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INFINITESIMAL DEFORMATIONS OF QUOTIENT SURFACE SINGULARITIES

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A quotient surface singularity is determined by a finite group G acting linearly on a two-dimensional complex vector space V or, equivalently, by the corresponding action of G on the local ring $\mathcal{O}_{V,0}$ of holomorphic functions at 0, respectively on the associated graded ring which is isomorphic to the polynomial ring $S = \mathbf{C}[u, v]$. As a general philosophy, all analytic, algebraic, geometric and topological invariants associated to the singularity $X = (V/G, 0)$ should be computable by those or by corresponding actions. For instance, Henry Pinkham has shown in [7] how one can describe the vector space T_X^1 of (isomorphism classes of first order) infinitesimal deformations of X by the invariants of the action of G on S and on the cohomology groups

$$H^1(V\setminus\{0\}, \mathcal{O}_V) \quad \text{and} \quad H^1(V\setminus\{0\}, \Theta_V)$$

(where Θ_V denotes the sheaf of germs of holomorphic vector fields on V), using a result of Schlessinger ([10]) on T_X^1 for an arbitrary normal surface singularity (X, x). He was able to present a basis for T_X^1 in the case of a cyclic group, reestablishing a dimension formula of the last author ([8]), and two of the authors of the present note used his method to find such a basis for the class of "dihedral" singularities ([2]). However, in the remaining cases it seemed to be hopeless to calculate T_X^1 by the same method, even in the case of the binary polyhedral groups $\boldsymbol{T}, \boldsymbol{O}, \boldsymbol{I}$ (i.e., the finite subgroups of $\mathrm{SL}(2, \boldsymbol{C})$), where the dual of T_X^1 can easily be gotten from a defining equation $R(x, y, z) = 0$ for X by the isomorphism

$$(1) \qquad (T_X^1)^* \cong \boldsymbol{C}[x, y, z]/\left(R, \frac{\partial R}{\partial x}, \frac{\partial R}{\partial y}, \frac{\partial R}{\partial z}\right).$$

In a first version of this paper written in 1983 it was shown how one can find a basis of the invariant vector spaces

$$H^1(V\backslash\{0\}, \mathcal{O}_V)^G \quad \text{and} \quad H^1(V\backslash\{0\}, \Theta_V)^G$$

for finite groups $G \subset \mathrm{GL}(2, \boldsymbol{C})$ in a canonical way via easily computable invariants like S^G. Here, we used the fact that the whole general linear group $\mathrm{GL}(2, \boldsymbol{C})$ acts on the cohomology groups envolved, especially by applying the Clebsch–Gordan formulas. With these bases and generators of S^G, it is possible to determine explicitly the linear mappings which enter the description of T_X^1 by Pinkham, making his result a little more concrete (Theorem 1). However, from the computational point of view, the systems of linear equations one obtains this way, are still not tractable, unless one has a priori bounds for the degrees of the base elements of T_X^1 or one knows the dimension of T_X^1 (as in the case of quotients by the binary polyhedral groups, i.e., for the rational double points).

It was remarked by Horst Knörrer that our results become much smoother, if one applies them to the dual space $(T_X^1)^*$: Using elementary duality for $\mathrm{GL}(2, \boldsymbol{C})$-modules, we deduce from Pinkham's result a description of $(T_X^1)^*$ as the cokernel of a natural mapping (Theorem 2). This result, however, is only a special case of a general formula for $(T_X^1)^*$ in the case of an arbitrary normal surface singularity (X, x) which was obtained by H. Knörrer and the first author in [1] by dualizing Schlessinger's description of T_X^1. Working on the resolution of the singularity (X, x), they were able to determine the dimension of T_X^1 for a fairly wide class of rational singularities including the quotient singularities with the exception of 63 individual cases with low embedding dimension. On the other hand, the second author found an algorithm for computing the dimension of T_X^1 for all series of quotient surface singularities, using only invariant theory ([5]). We include his tables which were obtained with the aid of a personal computer at the end of Section 7 of this paper.

Applying again the Clebsch–Gordan isomorphisms we get a natural splitting of $(T_X^1)^*$ into two parts (Theorem 3). For rational double points, the second part is trivial and the first part identifies (see Section 5) with

$$(2) \qquad S^G/\text{ideal generated by all Jacobians } J(f_1, f_2),\ f_1, f_2 \in S^G,$$

where

$$J(f_1, f_2) = \det \begin{bmatrix} \partial_u f_1 & \partial_v f_1 \\ \partial_u f_2 & \partial_v f_2 \end{bmatrix}.$$

Of course, formula (2) can easily be deduced from (1). Moreover, we should mention that (2) is only a special case of a formula due to J. Wahl ([15]) which holds for all quasi-homogeneous Gorenstein surface singularities (X, x): In this case, S^G has to be replaced by the dualizing module $\omega_{X,x}$ of the singularity and the ideal by the module generated by all exterior products $df_j \wedge df_k$ for functions $f_1, \ldots, f_e$ generating the maximal ideal $\mathfrak{m}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$. So, denoting by $\Omega^1_{X,x}$ the $\mathcal{O}_{X,x}$-module of Kähler differentials at $x \in X$, (2) should be interpreted as the canonical isomorphism

(3) $$(T^1_X)^* \cong \omega_{X,x}/\text{image of } \Omega^2_{X,x}.$$

Motivated by Wahl's formula we generalize our splitting result in Section 8 to some finite quotients of quasi-homogeneous Gorenstein surface singularities (including all quasi-homogeneous rational singularities of embedding dimension $e \geq 4$) and show that the second part has always a dimension equal to $\dim_{\mathbf{C}}(\omega_{X,x}/\mathfrak{m}_{X,x}\omega_{X,x})$, i.e., to the minimal number of generators for the dualizing module (which equals $e-2$ in the rational case).

In Section 6 we sketch two methods how one can deal, at least in principle, with the first part and apply the second one to the cyclic case. Finally, in Section 7, we compute this part for a complete series of quotients by the groups $\boldsymbol{T}\cdot(\boldsymbol{Z}/2m\boldsymbol{Z})$, $m = 6(b-2)+1$, $b \geq 3$, and list the results of [5].

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1. Some standard GL(2, C)-modules

For the convenience of the reader and in order to fix our notations, we describe some well-known representations of the general linear group $GL_2 = GL(2, \mathbf{C})$. We choose once and for all a basis e_1, e_2 of the vector space V and let the group element

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in GL_2$$

act on a vector $ue_1 + ve_2$ by

(4) $$g(ue_1 + ve_2) = (\alpha u + \beta v)e_1 + (\gamma u + \delta v)e_2 =: u'e_1 + v'e_2,$$

or, equivalently, by

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

This gives the standard left representation $G \times V \to V$ which in turn yields a left representation on the dual vector space V^* by the condition

$$\langle g(\lambda), g(\underline{w}) \rangle = \langle \lambda, \underline{w} \rangle, \quad \lambda \in V^*, \ \underline{w} \in V.$$

This representation is given by $(g, \lambda) \mapsto \lambda \circ g^{-1}$ and will be described via the dual basis e_1^*, e_2^* by the contragredient matrix

$${}^t g^{-1} = \frac{1}{\Delta(g)} \begin{bmatrix} \delta & -\gamma \\ -\beta & \alpha \end{bmatrix}, \quad \Delta(g) = \det g = \alpha\delta - \beta\gamma.$$

Denoting as above coordinates of V by u and v, we can interpret e_1^* and e_2^* as the coordinate functions which we will also denote by u and v, respectively. So, we identify V^* with the vector space of linear functions $au + bv$, a, $b \in \mathbf{C}$.

Having a representation on a vector space one can induce representations on the symmetric and exterior powers. In this note we use S_l for the lth symmetric power of V^*:

$$S_l = S_l(V^*)$$

which we identify with the vector space of homogeneous polynomials of degree l in u and v with the basis

$$e_{jk}^* = u^j v^k, \quad j + k = l.$$

The action of GL_2 on S_l is given by

$$g(u^j v^k) = \frac{1}{\Delta(g)^l} (\delta u - \beta v)^j (-\gamma u + \alpha v)^k \tag{5}$$

and induces a $\mathbf{C}$-algebra representation of GL_2 on the polynomial ring

$$S = \bigoplus_{l=0}^{\infty} S_l = \mathbf{C}[u, v].$$

More generally, if $f \in \mathcal{O}_{V,0}$ is a germ of a holomorphic function at the origin, we can define $g(f) = f \circ g^{-1} \in \mathcal{O}_{V,0}$ for all $g \in \mathrm{GL}_2$. In this way one gets a canonical left GL_2-representation which maps the lth power $\mathfrak{m}^l$ of the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{V,0}$ into itself. Hence, this action induces a left representation of GL_2 on the associated graded ring

$$\mathrm{Gr}\, \mathcal{O}_{V,0} = \bigoplus_{l=0}^{\infty} \mathfrak{m}^l / \mathfrak{m}^{l+1}$$

which is – as GL_2-algebra – isomorphic to S. One can look at this isomorphism also from another point of view: The multiplicative group C^* acts via

$$C^* \ni c \mapsto g_c = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \in \mathrm{GL}_2$$

on every GL_2-module M, such that we have the eigenspaces

$$M_l = \{m \in M:\ g_c(m) = c^{-l} m \text{ for all } c \in C^*\}, \qquad l \in Z,$$

of homogeneous elements of degree l. Since the operations of C^* and GL_2 commute, GL_2 operates also on the associated graded module

$$M_{\mathrm{gr}} = \bigoplus_{l=-\infty}^{\infty} M_l$$

and on the module

$$M_+ = \bigoplus_{l=1}^{\infty} M_l$$

consisting of homogeneous elements of positive degree. Of course, in the case above, $S_l \cong (\mathcal{O}_{V,0})_l$ and

$$(\mathcal{O}_{V,0})_{\mathrm{gr}} \cong \bigoplus_{l=0}^{\infty} S_l \cong C[u, v] \cong \mathrm{Gr}\, \mathcal{O}_{V,0}.$$

For certain purposes it is convenient to identify V^* with the cotangent space of V at 0 via

$$T^*_{V,0} \cong \mathfrak{m}/\mathfrak{m}^2 \cong S_1(V^*) \cong V^*$$

where the differential du is mapped to e_1^* and dv goes to e_2^*. If we provide $T^*_{V,0}$ with the natural GL_2-action

(6) $$g(d\lambda) = (d\lambda) \circ g^{-1} = d(\lambda \circ g^{-1}) = d(g(\lambda)),$$

$\lambda = au + bv$, $a, b \in C$, this isomorphism is GL_2-equivariant. Dualizing again, we get a GL_2-isomorphism

$$V \cong T_{V,0}$$

where $e_1 \mapsto \partial_u = \dfrac{\partial}{\partial u}$, $e_2 \mapsto \partial_v = \dfrac{\partial}{\partial v}$. We denote by

$$S^* = \bigoplus_{l=0}^{\infty} S_l^*, \qquad S_l^* = S_l(T_{V,0})$$

the algebra of linear differential operators with constant coefficients:

$$\partial \in S_l^* \quad \Leftrightarrow \quad \partial = \sum_{j+k=l} a_{jk}\, \partial_u^j \partial_v^k$$

with the GL_2-action given by

(7) $$g(\partial_u^j \partial_v^k) = (\alpha\partial_u + \gamma\partial_v)^j (\beta\partial_u + \delta\partial_v)^k.$$

S_l^* is in a canonical way the dual of S_l, the pairing being given by

$$\langle \partial_u^j \partial_v^k, u^\iota v^\varkappa \rangle = \begin{cases} j!\,k!, & \text{if } j = \iota,\ k = \varkappa, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, since

$$g(du \wedge dv) = g(du) \wedge g(dv) = \frac{1}{\Delta(g)^2}(\delta\, du - \beta\, dv) \wedge (-\gamma\, du + \alpha\, dv),$$

we get on $\Lambda^2 T_{V,0}^*$ the representation

(8) $$g(du \wedge dv) = \Delta(g)^{-1}\, du \wedge dv,$$

and on $\Lambda^2 T_{V,0}$ there is the dual representation

(9) $$g(\partial_u \wedge \partial_v) = \Delta(g)\, \partial_u \wedge \partial_v.$$

Now, for any GL_2-module M, there is a natural action of GL_2 on $M \otimes \Lambda^2 T_{V,0}^*$. Identifying this tensor product with M via $m \mapsto m \otimes (du \wedge dv)$, we get a new action on M which is given by

$$(g, m) \mapsto \chi(g) \cdot g(m).$$

Here χ stands for the character

(10) $$\chi\colon GL_2 \to C^*, \quad \chi(g) = \Delta(g)^{-1}.$$

To distinguish the new GL_2-module from the old one, we denote it by M_χ. In the same manner we construct M_{χ^k} for all $k \in Z$.

We notice that there exists a natural isomorphism $V^* \xrightarrow{\sim} V \otimes \Lambda^2 V^*$ given in coordinates by $e_1^* \mapsto e_2 \otimes (e_1^* \wedge e_2^*)$, $e_2^* \mapsto -e_1 \otimes (e_1^* \wedge e_2^*)$. Hence, it induces isomorphisms of GL_2-modules $S_l \xrightarrow{\sim} S_{l,\chi^l}^*$ by

(11) $$u^j v^k \mapsto (-1)^k \partial_u^k \partial_v^j, \quad j + k = l.$$

2. The GL_2-modules $H^1(V \setminus \{0\}, \mathcal{O}_V)$ and $H^1(V \setminus \{0\}, \Theta_V)$ and their duals

We realize cohomology classes on $U := V \setminus \{0\}$ by 1-cocycles via the canonical isomorphism

$$H^1(U, \mathscr{F}) \cong H^1(\mathscr{W}, \mathscr{F}) = Z^1(\mathscr{W}, \mathscr{F})/B^1(\mathscr{W}, \mathscr{F})$$

which holds for any coherent analytic sheaf $\mathscr{F}$ on U and for arbitrary Stein coverings $\mathscr{W} = \{W_\sigma\}_{\sigma \in \Sigma}$ of U. So, we write for a cohomology class $\xi \in H^1(U, \mathscr{F})$:

$$\xi = [f_{\sigma\tau}], \quad f_{\sigma\tau} \in H^0(W_\sigma \cap W_\tau, \mathscr{F}).$$

For $\mathscr{F} = \mathcal{O}_V$, GL_2 acts on the first cohomology group as follows: If $\xi = [f_{\sigma\tau}]$, $f_{\sigma\tau} \in H^0(W_\sigma \cap W_\tau, \mathcal{O}_V)$, then

$$g(\xi) = [f_{\sigma\tau} \circ g^{-1}] = [g(f_{\sigma\tau})]$$

where $g(f_{\sigma\tau}) \in Z^1(g(\mathscr{W}), \mathcal{O}_V)$, $g(\mathscr{W}) = \{g(W_\sigma)\}_{\sigma \in \Sigma}$. It is easily seen that this action is independent of the Stein covering $\mathscr{W}$. Using in the following the special covering $\mathscr{U} = \{U_1, U_2\}$ with

$$U_1 = \{(u, v) \colon u \neq 0\}, \qquad U_2 = \{(u, v) \colon v \neq 0\},$$

it is immediately clear that an element

$$g_1 = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} \in GL_2$$

acts on the special class

$$\xi_{jk} = \left[\frac{1}{u^{j+1} v^{k+1}}\right], \qquad j, k \geqslant 0$$

by

$$g_1(\xi_{jk}) = \alpha^{j+1} \delta^{k+1} \xi_{jk}. \tag{12}$$

So, by our conventions, ξ_{jk} is homogeneous of negative degree $-(j+k+2)$, and it is well known that the ξ_{jk} with fixed $l = j+k$ form a basis of the homogeneous part of $H^1(U, \mathcal{O}_V)$ of degree $-(l+2)$. In the following we write

$$H^1(U, \mathcal{O}_V)_{gr} := \bigoplus_{l=0}^{\infty} H^1(\mathcal{O}_U)_{-(l+2)}$$

and determine the structure of the GL_2-modules $H^1(\mathcal{O}_U)_{-(l+2)}$. To do this we have to compute the action of the special matrices

$$g_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad g_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

which together with the matrices g_1 generate the group GL_2. In the case of g_2 we have $g_2(\mathscr{U}) = \{U_2, U_1\}$, $g_2(u) = v$, $g_2(v) = -u$, and therefore

$$\begin{aligned} g_2(\xi_{jk}) &= (-1)^{k+1}\left[\frac{1}{u^{k+1} v^{j+1}}\right] \quad \text{(with respect to } g_2(\mathscr{U})) \\ &= (-1)^k \xi_{kj}. \end{aligned} \tag{13}$$

By the following calculation (which explains itself):

$$\begin{aligned} \frac{1}{(u-v)^{j+1} v^{k+1}} &= \frac{1}{u^{j+1} v^{k+1}} \sum_{\lambda=0}^{\infty} \binom{-j-1}{\lambda} (-1)^\lambda \frac{v^\lambda}{u^\lambda} \\ &= \frac{1}{u^{j+1} v^{k+1}} \sum_{\lambda=0}^{\infty} \binom{j+\lambda}{\lambda} \frac{v^\lambda}{u^\lambda}, \end{aligned}$$

we finally get

$$g_3(\xi_{jk}) = \left[\frac{1}{(u-v)^{j+1}v^{k+1}}\right] \quad \text{(with respect to } g_3(\mathcal{U})) \tag{14}$$

$$= \sum_{\lambda=0}^{k} \binom{j+\lambda}{\lambda} \xi_{j+\lambda,k-\lambda}.$$

We are now able to prove

LEMMA 1. *The C-linear isomorphism*

$$\varphi\colon S^*_{l,\chi-1} \to H^1(\mathcal{O}_U)_{-(l+2)}$$

defined by

$$\varphi(\partial_{jk}) = \xi_{jk}, \quad \partial_{jk} = \frac{1}{j!\,k!}\partial_u^j \partial_v^k \tag{15}$$

is GL_2*-equivariant.*

Proof. We have to show that $g_r \circ \varphi = \varphi \circ g_r$ for $r = 1, 2, 3$, which is trivial in the first two cases. For the last case this results from the following elementary computation:

$$\begin{aligned}(g_3 \circ \varphi)(\partial_u^j \partial_v^k) &= j!\,k! \sum_{\lambda=0}^{k} \binom{j+\lambda}{\lambda} \xi_{j+\lambda,k-\lambda} \\ &= \sum_{\lambda=0}^{k} \frac{j!\,k!}{(j+\lambda)!\,(k-\lambda)!} \binom{j+\lambda}{\lambda} \varphi(\partial_u^{j+\lambda} \partial_v^{k-\lambda}) \\ &= \varphi\Big(\sum_{\lambda=0}^{k} \binom{k}{\lambda} \partial_u^{j+\lambda} \partial_v^{k-\lambda}\Big) \\ &= \varphi(\partial_u^j (\partial_u + \partial_v)^k) \\ &= \varphi \circ g_3(\partial_u^j \partial_v^k).\end{aligned}$$

□

This result can be explained (and could have been derived) more naturally in the following way: Denote by Ω_V^2 the sheaf of germs of holomorphic 2-forms such that as GL_2-modules

$$H^0(V, \Omega_V^2) \cong H^0(V, \mathcal{O}_V) \otimes \Lambda^2 T^*_{V,0}$$

and

$$H^0(V, \Omega_V^2)_{\mathrm{gr}} \cong S \otimes \Lambda^2 T^*_{V,0} \cong \bigoplus_{l=0}^{\infty} S_{l,\chi} \cong \Big(\bigoplus_{l=0}^{\infty} S^*_{l,\chi-1}\Big)^*.$$

So, Lemma 1 establishes a concrete GL_2-equivariant pairing

$$H^0(V, \Omega_V^2)_{\mathrm{gr}} \times H^1(U, \mathcal{O}_V)_{\mathrm{gr}} \to C,$$

sending $(u^j v^k) \otimes (du \wedge dv) \times \xi_{\iota\varkappa}$ to 1, if $\iota = j+1$, $\varkappa = k+1$, and to 0 otherwise.

This pairing is induced, of course, by the natural residue pairing

$$H^0(V, \Omega_V^2) \times H^1(U, \mathcal{O}_V) \to C$$

associating to each pair $\omega = h\,du \wedge dv$, $\xi = [f_{12}]$ the residue of $f_{12} h\,du \wedge dv$ at the origin (in the sense of [4], p. 649 ff.) This pairing can be understood even in a more conceptual framework. By the long exact sequence for local cohomology and Theorem B for Stein manifolds, we get an exact sequence

$$0 = H^1(V, \mathcal{O}_V) \to H^1(U, \mathcal{O}_V) \to H^2_{\{0\}}(\mathcal{O}_V) \to H^2(V, \mathcal{O}_V) = 0,$$

and by local duality we have a canonical isomorphism

$$H^2_{\{0\}}(\mathcal{O}_V)^* \cong \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{O}_V, \Omega_V^2) \cong H^0(V, \Omega_V^2).$$

We remark that, by the same reasoning, one gets natural isomorphisms

$$H^1(X', \mathcal{F})^* \cong H^0(X', \mathcal{F}^* \otimes \Omega_{X'}^2)$$

where X is a normal two-dimensional Stein space with exactly one isolated singularity x, $X' = X \setminus \{x\}$ and $\mathcal{F}$ is a coherent reflexive sheaf on X.

In order to calculate the GL_2-module $H^1(U, \Theta_V)_{gr}$, we observe that

$$H^1(U, \Theta_V) \cong H^1(U, \mathcal{O}_V) \otimes T_{V,0}$$

as GL_2-modules. Hence, we have a GL_2-equivariant isomorphism

$$\psi\colon S^*_{\chi-1} \otimes S^*_1 \xrightarrow{\sim} H^1(U, \Theta_V)_{gr}$$

given by $\psi(P \otimes \partial) = \varphi(P) \otimes \partial$. But, by the classical Clebsch–Gordan isomorphisms (which can be found in [11] and explicitly in [12], e.g.), we have

$$S^*_{l-1,\chi-2} \oplus S^*_{l+1,\chi-1} \cong S^*_{l,\chi-1} \otimes S^*_1$$

for $l \geqslant 0$ where the isomorphism is given in the form $\delta \oplus \iota$ with

$$(16) \qquad \begin{aligned} \delta(\partial_u^j \partial_v^k) &= \partial_u^j \partial_v^{k+1} \otimes \partial_u - \partial_u^{j+1} \partial_v^k \otimes \partial_v, & j+k &= l-1, \\ \iota(\partial_u^j \partial_v^k) &= j\partial_u^{j-1} \partial_v^k \otimes \partial_u + k\partial_u^j \partial_v^{k-1} \otimes \partial_v, & j+k &= l+1. \end{aligned}$$

Since elements in S_l^* have degree $-l$, $H^1(U, \Theta_V)_{gr}$ has only negatively graded homogeneous parts:

$$H^1(U, \Theta_V)_{gr} := \bigoplus_{l=0}^{\infty} H^1(\Theta_U)_{-(l+1)}.$$

Combining our results we finally get

PROPOSITION 1. *The C-linear isomorphism*

$$D^* \oplus I^*\colon S^*_{l-1,\chi-2} \oplus S^*_{l+1,\chi-1} \to H^1(\Theta_U)_{-(l+1)},$$

given by

(17) $$D^*(\partial_{jk}) = (k+1)\xi_{j,k+1}\otimes\partial_u - (j+1)\xi_{j+1,k}\otimes\partial_v, \quad j+k = l-1,$$

and

(18) $$I^*(\partial_{jk}) = \xi_{j-1,k}\otimes\partial_u + \xi_{j,k-1}\otimes\partial_v, \quad j+k = l+1,$$

is GL_2*-equivariant for all* $l \geqslant 0$.

Again, it is interesting to note that the dual version is much easier to describe. We notice first that

$$H^1(U, \Theta_V)^* \cong (H^1(U, \mathcal{O}_V)\otimes T_{V,0})^* \cong H^0(V, \Omega_V^2)\otimes T_{V,0}^*$$
$$\cong H^0(V, \Omega_V^1\otimes\Omega_V^2),$$

where Ω_V^1 denotes the sheaf of germs of holomorphic 1-forms on V. Hence, by Proposition 1 there is a GL_2-equivariant isomorphism of vector spaces

$$H^0(V, \Omega_V^1\otimes\Omega_V^2)_{\mathrm{gr}} \xrightarrow{\sim} H^0(V, \Omega_V^2\otimes\Omega_V^2)_{\mathrm{gr}} \oplus H^0(V, \Omega_V^2)_+.$$

It is easy to check that this isomorphism is induced by an isomorphism

$$H^0(V, \Omega_V^1)_{\mathrm{gr}} \xrightarrow{\sim} H^0(V, \Omega_V^2)_{\mathrm{gr}} \oplus H^0(V, \mathcal{O}_V)_+$$

which is given, if represented in the form

$$\omega = f\,du + g\,dv \mapsto (D(\omega), I(\omega)),$$

by *differentiation*

$$D(\omega) = d\omega = df\wedge du + dg\wedge dv = (\partial_u g - \partial_v f)\,du\wedge dv$$

and *integration*

$$I(\omega) = uf + vg.$$

So, we have proved the following version of the Clebsch–Gordan isomorphism:

Proposition 2. *The* $\mathbf{C}$*-linear mappings*

$$H^0(V, \Omega_V^1)_{\mathrm{gr}} \to S_\chi \oplus S_+,$$
$$H^0(V, \Omega_V^1\otimes\Omega_V^2)_{\mathrm{gr}} \to S_{\chi^2} \oplus S_{+,\chi}$$

defined by

(19) $$(f\,du + g\,dv) \mapsto (\partial_u g - \partial_v f,\ uf + vg),$$
$$(f\,du + g\,dv)\otimes(h\,du\wedge dv) \mapsto (\partial_u(hg) - \partial_v(hf),\ h(uf+vg))$$

are GL_2*-equivariant isomorphisms.*

Of course, one can prove Proposition 2 directly. It is easy to see that the mappings are GL_2-equivariant. Moreover, after counting dimensions in any degree, it suffices to prove that the first mapping is injective. But, if f and g are homogeneous, f of degree l, say, then

$$\partial_u g - \partial_v f = 0 = uf + vg$$

implies by Euler's relation

$$0 = \partial_u(uf+vg) = f + u\partial_u f + v\partial_u g = f + (u\partial_u f + v\partial_v f) = (l+1)f,$$

hence $f = 0$, and also $g = 0$.

There is still another reason for this form of Clebsch–Gordan isomorphism: Since u, v form a regular sequence in $\mathcal{O}_{V,0}$, there exists a finite free resolution of the $\mathcal{O}_V$-module $\mathfrak{m}_{V,0}$ by the Koszul complex which might be written as

$$0 \to \Omega_V^2 \xrightarrow{E} \Omega_V^1 \xrightarrow{I} \mathfrak{m}_{V,0} \to 0, \tag{20}$$

where E and I are defined by contraction with the GL_2-invariant Euler derivation $u\partial_u + v\partial_v$, such that

$$E(h\,du \wedge dv) = h(u\,dv - v\,du) \tag{21}$$

is multiplication by the form $u\,dv - v\,du$ (which transforms under GL_2 by multiplication of the determinant) and I is given as above by integration:

$$I(f\,du + g\,dv) = uf + vg. \tag{22}$$

Hence, (20) is a GL_2-equivariant exact sequence of $\mathcal{O}_V$-modules. Moreover, the corresponding graded sequence splits as a sequence of $\operatorname{Gr}\mathcal{O}_{V,0}$-modules, since for homogeneous h of degree l we have

$$\begin{aligned} D \circ E(h\,du \wedge dv) &= D(hu\,dv - hv\,du) \\ &= (\partial_u(hu) + \partial_v(hv))\,du \wedge dv \\ &= (l+2)\,h\,du \wedge dv \end{aligned}$$

and

$$\begin{aligned} (I \circ d)(h) &= I((\partial_u h)\,du + (\partial_v h)\,dv) \\ &= u\partial_u h + v\partial_v h \\ &= lh. \end{aligned}$$

3. Reformulation of Pinkham's description of T_X^1

For any $f \in H^0(V, \mathcal{O}_V)$ there exists a canonical cup-product mapping

$$H^1(U, \mathcal{O}_V) \xrightarrow{\cup f} H^1(U, \mathcal{O}_V),$$

which can be defined by

(23) $$[f_{12}] \cup f = [f_{12} f].$$

If f is a homogeneous polynomial of degree d, then

$$(\bigcup f)\colon H^1(\mathcal{O}_U)_{-(l+2)} \to H^1(\mathcal{O}_U)_{-(l-d+2)},$$

and the composition

$$S^*_{l,\chi^{-1}} \xrightarrow{\sim} H^1(\mathcal{O}_U)_{-(l+2)} \to H^1(\mathcal{O}_U)_{-(l-d+2)} \xrightarrow{\sim} S^*_{l-d,\chi^{-1}}$$

is again denoted by $\bigcup f$. Obviously

(24) $$\partial_{jk} \cup u^r v^s = \partial_{j-r,k-s}, \quad j \geqslant r,\ k \geqslant s,$$

and zero otherwise. By the cup-product one can define a new mapping

$$\bigcup f\colon H^1(U, \Theta_V) \to H^1(U, \mathcal{O}_V),$$

setting

(25) $$(\xi \otimes \partial_u + \eta \otimes \partial_v) \cup f = \xi \cup (\partial_u f) + \eta \cup (\partial_v f).$$

Again, if $f \in S_d$, the cup-product respects the grading. We denote the composition

$$S^*_{\chi^{-2}} \xrightarrow{D^*} H^1(U, \Theta_V)_{\mathrm{gr}} \xrightarrow{\cup f} H^1(U, \mathcal{O}_V)_{\mathrm{gr}} \xrightarrow{\sim} S^*_{\chi^{-1}}$$

by $\bigcup_D f$, and the corresponding composition with I^* by

$$\bigcup_I f\colon S^*_{+,\chi^{-1}} = \bigoplus_{l=1}^{\infty} S^*_{l,\chi^{-1}} \to S^*_{\chi^{-1}}.$$

It is simply checked that

(26)
$$\partial_{jk} \bigcup_D u^r v^s = [(k+1)r - (j+1)s]\, \partial_{j-r+1,k-s+1},$$
$$\partial_{jk} \bigcup_I u^r v^s = (r+s)\, \partial_{j-r,k-s}.$$

Now let $G \subset \mathrm{GL}_2$ be any finite subgroup. Via the embedding in GL_2, G acts on every GL_2-module M. We denote by M^G the absolute invariants:

$$M^G = \{m \in M\colon g(m) = m \text{ for all } g \in G\}.$$

By the definition of M_{χ^k},

$$M^G_{\chi^k} := (M_{\chi^k})^G = \{m \in M\colon g(m) = \Delta(g)^k m \text{ for all } g \in G\}.$$

In particular, the invariant algebra S^G is finitely generated by a (minimal) system of homogeneous polynomials $P_1, \ldots, P_e$, and it is easily seen that

$$\bigcup P_\varepsilon\colon H^1(U, \Theta_V)^G \to H^1(U, \mathcal{O}_V)^G.$$

Pinkham proved in [7] that the vector space T_X^1 of first order infinitesimal deformations of the singularity of $X = V/G$ at the origin is equal to the intersection of the kernels of these mappings. Therefore we get

THEOREM 1. *Let $G \subset \mathrm{GL}_2$ be a finite subgroup, and let $P_1, \ldots, P_e$ be a minimal set of generators for the invariant algebra S^G. Then the vector space T_X^1 of infinitesimal deformations of the quotient singularity $X = V/G$ is isomorphic to the kernel of the C-linear mapping*

$$\alpha^*\colon (S^*_{\chi-2})^G \oplus (S^*_{+,\chi-1})^G \to \bigoplus_{\varepsilon=1}^{e} (S^*_{\chi-1})^G$$

which is defined by

$$\alpha^*(\partial_1, \partial_2) = (\delta_1, \ldots, \delta_e), \qquad \delta_\varepsilon = (\partial_1 \underset{D}{\cup} P_\varepsilon) + (\partial_2 \underset{I}{\cup} P_\varepsilon).$$

Since it is very easy to compute bases for all invariant spaces S^G_{l,χ^k} (see e.g. [9], [5]), the determination of the infinitesimal deformations of fixed degree is reduced to the solution of a concrete system of linear equations. So, if one knows an a priori bound for these degrees, Theorem 1 gives one an effective way to compute T_X^1 for a given group. We have done these computations for the binary polyhedral groups T, O and I, where we found the more or less expected results. However, such computations are much simpler, if one dualizes Theorem 1, as was remarked by H. Knörrer.

4. Dualization of the main result

For $f \in H^0(V, \mathcal{O}_V)$, the dual mapping

$$H^0(V, \Omega_V^2) \cong H^1(U, \mathcal{O}_V)^* \to H^1(U, \mathcal{O}_V)^* \cong H^0(V, \Omega_V^2)$$

is just multiplication by f, and

$$H^0(V, \Omega_V^2) \cong H^1(U, \mathcal{O}_V)^* \to H^1(U, \Theta_V)^* \cong H^0(V, \Omega_V^1 \otimes \Omega_V^2)$$

(the middle map in both lines being $(\cup f)^*$) is given by

$$\omega \mapsto df \otimes \omega,$$

$\omega \in H^0(V, \Omega_V^2)$, $df = (\partial_u f)\,du + (\partial_v f)\,dv$. So, dualizing Pinkham's description, we obtain

THEOREM 2. *Let $G \subset \mathrm{GL}_2$ be a finite subgroup, and let $P_1, \ldots, P_e$ be a (minimal) set of homogeneous generators of the invariant algebra S^G. Then, the dual $(T_X^1)^*$ is canonically isomorphic to the cokernel of the S^G-module homomorphism*

$$\bigoplus_{\varepsilon=1}^{e} H^0(V, \Omega_V^2)^G \to H^0(V, \Omega_V^1 \otimes \Omega_V^2)^G$$

given by

$$(\omega_1, \ldots, \omega_e) \mapsto \sum_{\varepsilon=1}^{e} dP_\varepsilon \otimes \omega_\varepsilon. \tag{27}$$

By Hartog's Theorem, $H^0(U, \mathscr{F}) \cong H^0(V, \mathscr{F})$ for all free $\mathcal{O}_V$-modules $\mathscr{F}$ of finite rank. Therefore, if we denote V/G by X and the regular part U/G of X by X', we get

$$H^0(V, \Omega_V^2)^G \cong H^0(X', \Omega_{X'}^2) \cong H^1(X', \mathcal{O}_X)^*$$

and

$$H^0(V, \Omega_V^1 \otimes \Omega_V^2)^G \cong H^0(X', \Omega_{X'}^1 \otimes \Omega_{X'}^2) \cong H^1(X', \Theta_{X'})^*,$$

such that Theorem 2 also could have been proven directly by dualizing Schlessinger's exact sequence for T_X^1 (see [1]).

Combining Propositions 1 and 2, we get the following dual version of Theorem 1:

PROPOSITION 3. *Let $G \subset \mathrm{GL}_2$ be a finite subgroup, and let $P_1, \ldots, P_e$ be a minimal set of homogeneous generators of the invariant algebra S^G, of degree $l_1, \ldots, l_e$, say. Then the dual vector space $(T_X^1)^*$ is canonically isomorphic to the cokernel of the* **C***-linear mapping*

$$\alpha\colon \bigoplus_{\varepsilon=1}^{e} S_\chi^G \to S_{\chi^2}^G \oplus S_{+,\chi}^G$$

where

$$\alpha(\omega_1, \ldots, \omega_e) = (J(\omega_1, \ldots, \omega_e), K(\omega_1, \ldots, \omega_e)),$$

$$J(\omega_1, \ldots, \omega_e) = \sum_{\varepsilon=1}^{e} J(P_\varepsilon, \omega_\varepsilon), \qquad J(P, \omega) = \partial_u P \cdot \partial_v \omega - \partial_v P \cdot \partial_u \omega, \tag{28}$$

$$K(\omega_1, \ldots, \omega_e) = \sum_{\varepsilon=1}^{e} l_\varepsilon P_\varepsilon \omega_\varepsilon.$$

In order to obtain concrete bases for $(T_X^1)^*$ or at least dimension formulas, it is necessary to analyze the mappings J and K in Proposition 3 a little further. Notice that K is a S^G-module homomorphism, but J is certainly not. However, the following is true:

THEOREM 3. *Under the assumptions of Theorem 2 the set*

$$\mathscr{J} = \{h \in S_{\chi^2}^G\colon h = J(\omega_1, \ldots, \omega_e), K(\omega_1, \ldots, \omega_e) = 0\}$$

is a S^G-submodule of $S_{\chi^2}^G$ which contains all elements

$$J(f, g)\omega, \quad f, g \in S^G, \quad \omega \in S_\chi^G.$$

$(T_X^1)^*$ fits as S^G-module into an exact sequence

$$0 \to S_{\chi^2}^G/\mathscr{J} \to (T_X^1)^* \to (S_{+,\chi}^G/\operatorname{im} K) \to 0. \tag{29}$$

Proof. Since $\mathscr{J} = \operatorname{im}(J \mid \ker K)$, the exactness of the S^G-sequence (29) follows from the first assertion. The mapping K may be, as S^G-module homomorphism, identified with the composition

$$\bigoplus_{\varepsilon=1}^{e} H^0(V, \Omega_V^2)^G \xrightarrow{\mu} H^0(V, \Omega_V^1 \otimes \Omega_V^2)^G \xrightarrow{I\otimes \mathrm{id}} H^0(V, \Omega_V^2)_+^G$$

where μ is the mapping in Theorem 2 and I is defined in Section 2. So, for $\omega = (\omega_1, \ldots, \omega_e)$ with $K(\omega) = 0$, there exists a unique $\eta \in H^0(V, \Omega_V^2 \otimes \Omega_V^2)$ with $(E\otimes\mathrm{id})(\eta) = \mu(\omega)$. Since the mapping J is the composition of μ with a left inverse to $E\otimes\mathrm{id}$, it follows that

$$J(\omega) = \eta.$$

The first claim is hence implied by the fact that $E\otimes\mathrm{id}$ is S^G-linear.

For the remaining part it is enough to show that $J(P_\varrho, P_\nu)\omega \in \mathscr{J}$ for all ϱ, ν and all homogeneous $\omega \in S_\chi^G$. Take $\omega_\varrho = -l_\nu P_\nu \omega$, $\omega_\nu = l_\varrho P_\varrho \omega$, $\varrho < \nu$, $\omega_\sigma = 0$, $\sigma \neq \varrho, \nu$, $\omega = (\omega_1, \ldots, \omega_e)$. Then we have

$$\begin{aligned}
\mu(\omega) &= (-l_\nu P_\nu dP + l_\varrho P_\varrho dP_\nu)\omega \\
&= [-(u\partial_u P_\nu + v\partial_v P_\nu)(\partial_u P_\varrho du + \partial_v P_\varrho dv) \\
&\quad + (u\partial_u P_\varrho + v\partial_v P_\varrho)(\partial_u P_\nu du + \partial_v P_\nu dv)]\omega \\
&= [(\partial_u P_\nu)(\partial_u P_\varrho) - (\partial_v P_\varrho)(\partial_v P_\nu)](v\,du - u\,dv)\omega \\
&= J(P_\nu, P_\varrho)(v\,du - u\,dv)\omega = (E\otimes\mathrm{id})(J(P_\nu, P_\varrho)\omega),
\end{aligned}$$

such that, up to a non-trivial constant, $J(\omega)$ is equal to

$$J(P_\nu, P_\varrho)\cdot\omega. \qquad \square$$

In Sections 6 and 7 we shall apply this result to the problem of the determination of $\dim T_X^1$ for all groups $G \subset \mathrm{GL}_2$. The special case of rational double points is treated in the following section.

5. $(T_X^1)^*$ and T_X^1 for rational double points

Suppose now that $G \subset \mathrm{SL}_2$ such that $S_\chi^G = S_{\chi^2}^G = S^G$. Moreover, since $1 \in S_\chi^G$, we have $P_\varepsilon \in \operatorname{im} K$ for all $\varepsilon = 1, \ldots, e$, and by the same reason the ideal $\mathscr{J} \subset S^G = S_{\chi^2}^G$ contains all elements $J(P_\nu, P_\mu)$. On the other hand, each element of $\mathscr{J}$ is of the form

$$\sum_{\varepsilon=1}^{e} J(P_\varepsilon, Q_\varepsilon), \qquad Q_\varepsilon \in S^G,$$

and therefore it is contained in the ideal generated by the elements $J(P_\nu, P_\mu)$. So we get

THEOREM 4. *For finite $G \subset \mathrm{SL}_2$, the S^G-module $(T_X^1)^*$ is canonically isomorphic to $S^G/\mathscr{J}$, where $\mathscr{J}$ is the ideal generated by the Jacobians $J(P_\nu, P_\mu)$ of a minimal set of generators $P_1, \ldots, P_e$.*

Of course, Theorem 4 is only remarkable by the fact that we proved it without any special knowledge of the invariant theory of the binary polyhedral groups. Indeed, using Felix Klein's results ([6]), the proof of Theorem 4 is a simple exercise: For a finite group $G \subset \mathrm{SL}_2$ the invariant ring S^G is generated by three homogeneous polynomials w_1, w_2, w_3 satisfying only one generating polynomial relation

$$R(w_1, w_2, w_3) = 0.$$

Hence, $S^G \cong \boldsymbol{C}[w_1, w_2, w_3]$/ideal generated by R. For suitably chosen w_1, w_2, w_3, there exists a constant $C \neq 0$ such that

$$J(w_\varepsilon, w_{\varepsilon+1}) = C\frac{\partial R}{\partial w_{\varepsilon+2}}, \qquad \varepsilon \in \boldsymbol{Z}_3. \tag{30}$$

(We prove this fact in Lemma 2 below). Hence,

$$S^G/\mathscr{J} \cong \boldsymbol{C}[w_1, w_2, w_3]\Big/\left(R, \frac{\partial R}{\partial w_1}, \frac{\partial R}{\partial w_2}, \frac{\partial R}{\partial w_3}\right),$$

and this is the description of $(T_X^1)^*$ for $X = \{(w_1, w_2, w_3) \in \boldsymbol{C}^3 : R(w_1, w_2, w_3) = 0\} \cong \boldsymbol{C}^2/G$ which we mentioned in the introduction.

LEMMA 2. *For $G \subset \mathrm{SL}_2$ there exist generators w_1, w_2, w_3 of S^G with a generating relation R such that* (30) *holds.*

Proof. For a cyclic group of order $n = r+1 \geqslant 2$, generated by the element

$$\begin{bmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{bmatrix},$$

ζ_n a nth primitive root of unity, we have obviously generators u^n, v^n and uv, satisfying $(u^n)(v^n) = (uv)^n$. By putting

$$w_1 = uv, \qquad w_2 = \tfrac{1}{2}(u^n + v^n), \qquad w_3 = \tfrac{1}{2}(u^n - v^n),$$

we get new generators satisfying the relation $R = 0$, where

$$R(w_1, w_2, w_3) = w_1^n - w_2^2 + w_3^2.$$

A simple calculation shows that

$$J(w_1, w_2) = \frac{n}{2}(v^n - u^n) = \frac{n}{2}\frac{\partial R}{\partial w_3},$$

$$J(w_2, w_3) = -\frac{n^2}{2}(uv)^{n-1} = \frac{n}{2}\frac{\partial R}{\partial w_1},$$

$$J(w_3, w_1) = \frac{n}{2}(u^n + v^n) = \frac{n}{2}\frac{\partial R}{\partial w_2}.$$

For the binary dihedral groups D_n of order $4n$, $n = r-2 \geqslant 2$, and the binary tetrahedral, octahedral and icosahedral group T, O and I respectively, there exist generators w_1, w_2, w_3 with the following degrees:

	$\deg w_1$	$\deg w_2$	$\deg w_3$
D_n	4	$2n$	$2n+2$
T	6	8	12
O	8	12	18
I	12	20	30

which satisfy a relation of the type

$$R(w_1, w_2, w_3) = w_3^2 + g(w_1, w_2) = 0.$$

Since $G \subset \mathrm{SL}_2$, $J(w_1, w_2) \in S^G$, and since obviously in all cases $\deg J(w_1, w_2) = \deg w_3$, there exists a constant C such that

$$J(w_1, w_2) = 2Cw_3 = C\frac{\partial R}{\partial w_3}. \tag{31}$$

(In fact, Klein found w_1, w_2, w_3 by constructing the invariant w_1 of lowest degree and by putting

$$w_2 = \mathrm{Hesse}(w_1) = \det\begin{bmatrix} \partial_u^2 w_1 & \partial_u \partial_v w_1 \\ \partial_v \partial_u w_1 & \partial_v^2 w_1 \end{bmatrix}, \qquad w_3 = J(w_1, w_2)).$$

Now, for all ε,

$$\begin{aligned} 0 &= J(R(w_1, w_2, w_3), w_\varepsilon) \\ &= \det\begin{bmatrix} \sum_\mu \frac{\partial R}{\partial w_\mu}\cdot\frac{\partial w_\mu}{\partial u} & \sum_\mu \frac{\partial R}{\partial w_\mu}\cdot\frac{\partial w_\mu}{\partial v} \\ \frac{\partial w_\varepsilon}{\partial u} & \frac{\partial w_\varepsilon}{\partial v} \end{bmatrix} = \sum_\mu \frac{\partial R}{\partial w_\mu} J(w_\mu, w_\varepsilon) \end{aligned}$$

which implies (30) using (31). □

Another advantage of our description of $(T_X^1)^*$ is the possibility to calculate a basis for T_X^1 itself. Recall that there exist GL_2-isomorphisms

$$S_l \to S^*_{l,\chi^l}$$

sending $u^j v^k$ to $(-1)^k \partial_u^k \partial_v^j$, $j+k=l$. Hence, for a binary polyhedral group $G \subset \mathrm{SL}_2$, these mappings induce isomorphisms

$$\gamma_l\colon S_l^G \xrightarrow{\sim} (S_l^*)^G.$$

For a polynomial $Q \in S_l^G$ we denote the corresponding differential operator $\gamma_l(Q)$ by $Q(\partial)$. γ_l gives rise to a non-degenerate bilinear form $(\ ,\ )$ on S_l^G by $(Q_1, Q_2) := \langle Q_1, \gamma_l(Q_2)\rangle = Q_2(\partial)Q_1$.

If now $\mathscr{J}_l = \mathscr{J} \cap S_l^G$, there exists an orthogonal complement of $\mathscr{J}_l$ in S_l^G with respect to this bilinear form which we call $\tilde{T}_l$:

$$\mathscr{J}_l \ominus \tilde{T}_l = S_l^G.$$

Then we get at once

COROLLARY 1. *Let $Q_1, \ldots, Q_r$ be a homogeneous basis of the finite dimensional vector space $\bigoplus_{l=0}^{\infty} \tilde{T}_l \subset \bigoplus_{l=0}^{\infty} S_l^G$. Then, a basis of T_X^1 in $H^1(U, \Theta_V)$ is given by the images of the operators $Q_1(\partial), \ldots, Q_r(\partial)$ under the mapping D^* of Proposition* 1.

To get a more precise result, let us first analyze the case of a cyclic group G of order $n = r+1 \geqslant 2$ in SL_2. The vector space S_l^G is generated by the elements

$$u^j v^k, \quad j-k \equiv 0 \bmod n, \quad j+k=l,$$

and generators for the ideal $\mathscr{J}$ are u^n, v^n, $(uv)^{n-1}$. Hence, for an element $u^j v^k \in S_l^G$:

(a) $j \neq k \Rightarrow u^j v^k \in \mathscr{J}$, since $j = k + \lambda n$, $\lambda \in \mathbf{Z} \setminus \{0\}$;
(b) $j = k \geqslant n-1 \Rightarrow u^j v^k \in \mathscr{J}$;
(c) $j = k < n-1 \Rightarrow u^j v^k \notin \mathscr{J}$.

So, we have

$$\text{(32)} \qquad \begin{aligned} \mathscr{J}_l &= S_l^G \quad && \text{for } l \geqslant 2n-2 \text{ or odd } l < 2n-2, \\ \mathscr{J}_l \oplus \langle u^\lambda v^\lambda \rangle &= S_l^G \quad && \text{for } l = 2\lambda \leqslant 2n-4. \end{aligned}$$

But for $u^j v^k \in \mathscr{J}_l$, $l = 2\lambda \leqslant 2n-4$, we have $j < \lambda$ or $k < \lambda$ such that

$$\partial_u^\lambda \partial_v^\lambda (u^j v^k) = 0.$$

Therefore, (32) is the orthogonal decomposition, and the Corollary 1 applies to the $r := n-1$ polynomials

$$Q_1 = 1, \quad Q_2 = uv, \ldots, Q_r = (uv)^{r-1},$$

i.e. to

$$w_1^0, \quad w_1^1, \ldots, w_1^{n-2}.$$

In the other cases, we use the same generators as in Lemma 2 such that a basis of $S^G/\mathscr{J}$ can be represented by elements of type $w_1^j w_2^k \in S^G$, only. These elements occur in different degrees such that again

(33) $$\mathscr{J}_l = S_l^G \quad \text{or} \quad \mathscr{J}_l \oplus \mathscr{J}_l' = S_l^G,$$

where $\mathscr{J}_l'$ is generated by one element of type $w_1^j w_2^k$, $j \deg w_1 + k \deg w_2 = l$. By using the explicit form of the generators w_1, w_2, w_3 and the relation R, the reader may check himself that indeed $\mathscr{J}_l' = \tilde{\bar{T}}_l$, i.e., that the elements $w_1^j w_2^k$ which are not in $\mathscr{J}$ form a basis $Q_1, \ldots, Q_r$ as in the Corollary 1.

6. Some remarks on the modules $S_{+,x}^G/\operatorname{im} K$, $\ker K$ and $S_{\chi^2}^G/\mathscr{J}$

From now on we assume $G \subset \mathrm{GL}_2$ to be small, i.e., G contains no pseudoreflections, and that $N = G \cap \mathrm{SL}_2$ is properly contained in G. Then, as is well known, the minimal number e of generators $P_1, \ldots, P_e$ of S^G is equal to the embedding dimension $\operatorname{emb} X$ of the quotient $X = V/G$ and $e \geqslant 4$. Henceforth, X cannot be realized as a hypersurface in C^3. In fact, $\binom{e-1}{2}$ equations are needed to describe X as a subspace of C^e such that X is never a complete intersection which makes the determination of T_X^1 a difficult task.

From the complete knowledge of Betti numbers ([13]) one can deduce, moreover, that the dualizing module ω_X is minimally generated at the singularity x by $e-2$ elements, i.e.,

$$\dim_C \omega_{X,x}/\mathfrak{m}_{X,x}\omega_{X,x} = e-2.$$

Returning to Theorem 3 again we see that in this case $1 \notin S_\chi^G$ such that

$$S_{+,\chi}^G = S_\chi^G = H^0(X', \Omega_{X'}^2) = H^0(X, \omega_X),$$

since ω_X is the reflexive hull of the sheaf Ω_X^2. But, by definition, $\operatorname{im} K$ equals the S^G-submodule of $H^0(X, \omega_X)$ generated by $P_1, \ldots, P_e$. After localization at x we conclude that

(34) $$\dim_C S_{+,\chi}^G/\operatorname{im} K = e-2.$$

It is clear, how one can get concrete infinitesimal deformations in $H^1(U, \Theta_V)$ from a generating set $\tilde{\omega}_1, \ldots, \tilde{\omega}_{e-2}$ of homogeneous elements in S_χ^G by duality.

So, we concentrate on the module $S_{\chi^2}^G/\mathscr{J}$. Denote by $\mathscr{R}$ the relation module of S_χ^G, i.e., the kernel of the epimorphism

$$\bigoplus_{j=1}^{e-2} S^G \to S_\chi^G$$

given by

$$(g_1, \ldots, g_{e-2}) \mapsto \sum_{j=1}^{e-2} g_j \tilde{\omega}_j,$$

where $\{\tilde{\omega}_1, \ldots, \tilde{\omega}_{e-2}\}$ denotes a minimal set of generators for S^G_χ. We grade $\mathscr{R}$ by

$$\deg(g_1, \ldots, g_{e-2}) = l \Leftrightarrow \deg g_j + \deg \tilde{\omega}_j = l, j = 1, \ldots, e-2.$$

We have a grading on $\ker K$, too: If

$$(\omega_1, \ldots, \omega_e) \in \ker K \subset \bigoplus_{\varepsilon=1}^{e} S^G_\chi,$$

then

$$\deg(\omega_1, \ldots, \omega_e) = l \Leftrightarrow \deg \omega_\varepsilon + l_\varepsilon = l, \ \varepsilon = 1, \ldots, e.$$

(Recall that $l_\varepsilon = \deg P_\varepsilon$). Finally, we denote by $\mathscr{T}$ the graded submodule of trivial relations $(\omega_1, \ldots, \omega_e)$ with $\omega_\nu = l_\mu P_\mu \omega$, $\omega_\mu = -l_\nu P_\nu \omega$, $\mu \neq \nu$ fixed, $\omega_\sigma = 0$, $\sigma \neq \mu, \nu$. Then, for $(g_1, \ldots, g_{e-2}) \in \mathscr{R}$, all g_j are in S^G_+, since $\tilde{\omega}_1, \ldots, \tilde{\omega}_{e-2}$ form a minimal set of generators. Hence,

$$g_j = \sum_{\varepsilon=1}^{e} l_\varepsilon P_\varepsilon R_{j\varepsilon}$$

for some $R_{j\varepsilon} \in S^G$. Because of

$$0 = \sum_{j=1}^{e-2} g_j \tilde{\omega}_j = \sum_{\varepsilon=1}^{e} l_\varepsilon P_\varepsilon \Big(\sum_{j=1}^{e-2} R_{j\varepsilon} \tilde{\omega}_j\Big),$$

we see that

$$\omega = (\omega_1, \ldots, \omega_e) \in \ker K,$$

if $\omega_\varepsilon = \sum_{j=1}^{e-2} R_{j\varepsilon} \tilde{\omega}_j$. Of course, ω is not well defined, but it is easily seen that it is unique modulo $\mathscr{R}$. On the other hand, it is clear that each relation

$$\sum_{\varepsilon=1}^{e} l_\varepsilon P_\varepsilon \omega_\varepsilon = 0$$

can be transformed into a relation

$$\sum_{j=1}^{e-2} g_j \tilde{\omega}_j = 0$$

by writing

$$\omega_\varepsilon = \sum_{j=1}^{e-2} Q_{j\varepsilon} \tilde{\omega}_j.$$

By this procedure we have constructed a surjective S^G-homomorphism:

$$\mathscr{R} \to \ker K/\mathscr{T}.$$

Since, in the proof of Theorem 3, we have shown that the image of $J \mid \mathscr{T}$ is the

S^G-module generated by the elements $J(P_\mu, P_\nu)\tilde{\omega}_j$, we finally get the following result:

PROPOSITION 4. *The submodule $\mathscr{J} \subset S^G_{\chi 2}$ can be written in the form $\mathscr{J} = \mathscr{J}' + \mathscr{J}''$, where $\mathscr{J}'$ is generated by the elements $J(P_\mu, P_\nu)\tilde{\omega}_j$, and $\mathscr{J}''$ is generated by the elements*

$$\sum_{\varepsilon=1}^{e} R^\tau_{j\varepsilon} J(P_\varepsilon, \tilde{\omega}_j), \quad j = 1, \ldots, e-2, \ \tau = 1, \ldots, t, \tag{35}$$

coming from a generating system of relations

$$\sum_{j=1}^{e-2} \Big(\sum_{\varepsilon=1}^{e} l_\varepsilon P_\varepsilon R^\tau_{j\varepsilon}\Big)\tilde{\omega}_j = 0, \qquad \tau = 1, \ldots, t. \tag{36}$$

Of course, we should mention that t is of order e^3, as is the number of generators for $\mathscr{J}'$ such that Proposition 4 gives no effective way for computing $S^G_{\chi 2}/\mathscr{J}$. However, we will see in the next section that, in concrete cases, we need much less relations.

It is interesting to note that one can drop the number of relations drastically, if one studies $\ker K$ before taking invariants. To be more precise, we define an S-module homomorphism

$$\tilde{K}\colon \bigoplus_{\varepsilon=1}^{e} S \to S$$

by

$$\tilde{K}(\omega_1, \ldots, \omega_e) = \sum_{\varepsilon=1}^{e} l_\varepsilon P_\varepsilon \omega_\varepsilon,$$

whose image is the ideal in S generated by the invariants $P_1, \ldots, P_e$. So, $\tilde{K}$ is the beginning of a projective resolution

$$\ldots \to \bigoplus_{\varepsilon=1}^{e} S \xrightarrow{\tilde{K}} S \to S/\operatorname{im}\tilde{K} \to 0$$

of the structure sheaf of the zero-dimensional fibre of the canonical cover $V \to V/G \cong X$ at the origin of V. Since this fibre is of codimension 2, $\operatorname{im}\tilde{K}$ can be written by a result of Hilbert as the maximal minors of a $(e-1)$ by e matrix. For instance, we have for rational double points the matrix

$$\begin{bmatrix} v & \partial_u w_1 & \partial_u w_2 \\ -u & \partial_v w_1 & \partial_v w_2 \end{bmatrix}.$$

From the Eagon–Northcott complex it follows that $\ker\tilde{K}$ is free of rank $e-1$ (each row of the matrix giving a relation). Of course,

$$\ker K = (\ker\tilde{K}) \cap (S^G_\chi)^e \quad \text{and} \quad \operatorname{im}(J\,|\ker K) = [\operatorname{im}(\tilde{J}\,|\ker\tilde{K})] \cap S^G_{\chi 2}, \tag{37}$$

where $\tilde{J}$ is given by

$$\tilde{J}(\omega_1, \ldots, \omega_e) = \sum_{\varepsilon=1}^{e} J(P_\varepsilon, \omega_\varepsilon),$$

since $\tilde{K}$ and $\tilde{J}$ are G-equivariant mappings.

We shall apply this method to the case of cyclic quotients in the rest of this section. The method works similarly for the dihedral singularities. For the exceptional cases, however, we did not make any attempt to find the matrices which describe $\operatorname{im}\tilde{K}$. (To our knowledge, they are not in the literature). Indeed, the most effective way for computing $(T_X^1)^*$ in these cases seems to be a compromise between taking all invariants and taking no invariants before computing $\ker K$, namely to take invariants under $N = G \cap SL_2$. We come back to this idea in the next section.

To apply our second method to cyclic quotient singularities we have to collect some facts and notations: Each cyclic quotient of order n is determined by n and another natural number q such that $0 < q < n$ and $gcd(n, q) = 1$; there exists a uniquely determined linear action of $\boldsymbol{Z}_n = \boldsymbol{Z}/n\boldsymbol{Z}$ on V such that the action of a generator of $\boldsymbol{Z}_n$ on $S = \boldsymbol{C}[u, v]$ is given by

$$u \mapsto \zeta_n u, \qquad v \mapsto \zeta_n^q v.$$

Hence, the invariant algebra $S^{\boldsymbol{Z}_n}$ under this action is generated by all monomials $u^j v^k$ with $j + qk \equiv 0 \mod n$. A minimal set of generators can be found in the following way (see e.g. [8] for the following or any text on toroidal embeddings): Develop first the rational number $n/(n-q)$ into a Hirzebruch–Jung continued fraction:

$$\frac{n}{n-q} = a_2 - \cfrac{1}{a_3 - \cfrac{1}{a_4 - \ddots \cfrac{1}{a_{e-1}}}}, \qquad a_\varepsilon \geqslant 2,$$

and define

$$j_1 = n, \quad j_2 = n-q, \ldots, j_{\varepsilon+1} = a_\varepsilon j_\varepsilon - j_{\varepsilon-1}, \ldots$$
$$k_1 = 0, \quad k_2 = 1, \ldots, k_{\varepsilon+1} = a_\varepsilon k_\varepsilon - k_{\varepsilon-1}, \ldots$$

(j_ε and k_ε are called i_ε and j_ε, resp., in [8]). Then

$$P_\varepsilon = u^{j_\varepsilon} v^{k_\varepsilon}, \qquad \varepsilon = 1, \ldots, e,$$

is a minimal set of generators of the invariant algebra. Since the sequence j_ε is strictly decreasing and k_ε is strictly increasing, we can write the ideal in S generated by $P_1, \ldots, P_e$ as the maximal minors of the matrix

$$\begin{bmatrix} u^{j_1-j_2} & v^{k_2-k_1} & 0 & & 0 \\ 0 & u^{j_2-j_3} & v^{k_3-k_2} & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & u^{j_{e-1}-j_e} & v^{k_e-k_{e-1}} \end{bmatrix}.$$

(Notice that $j_e = 0$ and $k_e = n$). Hence, $\ker \tilde{K}$ is generated by the $e-1$ relations

$$r_\varepsilon\colon\ u^{j_\varepsilon - j_{\varepsilon+1}} P_{\varepsilon+1} - v^{k_{\varepsilon+1}-k_\varepsilon} P_\varepsilon = 0, \qquad \varepsilon = 1, \ldots, e-1,$$

and $\operatorname{im}(\tilde{J} \mid \ker \tilde{K})$ is generated by the elements

$$\frac{1}{l_{\varepsilon+1}} J(u^{j_{\varepsilon+1}} v^{k_{\varepsilon+1}}, u^{j_\varepsilon - j_{\varepsilon+1}}) - \frac{1}{l_\varepsilon} J(u^{j_\varepsilon} v^{k_\varepsilon}, v^{k_{\varepsilon+1}-k_\varepsilon})$$
$$= \left[\frac{1}{l_{\varepsilon+1}} k_{\varepsilon+1}(j_\varepsilon - j_{\varepsilon+1}) + \frac{1}{l_\varepsilon} j_\varepsilon (k_{\varepsilon+1} - k_\varepsilon)\right] u^{j_\varepsilon - 1} v^{k_{\varepsilon+1}-1}.$$

Since the generator of Z_n acts on S_{χ^2} by

$$u^j v^k \mapsto \zeta_n^{j+qk} (\zeta_n^{1+q})^2 u^j v^k,$$

the invariant S^{Z_n}-module $S_{\chi^2}^{Z_n}$ is generated by all monomials

$$u^j v^k \quad \text{with} \quad (j+2) + q(k+2) \equiv 0 \bmod n.$$

Hence, the dimension of $S_{\chi^2}^{Z_n} / \operatorname{im}(\tilde{J} \mid \ker \tilde{K}) \cap S_{\chi^2}^{Z_n}$ is equal to the cardinality of

$$\{(j, k)\colon j, k \geqslant 0,\ (j+2) + q(k+2) \equiv 0 \bmod n,\ \forall \varepsilon\colon j < j_\varepsilon - 1 \text{ or } k < k_{\varepsilon+1} - 1\},$$

i.e., to the cardinality of

$$\{(j, k)\colon j \geqslant 2,\ k \geqslant 2,\ j + qk \equiv 0 \bmod n,\ \forall \varepsilon\colon j \leqslant j_\varepsilon \text{ or } k \leqslant k_{\varepsilon+1}\}.$$

But this set consists for $e \geqslant 4$ precisely of the pairs

$$(\alpha_\mu j_\mu, \alpha_\mu k_\mu)\colon \begin{cases} \alpha_\mu = 1, \ldots, a_\mu - 1, & \mu = 3, \ldots, e-2, \\ \alpha_\mu = 2, \ldots, a_\mu - 1, & \mu = 2, e-1. \end{cases}$$

(Notice that $k_2 = 1$, $j_{e-1} = 1$). Hence,

$$\dim(T_X^1)^* = (e-2) + \sum_{\varepsilon=2}^{e-1} (a_\varepsilon - 1) - 2$$
$$= (e-4) + \sum_{\varepsilon=2}^{e-1} (a_\varepsilon - 1).$$

But, if

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \ddots \cfrac{1}{b_r}}}, \qquad b_\varrho \geqslant 2,$$

then the numbers $-b_\varrho$ are the selfintersection numbers of the components of the exceptional set in a minimal resolution $\tilde{X}$ of $X = V/\mathbf{Z}_n$, and

$$\sum_{\varepsilon=2}^{e-1} (a_\varepsilon - 1) = \sum_{\varrho=1}^{r} (b_\varrho - 1)$$

(see [8]) can be interpreted as the dimension of the base space of a versal deformation of $\tilde{X}$, or, alternatively, as the dimension of the smooth component of the base space of a versal deformation of X consisting of simultaneously resolvable deformations (the Artin component).

In fact, one has for all quotient surface singularities of embedding dimension $e \geq 4$ the equation

$$\dim T_X^1 = (e-4) + \sum_{\varrho=1}^{r} (b_\varrho - 1), \tag{38}$$

where the numbers b_ϱ and the sum $\sum_{\varrho=1}^{r} (b_\varrho - 1)$ have the same meaning as above (see [5], [1] and [2] and the tables at the end of the next section).

7. Some examples and complete results for the exceptional cases

We now illustrate how one can use the results of the previous sections to determine a basis for $(T_X^1)^*$ in the exceptional cases. In all these cases (with the exception of precisely one series which, however, can be handled in a similar way, see [5]), the group G is the product $N \cdot \mathbf{Z}_{2m} \subset \mathrm{GL}_2$ of $N = G \cap \mathrm{SL}_2$ with the cyclic subgroup of order $2m$ in the center of GL_2 generated by $\zeta_{2m} \cdot E$, ζ_{2m} a $(2m)$th primitive root of unity. Then, if x, y, z (instead of w_1, w_2, w_3) denote a system of generators of the algebra S^N as in Section 5, $S_{\chi^l}^G$ is generated by all elements

$$x^a y^b \quad \text{with} \quad a \deg x + b \deg y + 2l \equiv 0 \bmod 2m \tag{39}$$

and

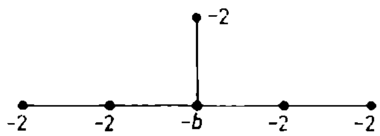
$$x^a y^b z \quad \text{with} \quad a \deg x + b \deg y + \deg z + 2l \equiv 0 \bmod 2m. \tag{40}$$

With respect to these bases the map J has a fairly simple description, since

$$(41) \quad J(x^a y^b z^c, x^{\bar a} y^{\bar b} z^{\bar c}) = \det \begin{bmatrix} a & b & c \\ \bar a & \bar b & \bar c \\ xJ(y, z) & yJ(z, x) & zJ(x, y) \end{bmatrix} x^{a+\bar a-1} y^{b+\bar b-1} z^{c+\bar c-1}.$$

Of course, formula (41) has the following interpretation: One has to develop the determinant, to multiply with the monomial on the utmost right side, to delete terms with at least one negative exponent, and, finally, to replace z^2 or z^3 by x and y via the relation R.

To have a concrete case let us concentrate on the series $T_m = T \cdot Z_{2m}$, $m = 6(b-2)+1$, $b \geqslant 2$. The corresponding singularity has a dual graph



and embedding dimension $e = b+1$. It is clear that the case $b = 2$ is a special one, namely the rational double point of type E_6. Unfortunately, the case $b = 3$ can also not be treated in a coherent manner together with all other cases (we shall see in a moment, why this is so). However, the case $b = 4$ shows all features of the rest of the series.

So, let us first determine for $b = 4$, i.e., for $m = 13$, the invariant algebra S^{T_m} (see also [9]). By (39) and (40) a basis is given in the first degrees by

					1					$0\,m$
				$\boxed{x^3y}$	⋮	$\boxed{xyz}$				$2\,m$
			x^6y^2	x^2y^5	⋮	x^4y^2z	$\boxed{y^5z}$			$4\,m$
	$\boxed{x^{13}}$	x^9y^3	x^5y^6	xy^9	⋮	$\boxed{x^{11}z}$	x^7y^3z	x^3y^6z		$6\,m$
$x^{16}y$	$x^{12}y^4$	x^8y^7	x^4y^{10}	y^{13}	⋮	$x^{14}yz$	$x^{10}y^4z$	x^6y^7z	$x^2y^{10}z$	$8\,m$
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

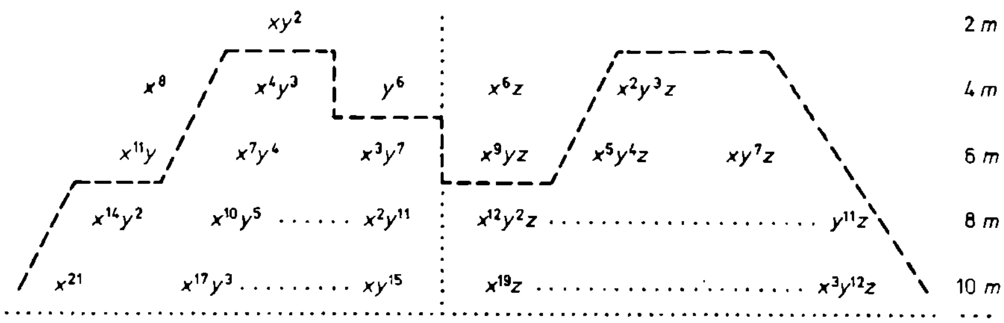
Here, $\deg x = 6$, $\deg y = 8$, $\deg z = 12$, $z^2+y^3+x^4 = 0$, and the $e = 5$ elements in boxes are algebra generators for S^{T_m}.

Similarly, we get for $S_\chi^{T_m}$:

			$\boxed{x^4}$	$\boxed{y^3}$	⋮	$\boxed{x^2z}$				$2\,m$
			x^7y	x^3y^4	⋮	x^5yz	xy^4z			$4\,m$
		$x^{10}y^2$	x^6y^5	x^2y^8	⋮	x^8y^2z	x^4y^5z	y^8z		$6\,m$
x^{17}	$x^{13}y^3$	x^9y^6	x^5y^9	xy^{12}	⋮	$x^{15}z$	$x^{11}y^3z$	x^7y^6z	x^3y^9z	$8\,m$
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

where now the $e-2 = 3$ elements in boxes are generators for the S^{T_m}-module $S_\chi^{T_m}$ contributing three dimensions to T_χ^1.

Finally, for $S_{\chi^2}^{T_m}$, we find



Of course, elements in degree $2m$ cannot lie in $\operatorname{im}(J\,|\ker K)$. We will show that 4 relations between the generators of S^{T_m} in S^T suffice to generate the module below, the broken line in the figure above. These relations are:

$$
\begin{aligned}
(x^3 y)z-(xyz)x^2 &= 0,\\
(2y^5 z)x-(xyz)2y^4 &= 0,\\
(3x^{11} z)y-(x^3 y)3x^8 z &= 0,\\
(x^{11} z)x^2-(x^{13})z &= 0.
\end{aligned}
\tag{42}
$$

Simple calculations show that

$$
\begin{aligned}
J(x^3 y, z)-J(xyz, x^2) &= \left\{\begin{vmatrix} 3 & 1 & 0\\ 0 & 0 & 1\\ 4x^4 & 3y^3 & 2z^2\end{vmatrix} - \begin{vmatrix} 1 & 1 & 1\\ 2 & 0 & 0\\ 4x^4 & 3x^3 & 2z^2\end{vmatrix}\right\} x^2\\
&= \left(-9y^3+4x^4+2(2z^2-3y^3)\right)x^2\\
&= \left(4x^4-15y^3-4(y^3+x^4)\right)x^2\\
&= -19x^2y^3,\\
J(y^5 z, x)-J(xyz, 2y^4) &= 29y^7+58x^4y^4,\\
J(x^{22} z, y)-J(x^3 y, 3x^8 z) &= -86x^{14}-43x^{10}y^3,\\
J(x^{11} z, x^2)-J(x^{13}, z) &= 45x^{10}y^2.
\end{aligned}
$$

Hence, x^2y^3, y^7, x^{14} and $x^{10}y^2$ are in the S^T-module $\operatorname{im}(J\,|\ker\tilde{K})$, where now $\ker\tilde{K}$ denotes relations in S^T. But then $\operatorname{im}(\tilde{J}\,|\ker K)$ contains all elements in $S^T_{x^2}{}^{13}$ with the possible exception of xy^2, x^8, y^6, x^6z, $x^{11}y$ and x^9yz. Hence,

$$\dim T_X^1 \leqslant 3+6 = 9 = 8+1,$$

where $8 = \sum_{\varrho=1}^{r}(b_\varrho-1)$ and $1 = e-4$. Since in (38) there always holds the $\geqslant$

sign by a result of J. Wahl (see [1]), we have shown that

$$\dim T_X^1 = 9 \quad \text{for} \quad X = C^2/T\cdot Z_{26}.$$

(It is also easily seen that the relations (42) are the only ones in low degrees, such that the dimension formula may be derived directly without Wahl's lower bound).

By the same procedure we see for general $b \geqslant 4$, $m = 6(b-2)+1$, that in degree $4m$ there are always precisely 3 and in degree $6m$ precisely 2 infinitesimal deformations coming from $S_{\chi^2}^{T_m}$. In addition, we have all elements in degree $2m$ in $S_{\chi^2}^{T_m}$ and all elements of degree $2m$ in $S_{\chi}^{T_m}$ (the generators). This last number is equal to

$$\dim S_{2m,\chi}^{T_m} + \dim S_{2m,\chi^2}^{T_m} = \dim S_{2m-2}^{T} + \dim S_{2m-4}^{T}.$$

It is easy to prove by invariant theory that

$$\dim S_l^T + \dim S_{l+2}^T = 1,\ 0,\ 1,\ 2,\ 1,\ 2,\ 3,\ 2,\ 3, \ldots$$

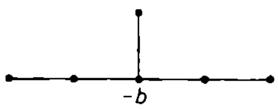
for $l = 0, 2, 4, 6, \ldots$. Hence, we get for $l = 2m-4 = 12(b-2)-2$ that this number is equal to $2b-4$. Altogether we have

$$\begin{aligned} \dim T_X^1 &= (2b-4)+5 = 2b+1 \\ &= (b+4)+(b+1-4) \\ &= \sum_{\varrho=1}^{r} (b_\varrho - 1)+(e-4). \end{aligned}$$

The formula and also the distribution of the degrees remain valid for $b = 3$; only the proof has to be modified, since in this case there is only one generator for S^{T_m} in degree $2m$ and therefore no relation for elements in this degree.

Without proof we now list for all exceptional cases the graphs of the corresponding singularities and the number and the degrees of all the basis elements of T_X^1. Here vertices without weights stand for (-2)-curves, b is always $\geqslant 2$. It should be mentioned that due to the algorithm described by the second author in [5] it is only necessary to compute T_X^1 for the cases $b = 2$, 3 and 4 in each individual series.

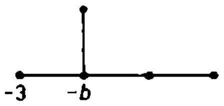
T_1



$m = 6(b-2)+1$
$e = b+1$

l	$\dim(T_X^1)_{-l}$ $b > 2$
$2m$	$2b-4$
$4m$	3
$6m$	2
Σ	$2b+1$

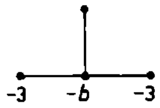
T_3



$m = 6(b-2)+3$
$e = b+2$

l	$b=2$	$b>2$
$2m$	$2b-3$	
$4m$	2	4
$6m$	1	1
$8m$	1	
$10m$	1	
Σ	$2b+2$	

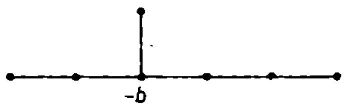
T_5



$m = 6(b-2)+5$
$e = b+3$

l	$b=2$	$b>2$
$2m$	$2b-2$	
$4m$	3	5
$6m$	2	
Σ	$2b+3$	

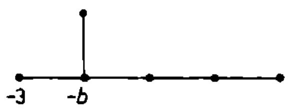
O_1



$m = 12(b-2)+1$
$e = b+1$

l	$b>2$
$2m$	$2b-4$
$4m$	3
$6m$	2
$8m$	1
Σ	$2b+2$

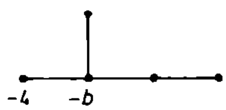
O_5



$m = 12(b-2)+5$
$e = b+2$

l	$b=2$	$b>2$
$2m$	$2b-3$	
$4m$	2	4
$6m$	1	1
$8m$	1	1
$10m$	1	
$12m$	1	
Σ	$2b+3$	

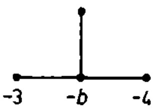
O_7



$m = 12(b-2)+7$
$e = b+3$

l	$b=2$	$b>2$
$2m$	$2b-3$	
$4m$	3	5
$6m$	2	2
$8m$	1	
$10m$	1	
Σ	$2b+4$	

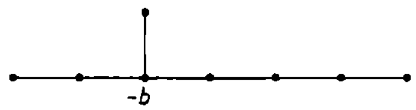
O_{11}



$m = 12(b-2)+11$
$e = b+4$

l	$b = 2$	$b > 2$
$2m$	$2b-2$	
$4m$	4	6
$6m$	3	1
Σ	$2b+5$	

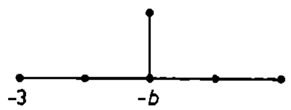
I_1



$m = 30(b-2)=1$
$e = b+1$

l	$b > 2$
$2m$	$2b-4$
$4m$	3
$6m$	2
$8m$	1
$10m$	1
Σ	$2b+3$

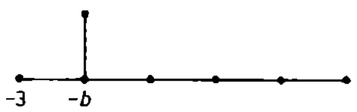
I_7



$m = 30(b-2)+1$
$e = b+1$

l	$b = 2$	$b > 2$
$2m$	$2b-3$	
$4m$	1	3
$6m$	1	3
$8m$	1	
$10m$	1	
$12m$	1	
$14m$	0	
$16m$	1	
Σ	$2b+3$	

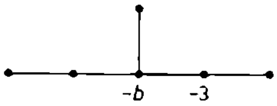
I_{11}



$m = 30(b-2)+11$
$e = b+2$

l	$b = 2$	$b > 2$
$2m$	$2b-3$	
$4m$	2	4
$6m$	1	1
$8m$	1	1
$10m$	1	1
$12m$	1	
$14m$	1	
Σ	$2b+4$	

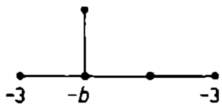
I_{13}



$m = 30(b-2)+13$
$e = b+2$

l	$b = 2$	$b > 2$
$2m$	$2b-3$	
$4m$	2	4
$6m$	1	1
$8m$	2	1
$10m$	1	
Σ	$2b+3$	

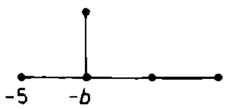
I_{17}



$m = 30(b-2)+17$
$e = b+3$

l	$b = 2$	$b > 2$
$2m$	$2b-2$	
$4m$	2	4
$6m$	2	2
$8m$	2	
Σ	$2b+4$	

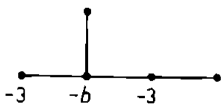
I_{19}



$m = 30(b-2)+19$
$e = b+4$

l	$b = 2$	$b > 2$
$2m$	$2b-3$	
$4m$	3	5
$6m$	3	3
$8m$	2	1
$10m$	1	
Σ	$2b+6$	

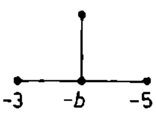
I_{23}



$m = 30(b-2)+23$
$e = b+3$

l	$b = 2$	$b > 2$
$2m$	$2b-2$	
$4m$	3	5
$6m$	2	0
$8m$	1	1
Σ	$2b+4$	

I_{29}



$m = 30(b-2)+29$
$e = b+5$

l	$b = 2$	$b > 2$
$2m$	$2b-2$	
$4m$	4	6
$6m$	4	2
$8m$	1	1
Σ	$2b+7$	

8. A generalization to quasi-homogeneous surface singularities

In this last section we want to show that our "splitting" method can be generalized to an effective method for computing the vector space $(T^1_X)^*$ for a much wider class of examples than the class of quotient surface singularities. Let for the rest of this manuscript (X, x) denote a *quasi-homogeneous normal surface singularity* and let d_E be the Euler derivation which associates to any homogeneous $f \in \mathcal{O}_{X,x}$ the element $(\deg f)\cdot f \in \mathcal{O}_{X,x}$. Denote by I the $\mathcal{O}_{X,x}$-homomorphism $\Omega^1_{X,x} \to \mathcal{O}_{X,x}$, making the diagram

$$\begin{array}{ccc} & \Omega^1_{X,x} & \\ {}^{d}\nearrow & & \searrow^{I} \\ \mathcal{O}_{X,x} & \xrightarrow{\quad d_E \quad} & \mathcal{O}_{X,x} \end{array}$$

commutative. Defining $E\colon \Omega^2_{X,x} \to \Omega^1_{X,x}$ by $E(\alpha \wedge \beta) := I(\alpha)\cdot\beta - I(\beta)\cdot\alpha$ for α, $\beta \in \Omega^1_{X,x}$ and extending everything to the (affine) neighbourhood X we get an exact sequence of $\mathcal{O}_{X'}$-sheaves over $X' = X - \{x\}$:

$$0 \to \Omega^2_{X'} \xrightarrow{E} \Omega^1_{X'} \xrightarrow{I} \mathcal{O}_{X'} \to 0. \tag{43}$$

Alternatively, view d_E outside x as a vector field and define I and E by interior multiplication with d_E. The exterior differentiation d gives a splitting of the associated (graded) sequence of sections, except in degree 0, where we have a cokernel of length 1 on the right. If we tensor this complex with the dualizing sheaf $\omega_{X'}$, we still have an exact sequence

$$0 \to \Omega^2_{X'} \otimes \omega_{X'} \xrightarrow{E\otimes \mathrm{id}} \Omega^1_{X'} \otimes \omega_{X'} \xrightarrow{I\otimes \mathrm{id}} \mathcal{O}_{X'} \otimes \omega_{X'} \to 0; \tag{44}$$

but in order to get a split-exact sequence of sections we have to assume a little more. Suppose that

(a) there exists a holomorphic connection $\nabla\colon \omega_{X'} \to \Omega^1_{X'} \otimes \omega_{X'}$ which gives a splitting of $I \otimes 1$: $(I\otimes 1)(\nabla\omega) = (\deg \omega)\cdot\omega$ for any homogeneous section $\omega \in H^0(X', \omega_{X'}) = H^0(X, \omega_X)$;

(b) the module $H^0(X, \omega_X)$ has no elements in degree 0.

Then, it is well known (and easy to check) that the C-linear map

$$\Delta\colon \Omega^1_{X'} \otimes \omega_{X'} \to \Omega^2_{X'} \otimes \omega_{X'}$$

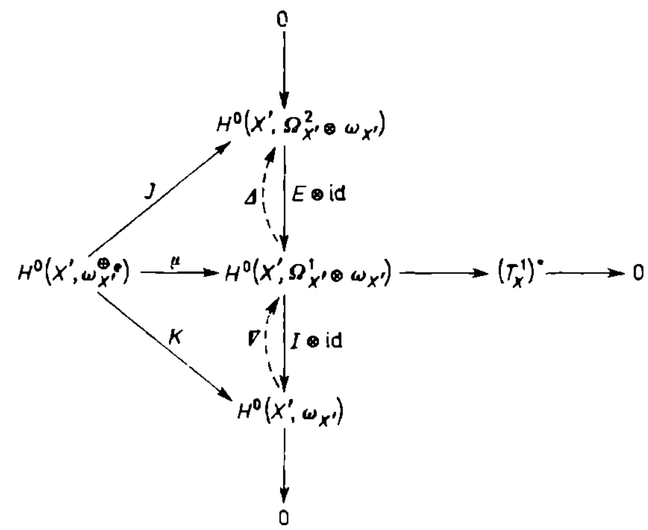
with $\Delta(\eta\otimes\omega) := d\eta\otimes\omega - \eta \wedge \nabla\omega$ is well defined, and that the pair (∇, Δ) gives a splitting of the sections of (44). For example, we show that – on the p-th graded piece of $H^0(X', \Omega^1_{X'} \otimes \omega_{X'})$ – the Lie derivative with recpect to ∇, which is defined by

$$L_\nabla = \nabla\circ(I\otimes \mathrm{id}) + (E\otimes \mathrm{id})\circ\Delta,$$

acts by multiplication with p.(1) Let η be a 1-form of degree k and ω a 2-form of degree l and let $\nabla\omega = \sum_i \xi_i \otimes \omega_i$. Then

$$\begin{aligned} L_\nabla(\eta\otimes\omega) &= dI(\eta)\otimes\omega + E(d\eta)\otimes\omega + I(\eta)\cdot\nabla\omega - (E\otimes\mathrm{id})(\eta\wedge\nabla\omega) \\ &= k\cdot\eta\otimes\omega + I(\eta)\cdot\sum_i \xi_i\otimes\omega_i - (E\otimes\mathrm{id})\Big(\sum_i \eta\wedge\xi_i\otimes\omega_i\Big) \\ &= k\cdot\eta\otimes\omega + \eta\otimes(I\otimes\mathrm{id})(\nabla\omega) \\ &= (k+l)\,\eta\otimes\omega. \end{aligned}$$

Now we have an exact vertical sequence in the following diagram:



The mappings μ, K and J are given by

$$\mu(\omega_1, \ldots, \omega_e) = \sum_{\varepsilon=1}^{e} df_\varepsilon \otimes \omega_\varepsilon,$$

(45) $$K(\omega_1, \ldots, \omega_e) = \sum_{\varepsilon=1}^{e} l_\varepsilon f_\varepsilon \omega_\varepsilon, \qquad l_\varepsilon = \deg f_\varepsilon,$$

$$J(\omega_1, \ldots, \omega_e) = -\sum_{\varepsilon=1}^{e} df_\varepsilon \wedge \nabla\omega_\varepsilon,$$

and the relations $K = (I\otimes\mathrm{id})\circ\mu$, $J = \Delta\circ\mu$ hold. Again, $\{f_1, \ldots, f_e\}$ denotes a set of homogeneous generators of the maximal ideal $\mathfrak{m}_{X,x}$. So, we get an exact sequence

(46) $$0 \to \mathrm{im}(J\,|\,\ker K) \to (T^1_X)^* \to (\omega_{X,x}/\mathfrak{m}_{X,x}\,\omega_{X,x}) \to 0$$

(1) As J. Wahl informed us the idea to prove exactness by means of Lie derivatives was already used by I. Naruki, *A note on isolated singularity II*, Proc. Japan Acad. 51 (1975), 380–383.

as in Theorem 3, such that the dualizing module $\omega_{X,x}$ contributes d independent infinitesimal deformations, if d is the cardinality of a minimal set of generators of $\omega_{X,x}$.

Remark. J. Wahl has shown ([15]) that the dual of the natural map on the right of (46) has a kernel of length one if X is Gorenstein and that it is injective otherwise. Thus, to get exactness of the sequence (46), we might replace (b) by the assumption that X is not Gorenstein. It is not known to us whether the connection in (a) always exists.

EXAMPLE. Let (Y, y) denote a quasi-homogeneous *Gorenstein* surface singularity, and let H be a finite group acting on (Y, y) such that

(i) *H acts freely on $Y' = Y - \{y\}$;*

(ii) *the action is compatible with the C^*-action on Y.*

Then the quotient $X = Y/H$ is a normal quasi-homogeneous surface singularity. The restriction of the Euler derivation d_E of $\mathcal{O}_{Y,y}$ to the invariant subring $\mathcal{O}^H_{Y,y} \cong \mathcal{O}_{X,x}$ is the Euler derivation of the graded algebra $\mathcal{O}_{X,x}$. Also, $\Omega^1_{X'} = (\pi_* \Omega^1_{Y'})^H$, where $\pi\colon Y \to X$ is the canonical quotient map. This implies immediately that the maps I and E on Y induce the corresponding maps I and E on X. Let, moreover, χ be the character attached to the action of H on $\omega_{Y,y}$. Then (44) is canonically isomorphic to the sequence

$$0 \to (\pi_* \Omega^2_{Y'})^H_\chi \xrightarrow{E} (\pi_* \Omega^1_{Y'})^H_\chi \xrightarrow{I} (\pi_* \mathcal{O}_{Y'})^H_\chi \to 0.$$

If we assume, moreover, that

(iii) *the character χ is not trivial,*

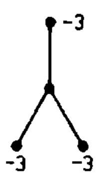
then the remarks above apply. The connection $\nabla\colon \omega_{X'} \to \Omega^1_{X'} \otimes \omega_{X'}$ is given by restricting the exterior differentiation on Y to the eigenspace

$$(\mathcal{O}_{Y,y})^H_\chi \cong \omega_{X,x}.$$

As an application we can take any quasi-homogeneous rational surface singularity (X, x) of embedding dimension $e \geqslant 4$ and Y the canonical Gorenstein cover of X (see [13]). Then (a), (b) and (c) hold; in particular $\dim T^1_X \geqslant e - 2$.

Remark. In [14] J. Wahl proved that a good C^*-action in the center of the maximal reductive group of automorphisms exists. For this action, the assumption (ii) is automatically satisfied.

We apply the preceding considerations to the quasi-homogeneous rational singularity (X, x) of embedding dimension $e = 6$ with dual graph



whose canonical Gorenstein cover Y is the simple elliptic singularity $\tilde{E}_6$ given by

$$z^3 = x^3 + y^3$$

in $\boldsymbol{C}^3$. The projection π: $Y \to X$ is induced by the $\boldsymbol{Z}_3$-action $\zeta \cdot (x, y, z) = (\zeta x, \zeta^2 y, \zeta z)$, ζ a third root of unity (see [13]). In fact, the invariant algebra

$$\mathcal{O}_{X,x} \cong \mathcal{O}_{Y,y}^H, \qquad H = \boldsymbol{Z}_3,$$

is generated by the 6 elements

$$f_1 = xy, \quad f_2 = yz,$$
$$f_3 = x^3, \quad f_4 = y^3, \quad f_5 = x^2 z, \quad f_6 = xz^2.$$

The dualizing sheaf of Y is free with basis element

$$\frac{dx \wedge dy}{z^2} = -\frac{dy \wedge dz}{x^2} = \frac{dx \wedge dz}{y^2}$$

such that ζ acts by multiplication, and the corresponding character χ is multiplication by ζ^2. Hence, a minimal set of generators for

$$\omega_{X,x} \cong (\mathcal{O}_{Y,y})_\chi^H$$

consists of the $e-2=4$ elements

$$\tilde{\omega}_1 = y,$$
$$\tilde{\omega}_2 = x^2, \quad \tilde{\omega}_3 = xz, \quad \tilde{\omega}_4 = z^2.$$

In the following table we list the elements of $(\mathcal{O}_{Y,y})_{\chi^2}^H$ of low degrees, the elements in boxes being the generators over $\mathcal{O}_{X,x} \cong (\mathcal{O}_{Y,y})^H$:

	$\boxed{x}$		$\boxed{z}$		1
		$\boxed{y^2}$			2
	x^2y	xyz	yz^2		3
x^4	x^3z	x^2z^2	xy^3	y^3z	4
x^3y^2	y^5	x^2y^2z		xy^2z^2	5

Obviously there are no elements of $\ker K$ in degree 1, 2 and 3. In degree 4 there are 6 linearly independent relations:

1. $(3f_3)\cdot 2\tilde{\omega}_1 - (2f_1)\cdot 3\tilde{\omega}_2 = 0,$
2. $(3f_5)\cdot 2\tilde{\omega}_1 - (2f_1)\cdot 3\tilde{\omega}_3 = 0,$

3. $(3f_5)\cdot 2\tilde{\omega}_1-(2f_2)\cdot 3\tilde{\omega}_2 = 0$,
4. $(3f_6)\cdot 2\tilde{\omega}_1-(2f_1)\cdot 3\tilde{\omega}_4 = 0$,
5. $(3f_6)\cdot 2\tilde{\omega}_1-(2f_2)\cdot 3\tilde{\omega}_3 = 0$,
6. $(3f_3)\cdot 2\tilde{\omega}_1+(3f_4)\cdot 2\tilde{\omega}_1-(2f_2)\cdot 3\tilde{\omega}_4 = 0$.

We compute the values of J for these relations:

1. $$df_3 \wedge 2\nabla\tilde{\omega}_1-df_1 \wedge 3\nabla\tilde{\omega}_2 = 3x^2\,dx \wedge (2\,dy)-(y\,dx+x\,dy) \wedge (3\cdot 2x\,dx) = 12x^2 z^2\cdot\frac{dx \wedge dy}{z^2}.$$

2. $$\begin{aligned} df_5 \wedge 2\nabla\tilde{\omega}_1-df_1 \wedge 3\nabla\tilde{\omega}_3 &= (2xz\,dx+x^2\,dz) \wedge 2\,dy \\ &\quad -(y\,dx+x\,dy)\wedge 3\,(z\,dx+x\,dz) \\ &= (7xz^3+5x^4-3xy^3)\frac{dx \wedge dy}{z^2} \\ &= (12x^4+4xy^3)\frac{dx \wedge dy}{z^2}. \end{aligned}$$

Similarly

3. $$df_5 \wedge 2\nabla\tilde{\omega}_1-df_2 \wedge 3\nabla\tilde{\omega}_2 = (12x^4+16xy^3)\frac{dx \wedge dy}{z^2},$$

4. $$df_6 \wedge 2\nabla\tilde{\omega}_1-df_1 \wedge 3\nabla\tilde{\omega}_4 = (12x^3 z-4y^3 z)\frac{dx \wedge dy}{z^2},$$

5. $$df_6 \wedge 2\nabla\tilde{\omega}_1-df_2 \wedge 3\nabla\tilde{\omega}_3 = (12x^3 z+8y^3 z)\frac{dx \wedge dy}{z^2},$$

6. $$df_3 \wedge 2\nabla\tilde{\omega}_1+df_4 \wedge 2\nabla\tilde{\omega}_1-df_2 \wedge 3\nabla\tilde{\omega}_4 = 12x^2 z^2\frac{dx \wedge dy}{z^2}.$$

These elements span the degree -4 part of $(\mathcal{O}_{Y,y})^H_{\chi^2}$, so that there is no nontrivial infinitesimal deformation of degree -4. Together with the $e-2 = 4$ deformations coming from $\omega_{X,x}$ we have

$1+2 = 3$ deformations in degree -1,

$3+1 = 4$ deformations in degree -2,

$0+3 = 3$ deformations in degree -3,

and no deformation of degree -4.

So we get 10 independent deformations, one more than the number

$$(e-4)+\sum_{\varrho=1}^{r}(b_\varrho-1)$$

would suggest. Of course, this was known long time ago by J. Wahl (see e.g. [2]).

We claim that in fact $\dim T^1_X = 10$. To prove the claim, observe that homogeneous generators of the local ring are in degrees 2 and 3, and equations, having quadratic initial forms, are in degrees 5 and 6. Since a minimal set of generating relations consists of linear relations, all entries of a homogeneous first order lifting of the matrix of relations must be homogeneous of degree 2 or 3. But from the minimality one concludes immediately that all deformations of negative degree at least 5 actually have to show up in any first order lifting of the relations.

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