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A Generalization of Kodaira's Embedding Theorem

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Kodaira's well known embedding theorem [2] can be formulated as follows:

A compact complex analytic manifold X is projective algebraic if (and only if) there exists a positive line bundle L on X.

In [1] Grauert and the author of this note introduced the concept of almost positive coherent analytic sheaves on complex analytic spaces. It was conjectured:

Conjecture I. *An irreducible normal compact complex analytic space X is Moišezon (i.e. the transcendence degree of the field of meromorphic functions on X is equal to the complex dimension of X) if and only if there exists an almost positive coherent analytic sheaf $\mathcal{S}$ on X.*

It was shown in [5] that a normal Moišezon space carries an almost positive torsion free coherent analytic sheaf of rank 1. The "if" part was proved for the case of isolated singularities.

Since each compact complex analytic space can be desingularized and coherent analytic sheaves can be made free (modulo torsion) by proper modifications, it would be enough to prove Conjecture I for the case X a manifold and $\mathcal{S}$ an almost positive invertible sheaf (associated to an almost positive line bundle L). The purpose of this note is to prove it under the additional assumption that X is a Kähler manifold. In that case, the assumptions on L may be weakend. We prove the following

Theorem. *An irreducible compact complex analytic Kähler manifold X with semipositive line bundle L, which is positive at at least one point $x_0 \in X$, is Moišezon (and hence projective algebraic due to Moišezon [3]).*

Before embarking on the proof, let us make some remarks:

1. Since a compact complex analytic manifold with a positive line bundle is necessarily Kähler, this theorem is in fact a generalization of Kodaira's embedding theorem. Of course, it is in general not possible to prove that L is ample.

2. The same result is true, if we replace the line bundle L by a vector bundle V (any definition of positivity).

3. Besides the verification of our theorem, the proof will show that Conjecture I is true if one would have the following generalization of Kodaira's vanishing theorem:

Conjecture II. *For an almost positive line bundle L on a compact complex analytic manifold X with canonical line bundle K, the cohomology groups*

$$H^{\nu}(X, L\otimes K), \quad \nu \geqq 1,$$

vanish.

On the other hand, if Conjecture I is true, i.e. if the existence of the almost positive line bundle L implies that X is Moišezon, this vanishing theorem follows from the main result of [1]. Hence Conjectures I and II are equivalent.

Here is the *proof* of our theorem: Let X be a compact complex analytic Kähler manifold and L be a semipositive complex analytic line bundle which is positive at at least one point $x_0 \in X$. That means: If L is given by transition functions g_{ij} with respect to a suitable covering $\{U_i\}$ of X, there are positive C^{∞} functions h_i on U_i such that

$$h_i = |g_{ji}|^2\, h_j$$

on $U_i \cap U_j$ for all i and j, and the form

$$\sum \frac{\partial^2(-\log h_i)}{\partial z_\mu \partial \bar{z}_\nu}\, dz_\mu d\bar{z}_\nu$$

is positive semidefinite on U_i for all i and positive definite at x_0 (and hence in a neighborhood U of x_0).

Adopting Kodaira's trick [2], we blow up x_0 in X and get a proper modification $\pi: \hat{X} \to X$; $\hat{X}$ is again a Kähler manifold. Let F be the line bundle on $\hat{X}$ associated to the divisor $\pi^{-1}(x_0)$. $F|\pi^{-1}(x_0)$ is the normal bundle of $\pi^{-1}(x_0)$ in $\hat{X}$ and hence negative. Therefore, it is easy to see that $F^{*(n+\mu)}$, $n = \dim X$, $\mu = 0, 1$, is positive in a neighborhood of $\pi^{-1}(x_0)$ and trivial outside $\pi^{-1}(U)$ (cf. [2, 5]). Since $\pi^* L$ is semipositive everywhere and positive on $\pi^{-1}(U - x_0)$, there exists a positive integer k_0 such that

$$\pi^* L^k \otimes F^{*(n+\mu)}$$

is semipositive everywhere and positive in $\pi^{-1}(U)$ for all $k \geqq k_0$ and $\mu = 0,1$. (If L is almost positive, i.e. semipositive everywhere and positive in a dense open subset of X, then $\pi^* L^k \otimes F^{*(n+\mu)}$ is also almost positive.)

Now, by the generalized vanishing theorem ([5], Theorem 6), we have

$$H^{\nu}(\hat{X}, \pi^* L^k \otimes F^{*(n+\mu)} \otimes K_{\hat{X}}) = 0, \quad \nu \geqq 1, \quad k \geqq k_0, \quad \mu = 0, 1,$$

(or by Conjecture II, if X is not assumed to be Kähler). With Kodaira's formula

$$K_{\hat{X}} = \pi^*(K_X) \otimes F^{n-1},$$

we get

$$H^1(\hat{X}, \pi^*(E) \otimes F^{*\mu}) = 0, \quad \mu = 1, 2,$$

for $E = L^{k_0} \otimes K_X$. Finally, this is equivalent to

$$H^1(X, E \otimes \mathfrak{m}^\mu) = 0, \quad \mu = 1, 2,$$

where $\mathfrak{m}$ denotes the maximal ideal sheaf associated to x_0.

From the exact sequences

$$0 \to \mathfrak{m}^\mu \to \mathcal{O}_X \to \mathcal{O}_X/\mathfrak{m}^\mu \to 0,$$

we deduce the exactness of

$$H^0(X, E) \to H^0(X, E \otimes (\mathcal{O}_X/\mathfrak{m}^\mu)) \to H^1(X, E \otimes \mathfrak{m}^\mu) = 0.$$

For $\mu = 1$ this means that the global holomorphic sections of E generate the stalk $E_{x_0} = H^0(X, E \otimes (\mathcal{O}_X/\mathfrak{m}))$. Hence the meromorphic mapping

$$\sigma: X \to \mathbb{P}^m,$$

induced by a basis $\sigma_0, \ldots, \sigma_m$ of $H^0(X, E)$, is holomorphic at $x_0 \in X$. For $\mu = 2$, the mapping

$$H^0(X, E) \to H^0(X, E \otimes (\mathcal{O}_X/\mathfrak{m}^2)) \cong \mathcal{O}_{X,x_0}/\mathfrak{m}^2$$

is surjective. This implies that σ has maximal rank at x_0, i.e. defines a closed embedding near x_0. Since then the image $\sigma(V)$ of a suitable neighborhood $V \subset X$ of x_0 is a n-dimensional complex analytic submanifold of an open subset of $\mathbb{P}^m$, there exist n meromorphic functions $\tilde{f}_1, \ldots, \tilde{f}_n$ on $\mathbb{P}^m$ such that

$$d(\tilde{f}_1 | \sigma(V)) \wedge \cdots \wedge d(\tilde{f}_n | \sigma(V))(\sigma(x_0)) \neq 0.$$

Since σ is meromorphic, the functions $\tilde{f}_1, \ldots, \tilde{f}_n$ can be lifted to meromorphic functions $f_1, \ldots, f_n$ on X with

$$df_1 \wedge \cdots \wedge df_n(x_0) \neq 0.$$

A theorem of Remmert [4] implies that $f_1, \ldots, f_n$ are analytically and algebraically independent on X, i.e. X is Moišezon. q.e.d.

References

1. Grauert, H., Riemenschneider, O.: Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen. Inventiones math. **11**, 263—292 (1970).
2. Kodaira, K.: On Kähler varieties of restricted type. Ann. of Math. **60**, (2) 28—48 (1954).

3. Moišezon, B. G.: On n-dimensional compact varieties with n algebraically independent meromorphic functions. I. Amer. Math. Soc. Translat. **63**, (2) 51—177 (1967). (Izvestija Akad. Nauk SSSR, Ser. Mat. **30**, 133—174 (1966).)
4. Remmert, R.: Meromorphe Funktionen in kompakten komplexen Räumen. Math. Ann. **132**, 277—288 (1956).
5. Riemenschneider, O.: Characterizing Moišezon spaces by almost positive coherent analytic sheaves. Math. Z. **123**, 263—284 (1971).

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