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Kähler, Erich

★**Mathematische Werke/Mathematical works. (German)** [[**Mathematical works**]]
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This volume marks the centenary of the birth in 1906 of Erich Kähler. He belongs thus to the generation of André Weil (an exact contemporary) and Henri Cartan and, like them both, he lived to a great age, dying only in 2000. (Weil died in 1998; Cartan, two years Kähler's senior, is still alive.) His fame, though, rests chiefly on work done before his thirtieth birthday, and above all on his 1932 paper [Abh. Math. Sem. Univ. Hamburg **9** (1932), 173–186; Zbl 0005.41301], which introduces what are now called Kähler metrics.

Kähler's name is extraordinarily ubiquitous. It, or its derivatives, occur in more than ten thousand entries in Mathematical Reviews, or nearly twelve thousand if one includes references to $K3$ surfaces, supposedly named after Kummer, Kähler and Kodaira. Even Galois barely does better. Yet Kähler himself remains anything but famous. Probably few of the authors of those ten thousand papers know much of his life, or even of the majority of his work. He founded no clearly identifiable school, although he had a number of successful students who have worked in widely differing areas of mathematics. He seems to have been a quiet man whose interest in mathematics persisted into his old age, but who allowed his work to speak for him.

The editors of this volume, one of them a student of Kähler, hope to explain and perhaps in some degree to change this state of affairs. The volume contains most of Kähler's published mathematical work (with a few notable exceptions described below). It also contains a selection of his writing on other subjects, mainly philosophy, in which mathematics is never far away; some short introductory articles describing Kähler's life, influence and personality; and several commentaries on Kähler's work written especially for this book.

One of the introductory articles is a short biography of Kähler, which when it does venture into mathematics does not always do so happily: for example, it is stated that "A Calabi-Yau manifold has a one-dimensional canonical bundle", which is badly wrong. But it does succeed in its primary purpose, which is to give an impression of the man, and the technical details receive their proper attention later. What the article does not explicitly say, but which becomes apparent, is that Kähler lost many of his best years to the Nazis and the war. After 1934 he published scarcely anything for two decades, although he had published a dozen papers, including the famous one, in the previous five or six years and was still in his twenties. Perhaps one should interpret this as a form of passive resistance. Little is said about his life during this time, but it seems quite clear that there is nothing to be hidden. Kähler evidently found the regime distasteful and was in no position to oppose it actively: he avoided criminal complicity and that, though perhaps unheroic, was more than many achieved and as much as could be realistically demanded. Later in life he came into conflict with the East German government, and this led to his move from Leipzig to Berlin and later to Hamburg, where he remained.

The survey of Kähler's mathematical work that follows is informative, for few people will have true familiarity with all aspects of the mathematics involved. He began as what we would now call an applied mathematician, working on the 3-body problem and related problems in mechanics, but these are always seen as problems in the theory of differential or integro-differential equations. The papers are reprinted here, but they do not seem to have had a major impact although they show high technical skill.

On completing his thesis he turned at once towards pure mathematics, initially in the form of complex function theory in two variables. Here he wrote a significant paper on the behaviour of algebraic functions near singularities whose lasting consequences for the topology of hypersurface singularities are well described in the commentary, later in the volume, by W. Neumann. Further investigation into the subject led him to consideration of the topological properties of complex surfaces, and in particular the topological interpretation of period relations.

From complex surfaces it was a short step to algebraic surfaces and contact with the Italian school of geometry, in particular with Severi in Rome. Moreover, around this time the work of Hodge on periods of algebraic varieties started to appear. By 1932 Kähler was writing about period relations for integrals on algebraic varieties: in some sense the same subject as before, but seen from a completely different point of view. The variety is no longer an auxiliary introduced in order to understand a function, but is an object of interest in its own right with an existence independent of the methods used to describe it. Now, too, differential forms and plurigenera of algebraic varieties make their appearance. A bridge is being built between analysis and geometry, and the language Kähler writes in at this period is Italian, not German as hitherto.

He reverts to German, however, for his most famous paper, which comes next. The names Kähler form, Kähler manifold, Kähler metric and many others allude to this paper of a dozen pages. The observation is simply that many naturally occurring $(1,1)$-forms are closed, and that it is therefore sensible to find out what the special properties of such forms and their associated metrics are. Kähler establishes the basic facts, but there is nothing in the paper that hints at how universal the occurrence of these forms would prove to be. But even if he was perhaps surprised by the extent of his invention's success, he does give clear geometric motivation for studying these forms particularly.

At around the same time Kähler published a short monograph, also in German, entitled *Einführung in die Theorie der Systeme von Differentialgleichungen (Introduction to the theory of systems of differential equations)* [Teubner, Leipzig, 1934; Zbl 0011.16103]. It contains what has become known as the Cartan-Kähler theorem, an important clarification and extension of the work of Élie Cartan in the first years of the twentieth century, using differential forms as tools to solve differential equations. It is not reprinted here, but some of its applications are described in the commentaries.

After that begins the long silence, broken by an application of the monograph to Maxwell's equations and a brief note on Pfaffian systems. During this time Kähler became increasingly interested in arithmetic and algebraic questions, in astronomy and in philosophy. From the early 1950s onwards he published a series of articles, in German, Italian and in one case French, in which he attempted to introduce arithmetic and algebra into the realms of analysis and geometry, with mixed success. The editors of this volume regard this work as Kähler's main achievement. The reviewer is not so sure, but there is certainly much that is of interest. Much of the work in these articles was superseded or even anticipated by Weil, Serre, Hirzebruch and Grothendieck, but Kähler's approach to differential forms in an algebraic context was largely adopted by Grothendieck. This shows, at the very least, that he was concerned with problems that occupied many of the best mathematicians of the time, but the real breakthroughs were now happening elsewhere.

This part of Kähler's work culminated in his immense *Geometria aritmetica*. This book, written in Italian, constituted an entire volume of Annali di matematica in 1958 [Ann. Mat. Pura Appl. (4) **45** (1958), ix+399 pp.; MR0105413]. It is not reprinted here, except for the short introduction. One must read Weil's review in MathSciNet, and the summary of the contents of the book in Berndt's commentary article on pp. 777–846 of this volume, as well as some other parts of the commentaries. Evidently the editors' evaluation of *Geometria aritmetica* differs from Weil's. The reviewer of this book is not rash enough to try to adjudicate. Certainly there are things in *Geometria aritmetica* that are worthy of wider attention than they have had, and a significant merit of this volume is that the commentaries exhibit some of them in a modern context and in English.

Two other topics, related to each other, occupy Kähler's last few papers. One is an interior calculus of differential forms, in which the ring of exterior differentials on a manifold M with a metric g is given a Clifford multiplication satisfying $dx^i \vee dx^k + dx^k \vee dx^i = g^{ik}$. This and the associated interior differentiation are used to give a formulation of the Dirac equation. It has attracted some interest but a more general Clifford algebra approach has proved to be more flexible. The Dirac equation appears again in the three papers, two in German and one in English, on the Poincaré group or new Poincaré group, as Kähler later called it. This is the group of isometries of the hyperbolic metric $(dx^2 + dy^2 + dz^2 + dt^2)/t^2$. The papers are open-ended and the last one, cryptically entitled "Raum-Zeit-Individuum" [Rend. Accad. Naz. Sci. XL Mem. Mat. (5) **16** (1992), 115–177; MR1205748; see also in *Mathematik aus Berlin*, 41–105, WEIDLER, Berlin, 1997; MR1607414], is extremely speculative. Nevertheless the study of this group and its discrete subgroups has proved to be fruitful, as Krieg's commentary elsewhere in the volume illustrates.

The commentaries have more helpful titles, and often go far to explain what the continuing importance of Kähler's work is. They are: Topology of hypersurface singularities (W. D. Neumann); The unabated vitality of Kählerian geometry (J.-P. Bourguignon); Some applications of the Cartan-Kähler theorem to economic theory (I. Ekeland); Kähler differentials and some applications in arithmetic geometry (R. Berndt); Why "Kähler" differentials? (E. Kunz); A neglected aspect of Kähler's work on arithmetic geometry: birational invariants of algebraic varieties over number fields (J.-B. Bost); Kähler's zeta function (R. Berndt); Panorama of zeta functions (A. Deitmar); Eisenstein series on Kähler's Poincaré group (A. Krieg); Supersymmetry, Kähler geometry and beyond (H. Nicolai). These articles provide an important part of the value of the book, since Kähler in his later years often did not work out full details of his ideas. They also collectively give some impression of the range of Kähler's work.

The last fifty pages of the book are an appendix consisting of a few of Kähler's philosophical writings and some further brief commentaries. The style and some of the contents, even though most of the writings are in Kähler's rather elegant Italian, recall Nietzsche: but Kähler's own personal interests, mathematics and Roman Catholicism, predominate, and it is hard to regard the articles as anything more than expressions of personal associations.

The editors of the book clearly hope to correct the reputation that Kähler probably has, of having had just one extraordinarily successful idea. They are partly successful, but perhaps it is more important to understand that Kähler did not just stumble on his major discovery: he knew exactly what he was doing, and although he was only aware of a small part of its importance he saw as much as anyone could at the time.

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