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DIHEDRAL SINGULARITIES: INVARIANTS, EQUATIONS AND INFINITESIMAL DEFORMATIONS

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In this note we give a short survey on joint work with K. Behnke; details will appear in [1] and [2].

Let n, q be positive integers with $2 \leqslant q < n$ and $\gcd(n, q) = 1$, $m = n - q$. We define elements $\phi_m, \psi_q, \eta \in \mathrm{GL}(2, \mathbf{C})$ by

$$\phi_m = \begin{pmatrix} \zeta_{2m} & 0 \\ 0 & \zeta_{2m} \end{pmatrix}, \quad \psi_q = \begin{pmatrix} \zeta_{2q} & 0 \\ 0 & \zeta_{2q}^{-1} \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where $i = \sqrt{-1}$ and $\zeta_k = \exp(2\pi i/k)$. The group $G_{n,q} \subset \mathrm{GL}(2, \mathbf{C})$ is generated by

(a) ϕ_m, ψ_q, η in case m odd,

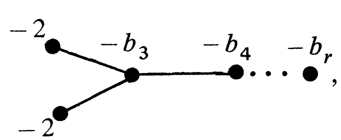
(b) $\psi_q, \eta \circ \phi_{2m}$ in case m even.

$G_{n,q}$ has finite order $4mq$; $G_{q+1,q}$ is the *binary dihedral group* of order $4q$.

$G_{n,q}$ acts on $\mathbf{C}^2$ in the usual way; the quotient $\mathbf{C}^2/G_{n,q}$ has precisely one (normal) complex-analytic singularity. We call it the *dihedral singularity of type* $D_{n,q}$. If we expand n/q into the modified continued fraction à la Hirzebruch-Jung,

$$n/q = b_3 - 1\rceil b_4 - \cdots - 1\rceil b_r, \qquad b_\rho \geqslant 2,\ r \geqslant 4,$$

it can be characterized by the dual graph of its minimal resolution (cf. [3]):



$, \quad \bullet \cong \mathbf{P}_1(\mathbf{C}).$

The equations are calculated by invariant theory. As in the cyclic group case [5], we put

$$n/m = a_2 - 1\rceil a_3 - \cdots - 1\rceil a_{e-1}, \qquad a_\epsilon \geqslant 2.$$

Further set $A_3 = a_3 + 1$, $A_\epsilon = a_\epsilon$, $\epsilon \neq 3$, and

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$$s_2 = 1,\ s_3 = 1, \qquad s_{\epsilon+1} = A_\epsilon s_\epsilon - s_{\epsilon-1}, \qquad 3 \leqslant \epsilon \leqslant e-1,$$

$$t_2 = a_2,\ t_3 = a_2 - 1,\ t_{\epsilon+1} = A_\epsilon t_\epsilon - t_{\epsilon-1}, \qquad 3 \leqslant \epsilon \leqslant e-1,$$

$$r_\epsilon = mt_\epsilon - qs_\epsilon, \qquad 2 \leqslant \epsilon \leqslant e-1.$$

Then we have

THEOREM 1. *A minimal set of generators for* $S_{n,q} = \mathbf{C}[u, v]^{G_{n,q}}$ *is formed by the polynomials*

$$z_1 = (uv)^{2m}, \quad z_\epsilon = (uv)^{r_\epsilon}(u^{2qs_\epsilon} + (-1)^{t_\epsilon} v^{2qs_\epsilon}), \qquad \epsilon = 2, \ldots, e.$$

After a (noncanonical) change of variables it is possible to find simple equations.

THEOREM 2. *The dihedral singularity of type* $D_{n,q}$ *is (minimally) described by* ½$(e-1)(e-2)$ *equations*

$$z_2^2 = z_1(z_3^2 + z_1^{a_2-1}),$$

$$z_1 z_\epsilon = z_2 z_3^{a_3-2} \cdots z_{\epsilon-2}^{a_{\epsilon-2}-2} z_{\epsilon-1}^{a_{\epsilon-1}-1}, \qquad \epsilon = 4, \ldots, e,$$

$$z_2 z_\epsilon = z_3^{a_3-2} \cdots z_{\epsilon-2}^{a_{\epsilon-2}-2} z_{\epsilon-1}^{a_{\epsilon-1}-1}(z_3^2 + z_1^{a_2-1}), \qquad \epsilon = 4, \ldots, e,$$

$$z_{\epsilon-1} z_{\epsilon+1} = z_\epsilon^{a_\epsilon}, \qquad \epsilon = 4, \ldots, e-1,$$

$$z_\delta z_\epsilon = z_{\delta+1}^{a_{\delta+1}-1} z_{\delta+2}^{a_{\delta+2}-2} \cdots z_{\epsilon-2}^{a_{\epsilon-2}-2} z_{\epsilon-1}^{a_{\epsilon-1}-1},$$

$$4 \leqslant \delta + 1 < \epsilon - 1 \leqslant e - 1.$$

In the case $e = 4$ *these equations are given by the maximal subdeterminants of the* 3×2*-matrix*

$$\begin{pmatrix} z_1 & z_2 & z_3^{a_3-1} \\ z_2 & z_3^2 + z_1^{a_2-1} & z_4 \end{pmatrix}.$$

This is in accordance with (and proved by) Wahl's theorem on equations defining rational singularities [6].

For the computation of T^1, the vector space of infinitesimal deformations, we use Pinkham's method [4]. In [1] we reduce the problem to the solution of a (large) system of linear equations and give some examples. A general formula for the dimension of T^1 will be proved in [2]:

THEOREM 3.

$$\dim T^1 = \sum_{\epsilon=2}^{e-1} a_\epsilon + c,$$

where

$$c = \begin{cases} 1, & e = 3, \\ 2, & a_3 = 2, \\ 3, & a_3 \geqslant 3. \end{cases}$$

In another forthcoming manuscript we determine the invariants and equations for all remaining quotient surface singularities.

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