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Riemenschneider, Oswald
Special surface singularities: a survey on the geometry and combinatorics of their deformations.
Analytic varieties and singularities (Japanese) (Kyoto, 1992).
Sūrikaisekikenkyūsho Kōkyūroku No. 807 (1992), 93–118.

This article surveys recent developments and progress in the study of the rich geometrical and combinatorial structure of deformations of so-called special surface singularities.

Although the term special (surface) singularities is not perfectly well defined, people working in that area have a rather good idea what they are talking about: A class of singularities is special if there is a natural construction for its objects in terms of "classical" mathematical objects like groups (finite, discrete, or continuous), symmetric spaces, or combinatorial objects like graphs. Just to mention a few: quotient singularities (finite groups acting on $\mathbf{C}^2$), cusp singularities of modular groups, and so-called triangle singularities, which are special cases of spectra of rings of automorphic forms.

Experience and numerous results during the past 20 or so years have shown that in particular the deformations of special singularities exhibit a rich geometry which reflects geometrical and combinatorial data of the objects defining them in many different, and often unexpected, ways.

The author, one of the principal contributors to this field for more than two decades now, summarizes work by some of his former students and related recent results. He also sketches his own recent study of the global monodromy group for the versal deformation of cyclic quotient surface singularities.

Somewhat more precisely, his topics are: The (reduced) components of the semiuniversal deformation of a quotient surface singularity and their local monodromy groups are now quite well understood due to work of Kollár-Shepherd-Barron, Christophersen, Stevens, De Jong-Van Straten, and Behnke-Christophersen. For cyclic quotient surface singularities the local monodromy groups on the components of the versal base space patch together to form a global monodromy on the versal base space. The group and its representation are completely and purely combinatorially defined in terms of the original group action (work of the author). The author outlines the work of De Jong-Van Straten on the versal deformations of rational quadruple points, using generic projections into three-space and versal deformation theory for nonisolated, weakly normal surface singularities. S. Brohme has applied this technique to finding the versal deformation spaces for certain minimally elliptic singularities of multiplicity 6 (those with reduced tangent cone, i.e. reduced fundamental cycle on the minimal resolution). Starting from P. Kronheimer's work, who a couple of years ago gave a construction of the versal deformation of a rational double point singularity (which is isomorphic to a quotient of $\mathbf{C}^2$ by a finite subgroup of $\mathrm{SL}_2(\mathbf{C})$, and hence a special singularity in the sense defined above) through the representations of the defining finite group, and who thus filled the last gap in the picture of deformations of rational double points, H. Cassens, W. Ebeling, and P. Slodowy have carried out a program to construct that deformation from the representations of the McKay quiver of the group, using only representation theory of quivers. Finally a completely different line of attack for finding the deformations of special surface singularities is outlined. A. Röhr has introduced and studied the concept of formats of equations for rational singularities. A format is a generic structure of the equations that generalizes the well-known determinantal structures of deformations. Formats provide good candidates for components of the versal base space of rational singularities, and they are possibly a very general concept that can be applied to a wide variety of singularities.

{For the collection containing this paper see MR1254038}

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