Math. Ann. 286, 529–535 (1990)

Mathematische Annalen

A vanishing theorem concerning the Artin component of a rational surface singularity

Oswald Riemenschneider[1], Ancus Röhr[1], and Jonathan M. Wahl[2,*]

[1] Mathematisches Seminar, Universität Hamburg, Bundesstrasse 55, 2000 Hamburg 13, Federal Republic of Germany
[2] Department of Mathematics, The University of North Carolina, Chapel Hill, NC 27514, USA

Dedicated to Professor Hans Grauert on his sixtieth birthday

Introduction

Let (X, x) be a rational surface singularity with minimal resolution $\pi: \tilde{X} \to X$, and let $\mathscr{X} \to S$ and $\tilde{\mathscr{X}} \to \tilde{S}$ be the versal deformations of X and of $\tilde{X}$, resp.. The latter blows down to a deformation $\mathscr{Y} \to \tilde{S}$ of X such that there exists a cartesian diagram

$$\begin{array}{ccc} \mathscr{Y} & \to & \mathscr{X} \\ \downarrow & & \downarrow \\ \tilde{S} & \to & S. \end{array}$$

It has been shown by Artin [1] that the image S_{art} of $\tilde{S}$ in S is an irreducible component of S (the *Artin component*), and by Lipman and the third author [4, 12] that the mapping $\tilde{S} \to S_{\mathrm{art}}$ can be identified with the quotient map associated to the action of a product $\prod W_\nu$ of Weyl groups on the affine space $\tilde{S}$, each Weyl group W_ν belonging to a maximal connected configuration $E^{(\nu)}$ of (-2)-curves in the exceptional set $E = \bigcup_{i=1}^{r} E_i \subset \tilde{X}$. Blowing down these configurations $E^{(\nu)}$, we get the *rational double point* resolution (RDP resolution) of X which will be denoted by $\hat{X}$. The resolution π factors through $\hat{X}$; the factorization will be written in the form $\pi = \tau \circ \sigma$.

The tangent spaces of the various base spaces can be identified with the corresponding vector spaces of (isomorphism classes of) infinitesimal deformations of first order:

$$\operatorname{Tang} S = T^1_X = \operatorname{Ext}^1_{\mathcal{O}_{X,x}}(\Omega^1_{X,x}, \mathcal{O}_{X,x}) \quad (\Omega^1_X = \text{Kähler differentials on } X),$$
$$\operatorname{Tang} \tilde{S} = T^1_{\tilde{X}} = H^1(\tilde{X}, \Theta_{\tilde{X}}) \qquad (\Theta_{\tilde{X}} = \text{tangent bundle of } \tilde{X}).$$

According to Lipman and the third author [loc. cit.], the tangent space of the Artin component is isomorphic to the space of first order deformations of the RDP resolution $\hat{X}$:

$$\operatorname{Tang} S_{\mathrm{art}} = T^1_{\hat{X}} = \operatorname{Ext}^1_{\hat{X}}(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}}).$$

* Research partially supported by the National Science Foundation

The crucial fact in connection with the above mentioned result $S_{\text{art}} \cong \tilde{S}/\prod W_v$ is the *injectivity* of the canonical map

$$T^1_{\hat{X}} \hookrightarrow T^1_X \tag{1}$$

resulting from blowing down deformations of $\hat{X}$ to those of X [4, Theorem], [12, Theorem 2.2]. In particular, the *smoothness* of the Artin component is a direct consequence of (1) [12, Theorem 1].

The purpose of the present note is to emphasize the usefulness of regarding the *dual* situation, i.e. the cotangent spaces T^{1*}_X, $T^{1*}_{\hat{X}}$, $T^{1*}_{\tilde{X}}$. As it turns out, each of these spaces can be described in terms of maps between sections of the sheaves

$$\mathscr{F}_Y := \Omega^1_Y \otimes \omega_Y \quad (\omega_Y = \text{dualizing sheaf of } Y)$$

for various spaces Y, viz. X, $\hat{X}$, $\tilde{X}$ and $X' = \tilde{X} \setminus E = X \setminus \{x\}$. To be more precise, we have (for a Stein representative X) the following sequence of canonical mappings:

$$H^0(X, \mathscr{F}_X) \to H^0(\hat{X}, \mathscr{F}_{\hat{X}}) \hookrightarrow H^0(\tilde{X}, \mathscr{F}_{\tilde{X}}) \hookrightarrow H^0(X', \mathscr{F}_{X'}) \tag{2}$$

in which

$$\operatorname{coker}(H^0(X, \mathscr{F}_X) \to H^0(X', \mathscr{F}_{X'})) \cong T^{1*}_X \tag{3}$$

by dualizing Schlessinger's description of T^1_X for normal surface singularities (cf. [9] and, for the dual version, [3]). Our main result (cf. Sect. 1) is the

Vanishing Theorem. $\quad H^1(\hat{X}, \Omega^1_{\hat{X}} \otimes \omega_{\hat{X}}) = 0$.

This will be proved by using (1). In fact, the Vanishing Theorem is *equivalent* to (1). Hence, a direct proof would be of independent value. In Sect. 2 we relate various quotients of modules occurring in (2) to $\mathbb{C}$-duals of deformation spaces. In particular, we show that

$$T^{1*}_{\hat{X}} \cong H^0(X', \mathscr{F}_{X'}) / H^0(\hat{X}, \mathscr{F}_{\hat{X}}).$$

Finally, in Sect. 3, we present explicit computations in the case of cyclic quotient singularities.

1. The vanishing of $H^1(\hat{X}, \mathscr{F}_{\hat{X}})$

Before we prove our main result, let us deal with two special cases. Of course, if X is a rational double point, then $X = \hat{X}$ and the vanishing is trivial. If, on the other hand, there is no (-2)-curve at all in the resolution, then $\hat{X} = \tilde{X}$ and we can use local duality:

$$H^1(\tilde{X}, \Omega^1_{\tilde{X}} \otimes \omega_{\tilde{X}})^* \cong H^1_E(\tilde{X}, \Theta_{\tilde{X}}).$$

But, by a result of the third author [11, Theorem 6.1], we have for any rational singularity X the identity

$$\dim H^1_E(\tilde{X}, \Theta_{\tilde{X}}) = \varrho = \text{number of } (-2)\text{-curves in } E; \tag{4}$$

so we are done.

Besides the injectivity mentioned in the introduction, the last result is the main ingredient for the proof of $H^1(\hat{X}, \mathscr{F}_{\hat{X}})=0$ in the general case. We set $D=\sigma(E)=\tau^{-1}(x)$ and investigate the following part of the long exact cohomology sequence with support in D:

$$H^0(\hat{X}, \mathscr{F}_{\hat{X}}) \xrightarrow{\alpha} H^0(X', \mathscr{F}_{X'}) \xrightarrow{\varphi} H^1_D(\hat{X}, \mathscr{F}_{\hat{X}}) \to H^1(\hat{X}, \mathscr{F}_{\hat{X}}) \xrightarrow{\psi} H^1(X', \mathscr{F}_{X'}). \tag{5}$$

The claim follows from the following two Propositions:

Proposition 1. *φ is surjective.*

Proposition 2. *ψ is the zero map.*

Proof of Proposition 1. By local duality and the fact that the sheaf $\omega_{\hat{X}}$ is invertible ($\hat{X}$ has only hypersurface singularities), it follows that

$$H^1_D(\hat{X}, \Omega^1_{\hat{X}} \otimes \omega_{\hat{X}}) \cong \mathrm{Ext}^1_{\hat{X}}(\Omega^1_{\hat{X}} \otimes \omega_{\hat{X}}, \omega_{\hat{X}})^* \cong \mathrm{Ext}^1_{\hat{X}}(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}})^* = T^{1*}_{\hat{X}}.$$

By (3) and (5) we have a surjection of T^{1*}_X on the cokernel of α; composed with φ this yields the map $T^{1*}_X \to T^{1*}_{\hat{X}}$ which must be surjective due to (1).

Proof of Proposition 2. Let E' be the union of all (-2)-curves in E. Any local section of $\Omega^1_{\tilde{X}}$ on $\hat{X} \backslash \sigma(E') = \tilde{X} \backslash E'$ can be extended across E' to a local section of $\Omega^1_{\hat{X}}$, since the singularities of $\hat{X}$ are rational double points (cf. Steenbrink [10]). Hence there is a canonical map $\mathscr{F}_{\hat{X}} \to \sigma_* \mathscr{F}_{\tilde{X}}$ which induces a factorization of ψ over $H^1(\hat{X}, \sigma_* \mathscr{F}_{\tilde{X}})$. This module is zero, as we will show now.

Consider the five term exact sequence associated to the spectral sequence $R^j \tau_*(R^k \sigma_* \mathscr{F}_{\tilde{X}}) \Rightarrow R^{j+k} \pi_* \mathscr{F}_{\tilde{X}}$:

$$0 \to R^1 \tau_*(\sigma_* \mathscr{F}_{\tilde{X}}) \to R^1 \pi_* \mathscr{F}_{\tilde{X}} \to \tau_*(R^1 \sigma_* \mathscr{F}_{\tilde{X}}) \to R^2 \tau_*(\sigma_* \mathscr{F}_{\tilde{X}}) = 0$$

which implies the exactness of the sequence

$$0 \to H^0(X, R^1 \tau_*(\sigma_* \mathscr{F}_{\tilde{X}})) \to H^0(X, R^1 \pi_* \mathscr{F}_{\tilde{X}}) \to H^0(X, \tau_*(R^1 \sigma_* \mathscr{F}_{\tilde{X}})) \to 0.$$

By (4), the $\mathbb{C}$-dimension of $H^0(X, R^1 \pi_* \mathscr{F}_{\tilde{X}}) = H^1(\tilde{X}, \mathscr{F}_{\tilde{X}})$ is equal to ϱ, the number of (-2)-curves in E, and

$$\begin{aligned} H^0(X, \tau_*(R^1 \sigma_* \mathscr{F}_{\tilde{X}})) &\simeq H^0(\hat{X}, R^1 \sigma_* \mathscr{F}_{\tilde{X}}) \\ &\simeq \bigoplus H^0(\hat{X}_\nu, R^1 \sigma_* \mathscr{F}_{\tilde{X}}) \\ &\simeq \bigoplus H^1(\tilde{X}_\nu, \mathscr{F}_{\tilde{X}}), \end{aligned}$$

where $\hat{X}_\nu$ are suitable Stein neighbourhoods of the singular points in $\hat{X}$ and $\tilde{X}_\nu = \sigma^{-1}(\hat{X}_\nu)$ are strongly pseudoconvex neighbourhoods of the sets $E^{(\nu)}$. Therefore, by the same result, $\dim H^0(X, \tau_*(R^1 \sigma_* \mathscr{F}_{\tilde{X}})) = \varrho$, such that

$$H^1(\hat{X}, \sigma_* \mathscr{F}_{\tilde{X}}) = H^0(X, R^1 \tau_*(\sigma_* \mathscr{F}_{\tilde{X}})) = 0.$$

This ends our proof of the Propositions and the Vanishing Theorem.

Remark. From the proof of Proposition 1 it is easily deduced that the vanishing of $H^1(\hat{X}, \Omega^1_{\hat{X}} \otimes \omega_{\hat{X}})$ in turn implies the surjectivity of the map $T^{1*}_X \to T^{1*}_{\hat{X}}$.

2. The description of $T_{\hat{X}}^{1*}$

Since the singularities of $\hat{X}$ are complete intersections, $\Omega_{\hat{X}}^1$ is torsion-free. Hence the map α in (5) is injective, which implies

Proposition 3. $T_{\hat{X}}^{1*} \simeq H^0(X', \mathscr{F}_{X'})/H^0(\hat{X}, \mathscr{F}_{\hat{X}})$.

We have a canonical, restriction-induced mapping from the subspace

$$H^0(\tilde{X}, \mathscr{F}_{\tilde{X}})/H^0(\hat{X}, \mathscr{F}_{\hat{X}})$$

of $T_{\hat{X}}^{1*}$ to

$$\oplus H^0(\tilde{X}_\nu, \mathscr{F}_{\tilde{X}_\nu})/H^0(\hat{X}_\nu, \mathscr{F}_{\hat{X}_\nu}) \simeq \oplus T_{\hat{X}_\nu}^{1*}$$

which is clearly injective. It is surjective, too, as can be seen from the well-known local-global relation for infinitesimal deformations [5], which in our case for the space $\hat{X}$ comes down to the exact sequence

$$0 \to H^1(\hat{X}, \Theta_{\hat{X}}) \to T_{\hat{X}}^1 \to \oplus T_{\hat{X}_\nu}^1 \to 0\,. \tag{6}$$

We claim that $H^1(\hat{X}, \Theta_{\hat{X}})$ is just the image of the blowing-down map $\beta: T_{\tilde{X}}^1 \to T_{\hat{X}}^1$. This map is zero for rational double points, hence, in view of (6), β factorizes over $H^1(\hat{X}, \Theta_{\hat{X}})$. By the results of [11], the kernel of β has the same dimension as the cokernel of $H^1(\hat{X}, \Theta_{\hat{X}}) \hookrightarrow T_{\hat{X}}^1$, whence the claim (remember that $T_{\tilde{X}}^1$ and $T_{\hat{X}}^1$ have the same dimension).

Let us summarize our results:

Proposition 4. $H^0(\tilde{X}, \mathscr{F}_{\tilde{X}})/H^0(\hat{X}, \mathscr{F}_{\hat{X}}) \simeq \oplus T_{\hat{X}_\nu}^{1*}$.

Proposition 5. $H^1(\hat{X}, \Theta_{\hat{X}}) \simeq \mathrm{im}(T_{\tilde{X}}^1 \to T_{\hat{X}}^1)$.

Finally, a consequence of (6) and Propositions 3, 4, and 5 is

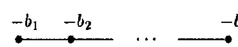
Proposition 6. *$H^0(X', \mathscr{F}_{X'})/H^0(\tilde{X}, \mathscr{F}_{\tilde{X}})$ is canonically isomorphic to the $\mathbb{C}$-dual of* $\mathrm{im}(T_{\tilde{X}}^1 \to T_{\hat{X}}^1)$.

3. Application to cyclic quotient singularities

In this section we shall apply Proposition 3 in order to compute

$$T_{\hat{X}}^{\perp} = H^0(\hat{X}, \mathscr{F}_{\hat{X}})/\mathrm{im}\, H^0(X, \mathscr{F}_X)$$

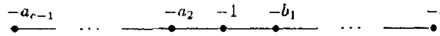
for two-dimensional cyclic quotient singularities $X = A_{n,q}$. Recall that if X has a dual graph



then a resolution $\tilde{X}$ is given by $r+1$ coordinate patches $M_0, \ldots, M_r$ ($M_i \simeq \mathbb{C}^2$, with coordinates u_i, v_i) glued together according to

$$(u_1, v_1) = (u_0^{-1}, u_0^{b_1} v_0), \quad (u_2, v_2) = (u_1 v_1^{b_2}, v_1^{-1}), \quad (u_3, v_3) = (u_2^{-1}, u_2^{b_3} v_2) \quad \text{etc.},$$

and functions $z_1=v_0$, $z_2=u_0v_0$, $z_{\varepsilon+1}=z_\varepsilon^{a_\varepsilon}z_{\varepsilon-1}^{-1}$, $2\leqq\varepsilon\leqq e-1$, which blow down $\tilde{X}$ to $A_{n,q}$. The numbers $a_2,\ldots,a_{e-1}$ are characterized by the property that the configuration



can be blown down to a smooth point. Another way of describing the relation between the a_ε's and the b_i's is the following: in an array with r columns and $e-2$ rows (the rows numbered $2,\ldots,e-1$) mark the entry (ε,i) with a dot if z_ε vanishes to first order along the i-th irreducible curve in the exceptional set of $\tilde{X}$. This yields the first author's "Punkteschema" [7] with b_i-1 dots in the i-th column, $a_\varepsilon-1$ dots in the row number ε and such that the first dot in the $(i+1)$-st column is adjacent to the last dot in the i-th.

The local algebra of $A_{n,q}$ is generated by the invariants $z_\varepsilon=u^{i_\varepsilon}v^{j_\varepsilon}$, $1\leqq\varepsilon\leqq e$; in particular $z_1=u^n$, $z_2=u^{n-q}v$ (here we view $A_{n,q}$ as a quotient of $\mathbb{C}^2$ with coordinates u,v). They satisfy the relations $z_\delta z_\varepsilon=z_{\delta+1}z_{\varepsilon-1}\prod_{\kappa=\delta+1}^{\varepsilon-1} z_\kappa^{a_\kappa-2}$, $2\leqq\delta+1\leqq\varepsilon-1\leqq e-1$. Cf. [7] for further information.

In [2] (cf. also [6]) it is shown that in the case $e=\operatorname{embdim}(X)\geqq 4$ the invariants

$$\lambda_\varepsilon^\alpha=z_\varepsilon^\alpha(U-V),$$
$$(\varepsilon,\alpha)\in\{(\varepsilon,1),\ldots,(\varepsilon,a_\varepsilon-1),3\leqq\varepsilon\leqq e-2\}\setminus\{(2,1),(e-1,1)\},$$
$$\lambda_2=z_2U,\quad \lambda_\varepsilon=z_\varepsilon(i_\varepsilon U+j_\varepsilon V),\ 3\leqq\varepsilon\leqq e-2,\quad \lambda_{e-1}=z_{e-1}V,$$

where

$$U=\frac{du}{u}\otimes\frac{du\wedge dv}{uv},\qquad V=\frac{dv}{v}\otimes\frac{du\wedge dv}{uv},$$

represent a basis of T_X^{1*}. The following Lemma tells us which of these elements are holomorphic on $\tilde{X}$.

Lemma. *Suppose* $2\leqq\varepsilon\leqq e-1$ *and* $(A,B)\in\mathbb{C}^2\setminus\{0\}$. *Then the pull-back of the invariant* $\lambda=z_\varepsilon^\alpha(AU+BV)$ *is in* $H^0(\tilde{X},\mathscr{F}_{\tilde{X}})$ *if and only if either of the following two conditions is fulfilled:*

1) $\alpha=1$, $3\leqq\varepsilon\leqq e-2$, $a_\varepsilon=2$, $A\hat{j}_\varepsilon+B\hat{i}_\varepsilon=0$,
2) $\alpha\geqq 2$.

Here the numbers $\hat{i}_\varepsilon$ *and* $\hat{j}_\varepsilon$ *are defined as* $\hat{i}_\varepsilon=i_\varepsilon-i_{\varepsilon+1}$, $\hat{j}_\varepsilon=j_{\varepsilon+1}-j_\varepsilon$.

Proof. On M_i we have $z_\varepsilon=u_i^{\beta_{\varepsilon i}}v_i^{\gamma_{\varepsilon i}}$ with nonnegative exponents. Moreover $\beta_{\varepsilon i}$, $\gamma_{\varepsilon i}\geqq 1$ if $\varepsilon\in\{2,\ldots,e-1\}$. Since $AU+BV=A_iU_i+B_iV_i$ on M_i (where U_i, V_i are defined like U,V with u_i,v_i instead of u,v), λ is holomorphic if $\alpha\geqq 2$.

Now consider the case $\alpha=1$. If $\varepsilon=2$ or $\varepsilon=e-1$, λ is not holomorphic on M_0 or M_r respectively, so we restrict to $3\leqq\varepsilon\leqq e-2$.

Suppose $a_\varepsilon\geqq 3$. Then there are two adjacent curves in the exceptional set along which z_ε vanishes to first order. Hence $z_\varepsilon=u_iv_i$ for some i, and λ cannot be holomorphic on M_i. Now if $a_\varepsilon=2$ we still have $\beta_{\varepsilon i}=1$ or $\gamma_{\varepsilon i}=1$ for some i. Then λ is not holomorphic on M_i unless $A_i=0$ (resp. $B_i=0$). Hence A and B have to satisfy a nontrivial linear condition. There is no other condition since

$$\lambda' := z_\varepsilon(\hat{i}_\varepsilon U - \hat{j}_\varepsilon V) \sim u_i^{\beta_{\varepsilon i}} v_i^{\gamma_{\varepsilon i}}((\beta_{\varepsilon+1,i} - \beta_{\varepsilon i})U_i + (\gamma_{\varepsilon+1,i} - \gamma_{\varepsilon i})V_i)$$

is holomorphic on $\tilde{X}$: for all i we have

$$\beta_{\varepsilon-1,i} + \beta_{\varepsilon+1,i} = a_\varepsilon \beta_{\varepsilon i} = 2\beta_{\varepsilon i},$$

hence $\beta_{\varepsilon i} \geqq 2$ or $\beta_{\varepsilon+1,i} - \beta_{\varepsilon i} = 0$, since all β's involved are $\geqq 1$; and the same holds for the γ's. This proves the Lemma.

It is easy to see that the invariant 2-forms $z_\varepsilon(uv)^{-1} du \wedge dv$, $2 \leqq \varepsilon \leqq e-1$, generate ω_X. Up to scalar factors, their pull-backs to $\tilde{X}$ are equal to $\omega_\varepsilon := z_\varepsilon(u_0 v_0)^{-1} du_0 \wedge dv_0$.

Proposition 7. *The tensors* $\hat{\lambda}_\varepsilon = d(z_{\varepsilon+1} z_\varepsilon^{-1}) \otimes \omega_{\varepsilon-1}$, $3 \leqq \varepsilon \leqq e-2$, *represent a basis of* $T_{\tilde{X}}^{\perp}$.

Proof.
We have

$$\hat{\lambda}_\varepsilon = \frac{z_{\varepsilon+1} z_{\varepsilon-1}}{z_\varepsilon} \left(\frac{dz_{\varepsilon+1}}{z_{\varepsilon+1}} - \frac{dz_\varepsilon}{z_\varepsilon} \right) \otimes \frac{\omega_2}{z_2} = z_\varepsilon^{a_\varepsilon - 1} \left(\frac{dz_{\varepsilon+1}}{z_{\varepsilon+1}} - \frac{dz_\varepsilon}{z_\varepsilon} \right) \otimes \frac{du_0 \wedge dv_0}{u_0 v_0}$$

hence $\hat{\lambda}_\varepsilon \in H^0(\tilde{X}, \mathscr{F}_{\tilde{X}})$ by the Lemma. By Proposition 4, we first have to show that, for each ν, $\hat{\lambda}_\varepsilon$ represents the zero element of $T_{\hat{X}_\nu}^{1*}$.

Before doing so, we shall have a closer look at the dot diagram described above. Think of each dot (ε, i) as replaced by the coordinate representation of z_ε in the two coordinate patches covering the i-th exceptional curve E_i (i.e. M_{i-1} and M_i). Then in columns with one dot only [corresponding to (-2)-curves] all exponents are 1 whereas in each of the remaining columns the exponents of either the u's (if i is odd) or the v's change by ± 1 from one row to the next. Remember also that $z_1 = v_0$; similarly, z_e is either u_r or v_r, depending on which one vanishes along E_r.

Now look at a maximal (-2)-configuration $\tilde{X}_\nu$ which does not contain either E_1 or E_r. We know explicitly how to blow down this configuration since we know this in general for cyclic quotients. From our discussion of the z_ε, we infer that in fact for a suitable δ the functions $(x, y, z) = (z_{\delta-1} z_\delta^{-1}, z_\delta, z_{\delta+1} z_\delta^{-1})$ blow down $\tilde{X}_\nu$ to the rational double point $\hat{X}_\nu = \{xz = y^{a_\delta - 2}\}$. Now, if $\delta \leqq \varepsilon$, then $\hat{\lambda}_\varepsilon \in H^0(\hat{X}_\nu, \mathscr{F}_{\hat{X}_\nu})$ follows from the fact that $z_{\varepsilon+1} z_\varepsilon^{-1}$ is a holomorphic function in x, y, z: for $\varepsilon = \delta$, this is clear, and, if $\delta < \varepsilon$, we have

$$\frac{z_{\varepsilon+1}}{z_\varepsilon} = z \frac{z_\delta z_{\varepsilon+1}}{z_{\delta+1} z_\varepsilon} = z \prod_{\kappa=\delta+1}^{\varepsilon} z_\kappa^{a_\kappa - 2}.$$

If $\varepsilon < \delta$, we use the same argument for $d(z_{\varepsilon-1} z_\varepsilon^{-1}) \otimes \omega_{\varepsilon+1}$ which is equal to $-\hat{\lambda}_\varepsilon$ in T_X^{1*}. Finally, if E_1 (resp. E_r) belongs to $\tilde{X}_\nu$, we have blowing-down functions $(x, y, z) = (z_1, z_2, z_3 z_2^{-1})$ [resp. $(z_{e-2} z_{e-1}^{-1}, z_{e-1}, z_e)$] and we can argue as above.

Using the basis of T_X^{1*}, one can show without difficulty that $\hat{\lambda}_\varepsilon$ is a (nonzero) $\mathbb{C}$-linear combination of the elements $\lambda_\varepsilon^{a_\varepsilon - 1}$ and λ_ε [modulo $\operatorname{im} H^0(X, \mathscr{F}_X)$], and that the $\hat{\lambda}_\varepsilon$ actually generate $T_{\tilde{X}}^{\perp}$ as a vector space.

Remark. Appending two more coordinate patches M_+, M_- according to

$$u_+ = (1+u_0)v_0^2, \quad u_- = (1-u_0)v_0^2, \quad v_+ = v_- = v_0^{-1}$$

to $\tilde{X}$, we get a manifold $\tilde{Y}$ which is blown down by the functions

$$y_1 = hz_2^2, \quad y_2 = h^{l_2}z_2z_3, \quad y_\varepsilon = h^{l_\varepsilon}z_\varepsilon, \quad 3 \leqq \varepsilon \leqq e,$$

where $h = 1 - u_0^{-2}$ and

$$l_2 = a_2 - [a_2/2], \quad l_3 = [a_2/2], \quad l_4 = a_3 l_3 - \frac{1+(-1)^{a_2}}{2},$$

$$l_{\varepsilon+1} = a_\varepsilon l_\varepsilon - l_{\varepsilon-1}, \quad 4 \leqq \varepsilon \leqq e-1,$$

to the dihedral singularity $D_{n,q}$. Putting

$$\eta_2 = \omega_2, \quad \eta_3 = y_1^{-1} y_2 \eta_2, \quad \eta_{\varepsilon+1} = y_{\varepsilon+1} y_\varepsilon^{-1} \eta_\varepsilon, 3 \leqq \varepsilon \leqq 3-2,$$

the elements $\hat{\kappa}_\varepsilon = d(y_{\varepsilon+1} y_\varepsilon^{-1}) \otimes \eta_{\varepsilon-1}$ represent a basis of $T_{\hat{Y}}^1$, where $\hat{Y}$ is the RDP resolution of $D_{n,q}$. Restricted to $\tilde{X}$, we have $\hat{\kappa}_\varepsilon = \hat{\lambda}_\varepsilon$ in T_X^{1*}. Cf. [8] for details.

References

1. Artin, M.: Algebraic construction of Brieskorn's resolutions. J. Algebra **29**, 330–348 (1974)
2. Behnke, K., Kahn, C., Riemenschneider, O.: Infinitesimal deformations of quotient surface singularities. In: Singularities. Banach Center Publ. **20**, 31–66. PWN-Polish Scient. Publ., Warsaw 1988
3. Behnke, K., Knörrer, H.: On infinitesimal deformations of rational surface singularities. Comp. Math. **61**, 103–127 (1987)
4. Lipman, J.: Double point resolutions of deformations of rational singularities. Compos. Math. **38**, 37–43 (1979)
5. Palamodov, V.P.: Deformations of complex spaces. Russ. Math. Surv. **31**, 129–197 (1976). Translated from: Usp. Mat. Nauk, **31**, 129–194 (1976)
6. Pinkham, H.: Deformations of quotient surface singularities. Proc. Symp. Pure Math. **30**, Part 1, 65–67 (1977)
7. Riemenschneider, O.: Deformationen von Quotientensingularitäten (nach zyklischen Gruppen). Math. Ann. **209**, 211–248 (1974)
8. Röhr, A.: Die infinitesimalen Deformationen der Diedersingularitäten. Diplomarbeit, Hamburg 1988
9. Schlessinger, M.: Rigidity of quotient singularities. Invent. Math. **14**, 17–26 (1971)
10. Steenbrink, J.: Mixed Hodge structure on the vanishing cohomology. In: Real and complex singularities, Oslo 1976. Proceedings of the Nordic Summer School NAVF
11. Wahl, J.: Vanishing theorems for resolutions of surface singularities. Invent. Math. **31**, 17–41 (1975)
12. Wahl, J.: Simultaneous resolution of rational singularities. Compos. Math. **38**, 43–54 (1979)

Received June 16, 1989