1)}, \; t_3^{(1)}, \ldots, t_3^{(a_3-1)}, \; t_4^{(1)}, \ldots, t_4^{(a_4-1)},$$

and let the group act on the affine space of the t-variables in the obvious way such that the variable $s_j^{(k)}$ is exactly the k-th elementary symmetric function in the variables $t_j^{(1)}, \ldots, t_j^{(a_j-1)}$. We then lift the versal deformation to the space of the t-variables via this covering. This clearly amounts to build up a deformation of the given singularity with the first equations completely decomposed into linear factors:

$$\begin{aligned} x_1(x_3+t_3) &= x_2(x_2+t_2^{(1)}) \cdot \ldots \cdot (x_2+t_2^{(a_2-1)}) \\ x_2x_4 &= (x_3+t_3)(x_3+t_3^{(1)}) \cdot \ldots \cdot (x_3+t_3^{(a_3-1)}) \\ x_3x_5 &= x_4(x_4+t_4^{(1)}) \cdot \ldots \cdot (x_4+t_4^{(a_4-1)}) \,. \end{aligned}$$

The relations which have to be imposed are necessarily the relations of the versal deformation lifted under the covering:

$$t_3 \prod_{k=1}^{a_3-1} t_3^{(k)} = \prod_{k=1}^{a_3-1} t_3^{(k)} \prod_{j=1}^{a_2-1} t_2^{(j)} = \prod_{k=1}^{a_3-1} t_3^{(k)} \prod_{j=1}^{a_4-1} t_4^{(j)} = 0 \,.$$

Consequently, the induced family has many components, namely

$$t_3^{(k)} = 0 \quad \text{for some} \quad k$$

and

$$t_3 = t_2^{(j)} = t_4^{(k)} = 0 \quad \text{for some} \quad j \text{ and } k \,.$$

However, the Galois group interchanges the components in each of these classes, and the stabilizer subgroup for either member is just

$$\mathfrak{S}_{a_2-1} \times \mathfrak{S}_{a_3-2} \times \mathfrak{S}_{a_4-1}$$

resp.

$$\mathfrak{S}_{a_2-2} \times \mathfrak{S}_{a_3-1} \times \mathfrak{S}_{a_4-2}$$

in accordance to our claim at the beginning of this section (recall that the only 3-chains representing 0 are (1,2,1) and (2,1,2)).

Notice that by this construction one finds several simultaneous resolutions of the Artin component in one family, and these are precisely all the nonisomorphic different resolutions which are possible.

Similarly, for embedding dimension $e = 6$ we make the following Ansatz for the first equations:

$$\begin{aligned} x_1(x_3+t_3) &= x_2(x_2^{a_2-1} + s_2^{(1)}x_2^{a_2-2} + \cdots + s_2^{(a_2-1)}) \\ x_2(x_4+t_4) &= (x_3+t_3)(x_3^{a_3-1} + s_3^{(1)}x_3^{a_3-2} + \cdots + s_3^{(a_3-1)}) \\ x_3x_5 &= (x_4+t_4)(x_4^{a_4-1} + s_4^{(1)}x_4^{a_4-2} + \cdots + s_4^{(a_4-1)}) \\ x_4x_6 &= x_5(x_5^{a_5-1} + s_5^{(1)}x_5^{a_5-2} + \cdots + s_5^{(a_5-1)}) \end{aligned}$$

having the right number of deformation parameters.

If all numbers $a_\varepsilon \geq 3$, then the base space is given by the following relations (cf. ARNDT):

$$\begin{array}{r|l}
t_3 s_3^{(a_3-1)} & t_4 s_4^{(a_4-1)} \\
s_2^{(a_2-1)} s_3^{(a_3-1)} & (s_3^{(a_3-1)} + t_3 s_3^{(a_3-2)}) s_4^{(a_4-1)} \\
s_3^{(a_3-1)} (s_4^{(a_4-1)} - t_4 \tilde{s}_4^{(a_4-2)}) & s_4^{(a_4-1)} s_5^{(a_5-1)} \\
s_2^{(a_2-1)} s_3^{(a_3-2)} s_4^{(a_4-1)} + s_2^{(a_2-2)} \left(s_3^{(a_3-1)}\right)^2 \tilde{s}_4^{(a_4-2)} & s_3^{(a_3-1)} s_4^{(a_4-2)} s_5^{(a_5-1)} + s_3^{(a_3-2)} \left(s_4^{(a_4-1)}\right)^2 s_5^{(a_5-2)} .
\end{array}$$

Here, the entries with a tilde are defined by the relation

$$\begin{aligned}
&(x_\delta + t_\delta)(x_\delta^{a_\delta - 1} + s_\delta^{(1)} x_\delta^{a_\delta - 2} + \cdots + s_\delta^{(a_\delta - 1)}) \\
&\quad = x_\delta((x_\delta + t_\delta)^{a_\delta - 1} + \tilde{s}_\delta^{(1)} (x_\delta + t_\delta)^{a_\delta - 2} + \cdots + \tilde{s}_\delta^{(a_\delta - 1)})
\end{aligned}$$

In the special case $a_2 = a_3 = a_4 = a_5 = 3$, which we now treat in full extend, we write s_3' instead of $s_3^{(1)}$ etc. So, we start with the following set of equations (together with the proposed covering):

$$\begin{array}{rcl|l}
x_1(x_3 + t_3) & = & x_2(x_2^2 + s_2' x_2 + s_2'') & \\
\text{resp.} & = & x_2(x_2 + t_2')(x_2 + t_2'') & s_2' = t_2' + t_2'' , \quad s_2'' = t_2' t_2'' \\
x_2(x_4 + t_4) & = & (x_3 + t_3)(x_3^2 + s_3' x_3 + s_3'') & \\
\text{resp.} & = & (x_3 + t_3)(x_3 + t_3')(x_3 + t_3'') & s_3' = t_3' + t_3'' , \quad s_3'' = t_3' t_3'' \\
x_3 x_5 & = & (x_4 + t_4)(x_4^2 + s_4' x_4 + s_4'') & \\
\text{resp.} & = & (x_4 + t_4)(x_4 + t_4')(x_4 + t_4'') & s_4' = t_4' + t_4'' , \quad s_4'' = t_4' t_4'' \\
x_4 x_6 & = & x_5(x_5^2 + s_5' x_5 + s_5'') & \\
\text{resp.} & = & x_5(x_5 + t_5')(x_5 + t_5'') & s_5' = t_5' + t_5'' , \quad s_5'' = t_5' t_5''
\end{array}$$

The equations of the versal base space are now given by the following list where $\tilde{s}_4' = s_4' - t_4$:

$$\begin{array}{r|l}
t_3 s_3'' & t_4 s_4'' \\
s_2'' s_3'' & (s_3'' + t_3 s_3') s_4'' \\
s_3''(s_4'' - t_4 \tilde{s}_4') & s_4'' s_5'' \\
s_2'' s_3' s_4'' + s_2' (s_3'')^2 \tilde{s}_4' & s_3'' s_4' s_5'' + s_3' (s_4'')^2 s_5' .
\end{array}$$

with the components

$$\begin{aligned}
&\{s_3'' = s_4'' = 0\} \\
&\{s_2'' = t_3 = s_4'' = \tilde{s}_4' = 0\} \\
&\{s_2'' = t_3 = s_4'' = t_4 = s_2' = s_5'' = 0\} \\
&\{s_5'' = t_4 = s_3'' = s_3' = 0\} \\
&\{s_5'' = t_4 = s_3'' = t_3 = s_5' = s_2'' = 0\}
\end{aligned}$$

We write down a complete list of the equations for the first three (base) components. It is clear that these correspond successively to the diagrams (c. f. [BR])

$$\begin{array}{ccccccc}
\bullet & & \bullet & & \bullet & & \bullet \\
& \times & & \times & & \times & \\
& & \circ & & \circ & & \\
& & & \circ & & &
\end{array}$$

$$\begin{array}{ccccccc}
\bullet & & \bullet & & \bullet & & \bullet \\
& \times & & \times & & \times & \\
& & \times & & \circ & & \\
& & & \circ & & &
\end{array}$$

$$\begin{array}{ccccccc}
\bullet & & \bullet & & \bullet & & \bullet \\
& \times & & \times & & \times & \\
& & \times & & \circ & & \\
& & & \times & & &
\end{array}$$

$$\begin{aligned}
x_1(x_3+t_3) &= x_2(x_2^2+s_2'x_2+s_2'') \\
x_2(x_4+t_4) &= x_3(x_3+t_3)(x_3+s_3') \\
x_3x_5 &= x_4(x_4+t_4)(x_4+s_4') \\
x_4x_6 &= x_5(x_5^2+s_5'x_5+s_5'') \\
x_1(x_4+t_4) &= (x_2^2+s_2'x_2+s_2'')x_3(x_3+s_3') \\
x_2x_5 &= (x_3+t_3)(x_3+s_3')x_4(x_4+s_4') \\
x_3x_6 &= (x_4+t_4)(x_4+s_4')(x_5^2+s_5'x_5+s_5'') \\
x_1x_5 &= (x_2^2+s_2'x_2+s_2'')(x_3+s_3')x_4(x_4+s_4') \\
x_2x_6 &= (x_3+t_3)(x_3+s_3')(x_4+s_4')(x_5^2+s_5'x_5+s_5'') \\
x_1x_6 &= (x_2^2+s_2'x_2+s_2'')(x_3+s_3')(x_4+s_4')(x_5^2+s_5'x_5+s_5'')
\end{aligned}$$

and

$$\begin{aligned}
x_1x_3 &= x_2^2(x_2+s_2') \\
x_2(x_4+t_4) &= x_3(x_3^2+s_3'x_3+s_3'') \\
x_3x_5 &= x_4(x_4+t_4)^2 \\
x_4x_6 &= x_5(x_5^2+s_5'x_5+s_5'') \\
x_1(x_4+t_4) &= x_2(x_2+s_2')(x_3^2+s_3'x_3+s_3'') \\
x_2x_5 &= (x_3^2+s_3'x_3+s_3'')x_4(x_4+t_4) \\
x_3x_6 &= (x_4+t_4)^2(x_5^2+s_5'x_5+s_5'') \\
x_1x_5 &= (x_2+s_2')(x_3^2+s_3'x_3+s_3'')^2x_4 \\
x_2x_6 &= (x_3^2+s_3'x_3+s_3'')(x_4+t_4)(x_5^2+s_5'x_5+s_5'') \\
x_1x_6 &= (x_2+s_2')(x_3^2+s_3'x_3+s_3'')^2(x_5^2+s_5'x_5+s_5'')
\end{aligned}$$

and

$$\begin{aligned}
x_1x_3 &= x_2^3 \\
x_2x_4 &= x_3(x_3^2+s_3'x_3+s_3'') \\
x_3x_5 &= x_4^2(x_4+s_4') \\
x_4x_6 &= x_5^2(x_5+s_5') \\
x_1x_4 &= x_2^2(x_3^2+s_3'x_3+s_3'') \\
x_2x_5 &= (x_3^2+s_3'x_3+s_3'')x_4(x_4+s_4') \\
x_3x_6 &= x_4(x_4+s_4')x_5(x_5+s_5') \\
x_1x_5 &= x_2(x_3^2+s_3'x_3+s_3'')^2(x_4+s_4') \\
x_2x_6 &= (x_3^2+s_3'x_3+s_3'')(x_4+s_4')x_5(x_5+s_5') \\
x_1x_6 &= (x_3^2+s_3'x_3+s_3'')^2(x_4+s_4')^2(x_5+s_5')
\end{aligned}$$

These examples show that insisting in producing the *miniversal* deformation destroys some symmetry in the equations. We therefore propose to increase at the beginning the number of extra variables to some extend and to make the families minimal only at the end by setting some of the parameters to zero. Such a *symmetric* Ansatz has also the advantage that we can start with only two equations and that we have to increase the number of variables only when we add a new equation in a new row.

In order to make that precise [Rie5], let us look in embedding dimension 4 at the two initial equations

$$x_1\,y_3 \;=\; x_2^{(1)}\cdot\ldots\cdot x_2^{(a_2)}\,, \qquad x_2\,y_4 \;=\; x_3^{(1)}\cdot\ldots\cdot x_3^{(a_3)}\,.$$

Then

$$x_1\,y_4 \;=\; \frac{\prod x_2^{(\ell)}}{x_2}\,\frac{\prod x_3^{(\ell)}}{y_3} \;=\; \sum_{j=0}^{a_2-1}\sigma_j\,(x_2;x_2^{(\ell)})\,x_2^{a_2-j-1}\cdot\sum_{j=0}^{a_3-1}\sigma_j\,(y_3;x_3^{(\ell)})\,y_3^{a_3-j-1}$$

under the side conditions

$$\sigma_{a_2}\,(x_2;x_2^{(\ell)}) \;=\; \prod_{\ell=1}^{a_2}(x_2-x_2^{(\ell)}) \;=\; 0\,,$$

$$\sigma_{a_3}\,(y_3;x_3^{(\ell)}) \;=\; \prod_{\ell=1}^{a_3}(y_3-x_3^{(\ell)}) \;=\; 0\,,$$

where in general

$$\sigma_j\,(z_k;\,x_k^{(\ell)})$$

denotes the j–th elementary symmetric function in the variables $z_k\,-\,x_k^{(\ell)}\,,\ \ell\,=\,1,\ldots,a_k\,.$ Minimalization just means to select one of the equations $x_2\,=\,x_2^{(\ell)}$, say $x_2\,=\,x_2^{(a_2)}$, and correspondingly $y_3=x_3^{(a_3)}$. This leads to

$$x_1y_3 \;=\; x_2x_2^{(1)}\cdot\ldots\cdot x_2^{(a_2-1)}\,, \qquad x_2y_4 \;=\; y_3x_3^{(1)}\cdot\ldots\cdot x_3^{(a_3-1)}\,,$$

$$x_1y_4 \;=\; x_2^{(1)}\cdot\ldots\cdot x_2^{(a_2-1)}\cdot x_3^{(1)}\cdot\ldots\cdot x_3^{(a_3-1)}\,.$$

The obvious action of $\mathfrak{S}_{a_2-1} \times \mathfrak{S}_{a_3-1}$ and introduction of new coordinates produces the versal family.

For embedding dimension 5, we have to add a third initial equation

$$x_3\, y_5 \;=\; x_4^{(1)} \cdot \ldots \cdot x_4^{(a_4)}$$

and the equation

$$x_2\, y_5 \;=\; \frac{\prod x_3^{(\ell)}}{x_3} \cdot \frac{\prod x_4^{(\ell)}}{y_4} \;=\; \sum_{j=0}^{a_3-1} \sigma_j(x_3; x_3^{(\ell)})\, x_3^{a_3-j-1} \sum_{j=0}^{a_4-1} \sigma_j(y_4; x_4^{(\ell)})\, y_4^{a_4-j-1}$$

together with the conditions

$$\sigma_{a_3}(x_3;\, x_3^{(\ell)}) \;=\; 0\,, \quad \sigma_{a_4}(y_4;\, x_4^{(\ell)}) \;=\; 0\,.$$

The last equation is formally given due to our general scheme by

$$x_1\, y_5 \;=\; \frac{\prod x_2^{(\ell)}}{x_2} \cdot \frac{\prod x_3^{(\ell)}}{x_3\, y_3} \cdot \frac{\prod x_4^{(\ell)}}{y_4}\,.$$

Before we can proceed further, we have to prepare the middle term. We write

$$\begin{aligned} \frac{\prod x_3^{(\ell)}}{y_3} &= \sum_{j=0}^{a_3-1} \sigma_j\,(y_3;\, x_3^{(\ell)})\, y_3^{a_3-j-1} \;=\; \sum_{j=0}^{a_3-1} \sigma_j\,(y_3;\, x_3^{(\ell)})\,(x_3 + (y_3 - x_3))^{a_3-j-1} \\ &= \sum_{k=0}^{a_3-1} \tilde{\sigma}_k\,(x_3,\, y_3;\, x_3^{(\ell)})\, x_3^{a_3-k-1} \end{aligned}$$

with the obvious definition for the coefficients

$$\tilde{\sigma}_k\,(x_3,\, y_3;\, x_3^{(\ell)}) \;=\; \sum_{j=0}^{k} \binom{a_3-j-1}{k-j} \sigma_j(y_3;\, x_3^{(\ell)})\,(y_3 - x_3)^{k-j}\,, \qquad k = 0, \ldots, a_3 - 1\,.$$

We next introduce the following two relations

$$\sigma_{a_2-1}\,(x_2;\, x_2^{(\ell)}) \cdot \tilde{\sigma}_{a_3-1}\,(x_3,\, y_3;\, x_3^{(\ell)}) \;=\; \tilde{\sigma}_{a_3-1}\,(x_3,\, y_3;\, x_3^{(\ell)}) \cdot \sigma_{a_4-1}(y_4;\, x_4^{(\ell)}) \;=\; 0\,.$$

Remark that

$$\tilde{\sigma}_{a_3-1}\,(x_3,\, y_3;\, x_3^{(\ell)}) \;=\; \sum_{j=0}^{a_3-1} (y_3 - x_3)^{a_3-j-1}\, \sigma_j\,(y_3;\, x_3^{(\ell)})\,.$$

With this proviso in mind, we see that the right side of $x_1\, y_5$ is indeed regular (we use some obvious abbreviations):

$$\begin{aligned} x_1\, y_5 &= \sum_{j=0}^{a_2-1} \sigma_{2j}\, x_2^{a_2-j-1} \sum_{j=0}^{a_3-1} \tilde{\sigma}_{3j}\, x_3^{a_3-j-2} \sum_{j=0}^{a_4-1} \sigma_{4j}\, y_4^{a_4-j-1} \\ &= x_2\, y_4 \sum_{j=0}^{a_2-2} \sum_{j=0}^{a_3-1} \sum_{j=0}^{a_4-2} - s_{2,a_2-1}\, s_{4,a_4-1} \sum_{j=0}^{a_3-2} + s_{4,a_4-1} \sum_{j=0}^{a_2-1} \sum_{j=0}^{a_3-2} + s_{2,a_2-1} \sum_{j=0}^{a_3-2} \sum_{j=0}^{a_4-1}\,. \end{aligned}$$

Now, $x_2\, y_4 = \prod x_3^{(\ell)}$ contains the factor x_3 by the third relation, such that the right side is indeed regular.

To minimalize the construction, we have to put

$$y_3 = x_3^{(a_3)}\,, \quad x_2 = x_2^{(a_2)}\,, \quad y_4 = x_4^{(a_4)}\,.$$

Then, from the first four relations only one remains, namely

$$(x_3 - y_3) \prod_{\ell=1}^{a_3-1} (x_3^{(\ell)} - x_3) = 0\,,$$

i. e.

$$(x_3 - y_3)\, \sigma'_{a_3-1}(x_3;\, x_3^{(\ell)}) = 0\,,$$

if $\sigma'_j(x_3;\, x_3^{(\ell)})$ now denotes the j-th symmetric function in the variables $x_3^{(\ell)} - x_3$, $\ell = 1, \ldots, a_3 - 1$, only, and similarly with respect to the other lower indices. Then,

$$\sigma_{a_2-1}(x_2;\, x_2^{(\ell)}) = \sigma'_{a_2-1}(x_2;\, x_2^{(\ell)}) = \prod_{\ell=1}^{a_2-1} (x_2^{(\ell)} - x_2)$$

and

$$\sigma_{a_4-1}(y_4;\, x_4^{(\ell)}) = \prod_{\ell=1}^{a_4-1} (x_4^{(\ell)} - y_4)\,.$$

Finally, we get

$$\tilde{\sigma}_{a_3-1}(x_3,\, y_3;\, x_3^{(\ell)}) = \sum_{j=0}^{a_3-1} (y_3 - x_3)^{a_3-j-1}\, \sigma'_j(x_3^{(\ell)} - y_3) = \sigma'_{a_3-1}(x_3;\, x_3^{(\ell)})\,.$$

Hence, we have exactly the relations introduced earlier in this section and the correct action of the group $\mathfrak{S}_{a_2-1} \times \mathfrak{S}_{a_3-1} \times \mathfrak{S}_{a_4-1}$. Dividing out this group action yields the versal deformation.

6 The McKay correspondence

McKay linked together *binary polyhedral groups* and the (extended) CDW–diagrams by the following calculation: Take ρ_0 the trivial representation and $\rho_1, \ldots, \rho_k$ the nontrivial irreducible complex representations of a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$, which we here call Γ instead of G for certain reasons which will be become transperent in the next section. Denote further by c the natural representation induced by the embedding $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$. Then decompose the representation $\rho_i \otimes c$ into irreducibles:

$$\rho_i \otimes c = \sum_j n_{ji} \rho_j$$

and realize that

$$n_{ji} = 1 \quad \text{or} \quad 0$$

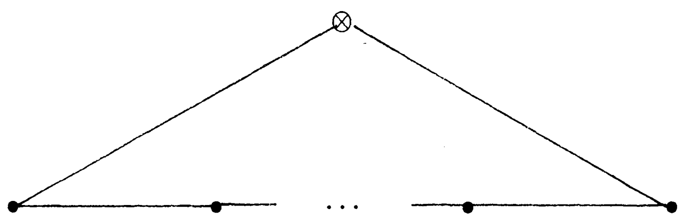
with

$$n_{ii} = 0$$

and

$$2E_{k+1} - (n_{ji}) = \text{Cartan matrix of the extended CDW diagram of type } \widetilde{\mathrm{A}}\widetilde{\mathrm{D}}\widetilde{\mathrm{E}} ,$$

where the extra vertex $\otimes$ belongs to the trivial representation. The following diagram shows the case $\widetilde{\mathrm{A}_k}$:



Moreover, it is easy to observe that the ranks of the representations equal the weights of *fundamental divisor* on the minimal resolution of the correponding Klein singularity and are also equal to the weights of the *highest root* of the associated CDW–diagram.

A geometric explanation for this phenomenon was given by Artin and Gonzales–Sprinberg, Artin and Verdier [AV] and H. Esnault [Es]: The irreducible representations are in 1:1–correspondence to the indecomposable reflexive modules on $X = \mathbb{C}^2/\Gamma$ which in turn are classified by certain (*full*) indecomposable vector bundles on the minimal resolution $\widetilde{X}$ whose *Chern divisor* intersect precisely one exceptional curve with multiplicity 1 (if they are nontrivial, of course). This result has been extended to a great part to all quotient surface singularities by H. Esnault and J. Wunram [Wu]:

Take a quotient singularity $\mathbb{C}^2/\Gamma$, $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$ a *small subgroup*. Then one should regard the McKay *quiver*. As above, $\rho_0, \ldots, \rho_r$ denote the irreducible complex representations of Γ with ρ_0 the trivial one, and take now the dual representation c^* instead of $c : \Gamma \hookrightarrow \mathrm{GL}(2, \mathbb{C})$. Then, as above, the decomposition

$$\rho_i \otimes c^* = \sum_{r=0}^{r} n_{ij} \rho_j$$

defines in a natural way a quiver which is identical to the Auslander–Reiten *quiver* if one replaces the representations by the associated reflexive modules.

It should be apparent that in this general case there are much more representations of Γ than curves in the minimal resolution of the quotient surface singularity. But there exists a 1:1–correspondence between the exceptional curves and some *special* nontrivial full vector bundles which have been characterized by J. Wunram. On the level of representations, the characterization is as follows[Rie4]: A nontrivial irreducible representation $\Gamma \to \mathrm{Aut} V$ is special if and only if the canonical homomorphism

$$(\Omega^2_{\mathbb{C}})^\Gamma \otimes (\mathcal{O}_{\mathbb{C}^2} \otimes V)^\Gamma \longrightarrow (\Omega^2_{\mathbb{C}} \otimes V)^\Gamma$$

is surjective.

7 The quiver construction

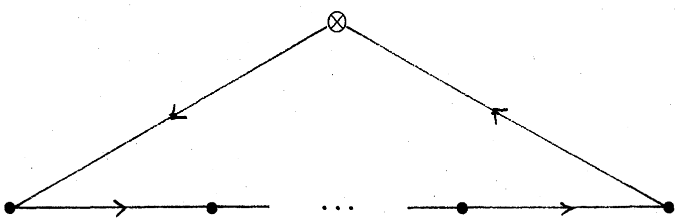
Using representations of the binary polyhedral groups $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ and the theory of *hyper–Kähler–quotients*, P. Kronheimer [K] constructed in an ingenious way the versal deformation (or rather its simultaneous resolution after lifting by the Weyl group W):

$$\mathcal{X}_T = \mathcal{X} \times_{T/W} T .$$

He realized himself that his construction can be formulated in terms of *representations of quivers*. The program has been carried out by W. Ebeling, P. Slodowy and a student in Hamburg, H. Cassens.

We give a short description of the results. Let $\widetilde{\Delta}$ be any quiver in the sense that with any arrow also the arrow in the opposite direction occurs:

(So, in this special situation, only McKay quivers of ADE–type are allowed but no quivers associated to arbitrary subgroups $G \subset \mathrm{GL}(2, \mathbb{C})$.) Then, choose an *oriented* subgraph P of $\widetilde{\Delta}$ such that $\widetilde{\Delta} = P \cup \overline{P}$, where of course $\overline{P}$ denotes the graph P with all arrows reversed. For instance, the graph P can in the case $\widetilde{A}_k$ be choosen as follows:



Associate now to any vertex i a complex vector space V_i of fixed dimension d_i and call $d = (d_0, \ldots, d_k)$, $d_i = \dim V_i$, the *dimension vector*. We always assume that the index 0 belongs to the trivial representation, that is to the extra vertex $\otimes$ in the McKay quiver.

A *representation* of the quiver $\widetilde{\Delta}$ (with respect to the given dimension d) is just a system of homomorphisms

$$f^i_j : V_i \longrightarrow V_j\,, \quad (i \to j) \in \widetilde{\Delta}\,.$$

The representation space is then

$$\mathcal{Y}_d = \bigoplus_{(i \to j) \in P} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{(i \to j) \in P} \mathrm{Hom}(V_j, V_i)\,.$$

Isomorphism classes of such representations are orbits with respect to the conjugation operation of the group

$$\widetilde{G} = \prod \mathrm{GL}\,(V_j)\,.$$

Clearly, the diagonally embedded subgroup $\mathbb{C}^* \subset \widetilde{G}$ operates trivially on the representation space, such that the group $G = \widetilde{G}/\,\mathbb{C}^*$ is operating effectively. Now, put

$$V = \bigoplus V_j\,,$$

and regard

$$\bigoplus \mathrm{Hom}\,(V_i,\, V_j) \quad \text{as a subgroup of} \quad \mathrm{End}\, V\,.$$

In other words, we interpret elements in $\mathcal{Y}_d$ as pairs $(\alpha,\, \beta) \in \mathrm{End}\, V \oplus \mathrm{End}\, V$. In these terms, the *momentum map* can be regarded as

$$\begin{cases} \mathcal{Y}_d & \longrightarrow \qquad \mathfrak{g} = \mathrm{Lie}\, G \\ (\alpha,\, \beta) & \longmapsto \quad [\alpha, \beta] = \alpha \circ \beta - \beta \circ \alpha\,. \end{cases}$$

Denote now by T the center $C(\mathfrak{g})$ of $\mathfrak{g}$ and remark that T is a manifold of the correct dimension if all $d_i \geq 1$. Moreover, let $\mathcal{Y}_T$ be the preimage of T under the momentum map $\mathcal{Y}_d \to \mathfrak{g}$. Then the group G acts on the fibers of this family such that one can form

$$\mathcal{X}_T = \mathcal{Y}_T/\!/\,G \longrightarrow T = C(\mathfrak{g})\,.$$

Theorem 7.1 (H. Cassens) *If the dimension vector is choosen appropriately, i. e. if d_i equals the multiplicity of E_i in the fundamental cycle of the Klein singularity $X = \mathbb{C}^2/\,\Gamma$, then*

$$\mathcal{X}_T = \mathcal{Y}_T/\!/\,G \longrightarrow T$$

is the versal deformation of X after lifting under the natural action of the Weyl group W.

Remarks.

1. H. Cassens is also able to construct the simultaneous resolution of this family.

2. He can identify nearby fibers, even nongenerically.

3. There are examples that this construction works for other McKay quivers. At least, one finds the *correct* singularity and its resolution by blowing up subvarieties of the representation space for some cyclic quotients.

The results on deformations in the non RDP–case are not satisfactory at the moment. We nevertheless still hope to find by this procedure all (or at least the Artin) component(s) for cyclic quotient singularities.

Let us give some hints to the *proof* of Cassen's result: The special fiber has to be calculated which is quite easy in all cases except for the CDW–diagram of type E_8. Flatness of the family $\mathcal{X}_T \to T$ reduces to that of $\mathcal{Y}_d \to \mathfrak{g}$: Since G is linearly reductive, $\mathcal{O}(\mathcal{X})$ is a direct summand of $\mathcal{O}(\mathcal{Y}_T)$ as a $\mathcal{O}(T)$–module. Hence it suffices to prove flatness of $\mathcal{O}(\mathcal{Y}_T)$ over $\mathcal{O}(T)$. But flatness is preserved by base change such that we are reduced to the claim. Now $\mathcal{Y}_d \to \mathfrak{g}$ is a $\mathbb{C}^*$–equivariant mapping of smooth affine spaces with a *good* $\mathbb{C}^*$–action which is easily seen to be flat if and only if the dimension of the special fibre equals the difference of the dimensions of the source and the target. By a result of LUSZTIG, the dimension of the special fibre is equal to $\operatorname{ord}\Gamma + 1$. On the other hand,

$$\dim G = \sum d_i^2 - 1 = \operatorname{ord}\Gamma - 1$$

and

$$\dim \mathcal{Y}_d = \sum n_{ij} d_i d_j = 2 \operatorname{ord}\Gamma ,$$

q. e. d. □

As an *example* we treat the case $\widetilde{A}_k$: Here $V_0, \ldots, V_k$ are all of dimension 1. So the mappings represented by arrows are just numbers:

$$V_i \xrightarrow{\alpha_i} V_{i+1} ,$$
$$V_i \xleftarrow{\beta_i} V_{i+1} .$$

Moreover, $\mathcal{Y}_d = \mathbf{C}^{2k+2}$, and $G = (\mathbf{C}^*)^{k+1}/\mathbf{C}^* \cong$ maximal torus

$$\{(t_0, \ldots, t_k) \in (\mathbf{C}^*)^{k+1} : \prod t_\kappa = 1\} \subset \mathrm{SL}(k+1, \mathbf{C})$$

operates on

$$\mathcal{O}(\mathcal{Y}_d) = \mathbf{C}[\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_k]$$

by

$$t\,\alpha_i = t_i^{-1}\,\alpha_i , \quad t\,\beta_i = t_i\,\beta_i .$$

The invariants are readily computed:

$$\begin{aligned} \mathcal{O}(\mathcal{Y}_d)^G &= \mathbf{C}[\alpha_0\beta_0, \ldots, \alpha_k\beta_k , \alpha_0 \ldots \alpha_k , \beta_0 \ldots \beta_k] \\ &\cong \mathbf{C}[z_0, \ldots, z_k, x, y]/(z_0 \cdot \ldots \cdot z_k - xy) . \end{aligned}$$

Finally, we have

$$T \cong \{(t_0, \ldots, t_k) : \sum t_i = 0\}$$

and the relations

$$\beta_{i+1}\,\alpha_{i+1} - \alpha_i\,\beta_i = t_{i+1} .$$

Putting all $t_i = 0$ yields the equations

$$z_i = \alpha_i\,\beta_i = \alpha_{i+1}\,\beta_{i+1} = \cdots = z_0 .$$

Hence the special fiber has the equation

$$xy = z_0^{k+1} ,$$

a singularity of type A_k as it should be. More generally, if we set $z := \dfrac{1}{k+1} \sum \alpha_i\,\beta_i$, we get $z_i := \alpha_i\beta_i = z + t_i$, $t = (t_0, \ldots, t_k) \in T$ suitably choosen. The final equation then is

$$x\,y = \prod_{i=0}^{k} (z + t_k) , \quad \sum t_i = 0 ,$$

which is indeed the equation for the versal family after W-lifting. □

8 Formats of rational singularities

I want to finish this survey with some results due to A. RÖHR [Roe]. His investigations started from J. Wahl's theorem [W1] that the Artin component of a *determinantal* rational surface singularity is just given by varying the entries of a describing matrix generically.

Of course, determinantal equations are special in the sense that relations between the equations can be read off very easily. There are other "formats" like quasideterminantal ones introduced by the author which also give the Artin component by perturbation in the case of cyclic quotients. In this case, the other components are obviously caused by different formats.

A. Röhr now formalizes the concept of a format which I don't want to repeat here. More importantly, he shows that there exists at least one format in his abstract sense associated to any rational surface singularity X, namely (the germ of) the total space $F(X)$ of the deformation space of X over the Artin component "modulo smooth factors" which perhaps should be called the *Artin format*. His main result may be stated as follows:

Theorem 8.1 (A. Röhr) *Let X, X' be rational surface singularities. Then the following are equivalent:*

i) X' *is* of type $F(X)$;

ii) *there exists a complete intersection* $X' \to F(X)$;

iii) $F(X') \cong F(X)$.

Here, I don't want to make the first assumption precise. It is a rather technical definition saying intuitively that X' can be described by specializing the equations of $F(X)$. The precise definition includes ii) as a part.

An easy consequence of this theorem is the following

Corollary 8.1 *If X is (rational and) determinantal (resp. quasideterminantal) then so is $F(X)$.*

He is even able to characterize these singularities by their minimal resolution graph generalizing a result of J. Wahl.

Theorem 8.2 *Let X be a rational surface singularity of multiplicity $m \geq 3$.*
a) *X is determinantal if and only if the minimal resolution contains one exceptional curve of selfintersection number $-m$;*
b) *X is quasideterminantal if and only if the minimal resolution graph contains the (linear) graph of a cyclic quotient singularity of the same multiplicity.*

Since the graphs of quotient surface singularities are known [Br2] this implies the

Corollary 8.2 *Quotient surface singularities are quasideterminantal.*

Röhr also determines all possible Artin formats for small multiplicity. For $m = 3$, there exists only the determinantal one. For $m = 4$, there are of course the determinantal and the quasideterminantal one, but a third one is showing up which is described by the following interesting set of equations:

$$\begin{vmatrix} x_1 & x_2 \\ x_3 & x_4 \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ y_3 & y_4 \end{vmatrix} = 0$$

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} - \begin{vmatrix} y_1 & x_3 & y_3 \\ z_1 & 0 & z_3 \\ y_2 & x_4 & y_4 \end{vmatrix} = 0\,, \quad \begin{vmatrix} x_1 & x_2 \\ y_3 & y_4 \end{vmatrix} - \begin{vmatrix} y_1 & x_3 & y_3 \\ z_2 & 0 & z_3 \\ y_2 & x_4 & y_4 \end{vmatrix} = 0\,,$$

$$x_1^2 + \begin{vmatrix} z_1 & y_1 & z_2 \\ y_1 & x_3^2 & y_3 \\ z_2 & y_3 & z_3 \end{vmatrix} = 0\,, \quad x_1 x_2 + \begin{vmatrix} z_1 & y_1 & z_2 \\ y_2 & x_3 x_4 & y_4 \\ z_2 & y_3 & z_3 \end{vmatrix} = 0\,, \quad x_2^2 + \begin{vmatrix} z_1 & y_2 & z_2 \\ y_2 & x_4^2 & y_4 \\ z_2 & y_4 & z_3 \end{vmatrix} = 0\,.$$

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