Algebraic Topology, summer term 2025

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Contents

CHAPTER 1

Homology theory

1. Chain complexes

DEFINITION 1.1. A *chain complex* is a sequence of abelian groups, $(C_n)_{n\in\mathbb{Z}}$, together with homomorphisms $d_n\colon C_n \to C_{n-1}$ for $n \in \mathbb{Z}$, such that $d_{n-1} \circ d_n = 0$.

Let R be an associative ring with unit 1_R. A *chain complex of R-modules* can analoguously be defined as a sequence of R-modules $(C_n)_{n\in\mathbb{Z}}$ with R-linear maps $d_n\colon C_n \to C_{n-1}$ with $d_{n-1} \circ d_n = 0$.

DEFINITION 1.2.

- The d_n are *differentials* or *boundary operators*.
- The $x \in C_n$ are called *n-chains*.
- Is $x \in C_n$ and $d_n x = 0$, then x is an *n-cycle*.

$$Z_n(C) := \{x \in C_n | d_n x = 0\}.$$

- If $x \in C_n$ is of the form $x = d_{n+1}y$ for some $y \in C_{n+1}$, then x is an *n-boundary*.

$$B_n(C) := Im(d_{n+1}) = \{d_{n+1}y, y \in C_{n+1}\}.$$

Note that the cycles and boundaries form subgroups of the chains. As $d_n \circ d_{n+1} = 0$, we know that the image of d_{n+1} is a subgroup of the kernel of d_n and thus

$$B_n(C) \subset Z_n(C).$$

We'll often drop the subscript n from the boundary maps and we'll just write C_* for the chain complex.

DEFINITION 1.3. The abelian group $H_n(C) := Z_n(C)/B_n(C)$ is the *nth homology group of the complex* C_*.

Notation: We denote by $[c]$ the equivalence class of a $c \in Z_n(C)$.
If $c, c' \in C_n$ satisfy that $c - c'$ is a boundary, then c is *homologous* to c'. That's an equivalence relation.

Examples:

1) Consider

$$C_n = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

and let d_1 be the multiplication with $N \in \mathbb{N}$, then

$$H_n(C) = \begin{cases} \mathbb{Z}/N\mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2) Take $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}$ and

$$d_n = \begin{cases} \mathrm{id}_{\mathbb{Z}} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

What is the homology of this chain complex?

3) Consider $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}$ again, but let all boundary maps be trivial. What is the homology of this chain complex?

Definition 1.4. Let C_* and D_* be two chain complexes. A *chain map* $f\colon C_* \to D_*$ is a sequence of homomorphisms $f_n\colon C_n \to D_n$ such that $d_n^D \circ f_n = f_{n-1} \circ d_n^C$ for all n, *i.e.*, the diagram

$$\begin{array}{ccc} C_n & \xrightarrow{d_n^C} & C_{n-1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n-1}} \\ D_n & \xrightarrow{d_n^D} & D_{n-1} \end{array}$$

commutes for all n.

Such an f sends cycles to cycles and boundaries to boundaries. We therefore obtain an induced map

$$H_n(f)\colon H_n(C) \to H_n(D)$$

via $H_n(f)_*[c] = [f_n c]$.

There is a chain map from the chain complex mentioned in Example 1) to the chain complex D_* that is concentrated in degree zero and has $D_0 = \mathbb{Z}/N\mathbb{Z}$. Note, that $H_0(f)$ is an isomorphism on zeroth homology groups.

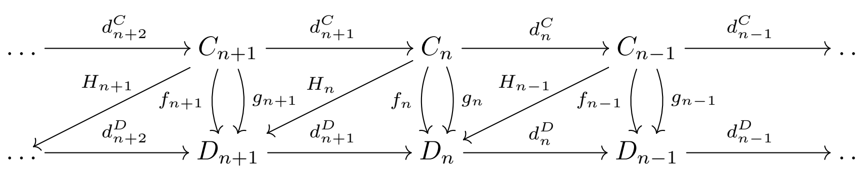
Are there chain maps between the complexes from Examples 2) and 3)?

Lemma 1.5. *If $f\colon C_* \to D_*$ and $g\colon D_* \to E_*$ are two chain maps, then $H_n(g) \circ H_n(f) = H_n(g \circ f)$ for all n.*

When do two chain maps induce the same map on homology?

Definition 1.6. A *chain homotopy* H between two chain maps $f, g\colon C_* \to D_*$ is a sequence of homomorphisms $(H_n)_{n\in\mathbb{Z}}$ with $H_n\colon C_n \to D_{n+1}$ such that for all n

$$d_{n+1}^D \circ H_n + H_{n-1} \circ d_n^C = f_n - g_n.$$



If such an H exists, then f and g are *(chain) homotopic*: $f \simeq g$.

We will later see geometrically defined examples of chain homotopies.

Proposition 1.7.
(a) *Being chain homotopic is an equivalence relation.*
(b) *If f and g are homotopic, then $H_n(f) = H_n(g)$ for all n.*

Proof. (a) If H is a homotopy from f to g, then $-H$ is a homotopy from g to f. Each f is homotopic to itself with $H = 0$. If f is homotopic to g via H and g is homotopic to h via K, then f is homotopic to h via $H + K$.

(b) We have for every cycle $c \in Z_n(C_*)$:

$$H_n(f)[c] - H_n(g)[c] = [f_n c - g_n c] = [d_{n+1}^D \circ H_n(c)] + [H_{n-1} \circ d_n^C(c)] = 0.$$

□

Definition 1.8. Let $f\colon C_* \to D_*$ be a chain map. We call f a *chain homotopy equivalence*, if there is a chain map $g\colon D_* \to C_*$ such that $g \circ f \simeq \mathrm{id}_{C_*}$ and $f \circ g \simeq \mathrm{id}_{D_*}$. The chain complexes C_* and D_* are then *chain homotopically equivalent*.

Note, that such chain complexes have isomorphic homology. However, chain complexes with isomorphic homology do not have to be chain homotopically equivalent. (Can you find a counterexample?)

Definition 1.9. If C_* and C'_* are chain complexes, then their *direct sum*, $C_* \oplus C'_*$, is the chain complex with

$$(C_* \oplus C'_*)_n = C_n \oplus C'_n = C_n \times C'_n$$

with differential $d = d_\oplus$ given by

$$d_\oplus(c, c') = (dc, dc').$$

Similarly, if $(C_*^{(j)}, d^{(j)})_{j \in J}$ is a family of chain complexes, then we can define their direct sum as follows:

$$(\bigoplus_{j \in J} C_*^{(j)})_n := \bigoplus_{j \in J} C_n^{(j)}$$

as abelian groups and the differential $d_\oplus$ is defined via the property that its restriction to the jth summand is $d^{(j)}$.

2. Singular homology

Let $v_0, \ldots, v_n$ be $n+1$ points in $\mathbb{R}^{n+1}$. Consider the convex hull

$$K(v_0, \ldots, v_n) := \{\sum_{i=0}^n t_i v_i | \sum_{i=0}^n t_i = 1, t_i \geqslant 0\}.$$

Definition 2.1. If the vectors $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent, then $K(v_0, \ldots, v_n)$ is the *simplex generated by* $v_0, \ldots, v_n$. We denote such a simplex by $\mathrm{simp}(v_0, \ldots, v_n)$.

Example. The standard topological n-simplex is $\Delta^n := \mathrm{simp}(e_0, \ldots, e_n)$. Here, e_i is the vector in $\mathbb{R}^{n+1}$ that has a 1 in coordinate $i+1$ and is zero in all other coordinates. The first examples are: Δ^0 is the point e_0, Δ^1 is the line segment between e_0 and e_1, Δ^2 is a triangle in $\mathbb{R}^3$ and Δ^3 is homeomorphic to a tetrahedron.

The coordinate description of the n-simplex is

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | \sum t_i = 1, t_i \geqslant 0\}.$$

We consider Δ^n as $\Delta^n \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2} \subset \ldots$.

The boundary of Δ^1 consists of two copies of Δ^0, the boundary of Δ^2 consists of three copies of Δ^1. In general, the boundary of Δ^n consists of $n+1$ copies of Δ^{n-1}.

We need the following *face maps* for $0 \leqslant i \leqslant n$

$$d_i = d_i^{n-1} \colon \Delta^{n-1} \hookrightarrow \Delta^n; (t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

The image of d_i^{n-1} in Δ^n is the face that is opposite to e_i. It is the simplex generated by $e_0, \ldots, e_{i-1}$, $e_{i+1}, \ldots, e_n$.

Draw the examples of the faces in Δ^1 and Δ^2!

Lemma 2.2. *Concerning the composition of face maps, the following rule holds:*

$$d_i^{n-1} \circ d_j^{n-2} = d_j^{n-1} \circ d_{i-1}^{n-2}, \quad 0 \leqslant j < i \leqslant n.$$

Example: face maps for Δ^0 and composition into Δ^2: $d_2 \circ d_0 = d_0 \circ d_1$.

Proof. Both expressions yield

$$d_i^{n-1} \circ d_j^{n-2}(t_0, \ldots, t_{n-2}) = (t_0, \ldots, t_{j-1}, 0, \ldots, t_{i-2}, 0, \ldots, t_{n-2}) = d_j^{n-1} d_{i-1}^{n-2}(t_0, \ldots, t_{n-2}).$$

□

Let X be an arbitrary topological space, $X \neq \varnothing$.

Definition 2.3. A *singular n-simplex in X* is a continuous map $\alpha \colon \Delta^n \to X$.

Note, that α just has to be continuous, not smooth or anything!

Definition 2.4. Let $S_n(X)$ be the free abelian group generated by all singular n-simplices in X. We call $S_n(X)$ the *nth singular chain module of X*.

Elements of $S_n(X)$ are finite sums $\sum_{i\in I}\lambda_i\alpha_i$ with $\lambda_i = 0$ for almost all $i \in I$ and $\alpha_i \colon \Delta^n \to X$.

For all $n \geqslant 0$ there are non-trivial elements in $S_n(X)$, because we assumed that $X \neq \varnothing$: we can always take an $x_0 \in X$ and the constant map $\kappa_{x_0} : \Delta^n \to X$ as α. By convention, we define $S_n(\varnothing) = 0$ for all $n \geqslant 0$.

If we want to define maps from $S_n(X)$ to some abelian group then it suffices to define such a map on generators.

Example. What is $S_0(X)$? A continuous $\alpha \colon \Delta^0 \to X$ is determined by its value $\alpha(e_0) =: x_\alpha \in X$, which is a point in X. A singular 0-simplex $\sum_{i\in I}\lambda_i\alpha_i$ can thus be identified with the formal sum of points $\sum_{i\in I}\lambda_i x_{\alpha_i}$. For instance if you count the zeroes and poles of a meromorphic function with multiplicities then this gives an element in $S_0(X)$. In algebraic geometry a divisor is an element in $S_0(X)$.

Definition 2.5. We define $\partial_i \colon S_n(X) \to S_{n-1}(X)$ on generators

$$\partial_i(\alpha) = \alpha \circ d_i^{n-1}$$

and call it *the ith face of* α.

On $S_n(X)$ we therefore get $\partial_i(\sum_j \lambda_j\alpha_j) = \sum_j \lambda_j(\alpha_j \circ d_i^{n-1})$.

Lemma 2.6. *The face maps on* $S_n(X)$ *satisfy*

$$\partial_j \circ \partial_i = \partial_{i-1} \circ \partial_j, \quad 0 \leqslant j < i \leqslant n.$$

Proof. The proof follows from the one of Lemma 2.2. □

Definition 2.7. We define the boundary operator on singular chains as $\partial \colon S_n(X) \to S_{n-1}(X)$, $\partial = \sum_{i=0}^n (-1)^i\partial_i$.

Lemma 2.8. *The map* ∂ is *a boundary operator,* i.e., $\partial \circ \partial = 0$.

Proof. We calculate

$$\begin{aligned}\partial \circ \partial = (\sum_{j=0}^{n-1}(-1)^j\partial_j) \circ (\sum_{i=0}^{n}(-1)^i\partial_i) &= \sum\sum(-1)^{i+j}\partial_j \circ \partial_i \\ &= \sum_{0\leqslant j<i\leqslant n}(-1)^{i+j}\partial_j \circ \partial_i + \sum_{0\leqslant i\leqslant j\leqslant n-1}(-1)^{i+j}\partial_j \circ \partial_i \\ &= \sum_{0\leqslant j<i\leqslant n}(-1)^{i+j}\partial_{i-1} \circ \partial_j + \sum_{0\leqslant i\leqslant j\leqslant n-1}(-1)^{i+j}\partial_j \circ \partial_i = 0.\end{aligned}$$

□

We therefore obtain the *singular chain complex*, $S_*(X)$,

$$\ldots \longrightarrow S_n(X) \xrightarrow{\partial} S_{n-1}(X) \xrightarrow{\partial} \ldots \xrightarrow{\partial} S_1(X) \xrightarrow{\partial} S_0(X) \longrightarrow 0.$$

We abbreviate $Z_n(S_*(X))$ by $Z_n(X)$, $B_n(S_*(X))$ by $B_n(X)$ and $H_n(S_*(X))$ by $H_n(X)$.

Definition 2.9. For a space X, $H_n(X)$ is the *nth singular homology group of* X.

Note that $Z_0(X) = S_0(X)$.

As an example of a 1-cycle consider a 1-chain $c = \alpha + \beta + \gamma$ where $\alpha, \beta, \gamma \colon \Delta^1 \to X$ such that $\alpha(e_1) = \beta(e_0)$, $\beta(e_1) = \gamma(e_0)$ and $\gamma(e_1) = \alpha(e_0)$ and calculate that $\partial c = 0$.

We need to understand how continuous maps of topological spaces interact with singular chains and singular homology.

Let $f \colon X \to Y$ be a continuous map.

Definition 2.10. The map $f_n = S_n(f) \colon S_n(X) \to S_n(Y)$ is defined on generators $\alpha \colon \Delta^n \to X$ as

$$f_n(\alpha) = f \circ \alpha : \Delta^n \xrightarrow{\alpha} X \xrightarrow{f} Y.$$

Lemma 2.11. *For any continuous $f\colon X \to Y$ we have*

$$\begin{array}{ccc} S_n(X) & \xrightarrow{f_n} & S_n(Y) \\ \downarrow{\scriptstyle \partial^X} & & \downarrow{\scriptstyle \partial^Y} \\ S_{n-1}(X) & \xrightarrow{f_{n-1}} & S_{n-1}(Y), \end{array}$$

i.e., *$(f_n)_n$ is a chain map and hence induces a map $H_n(f)\colon H_n(X) \to H_n(Y)$.*

Proof. By definition

$$\partial^Y(f_n(\alpha)) = \sum_{i=0}^{n}(-1)^i(f \circ \alpha) \circ d_i = \sum_{i=0}^{n}(-1)^i f \circ (\alpha \circ d_i) = f_{n-1}(\partial^X \alpha).$$

□

Of course, the identity map on X induces the identity map on $H_n(X)$ for all $n \geqslant 0$ and if we have a composition of continuous maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

then $S_n(g \circ f) = S_n(g) \circ S_n(f)$ and $H_n(g \circ f) = H_n(g) \circ H_n(f)$. In categorical language, this says precisely that $S_n(-)$ and $H_n(-)$ are functors from the category of topological spaces and continuous maps into the category of abelian groups. Taking all $S_n(-)$ together turns $S_*(-)$ into a functor from topological spaces and continuous maps into the category of chain complexes with chain maps as morphisms.

One implication of Lemma 2.11 is that homeomorphic spaces have isomorphic homology groups:

$$X \cong Y \Rightarrow H_n(X) \cong H_n(Y) \text{ for all } n \geqslant 0.$$

Our first (not too exciting) calculation is the following:

Proposition 2.12. *The homology groups of a one-point space* pt *are trivial but in degree zero,*

$$H_n(\mathrm{pt}) \cong \begin{cases} 0, & \text{if } n > 0, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. For every $n \geqslant 0$ there is precisely one continuous map $\alpha\colon \Delta^n \to \mathrm{pt}$, namely the constant map. We denote this map by κ_n. Then the boundary of κ_n is

$$\partial \kappa_n = \sum_{i=0}^{n}(-1)^i \kappa_n \circ d_i = \sum_{i=0}^{n}(-1)^i \kappa_{n-1} = \begin{cases} \kappa_{n-1}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

For all n we have $S_n(\mathrm{pt}) \cong \mathbb{Z}$ generated by κ_n and therefore the singular chain complex looks as follows:

$$\ldots \xrightarrow{\partial=0} \mathbb{Z} \xrightarrow{\partial=\mathrm{id}_{\mathbb{Z}}} \mathbb{Z} \xrightarrow{\partial=0} \mathbb{Z}.$$

□

3. H_0 and H_1

Before we calculate anything, we define a map.

Proposition 3.1. *For any topological space X there is a homomorphism $\varepsilon\colon H_0(X) \to \mathbb{Z}$ with $\varepsilon \neq 0$ for $X \neq \varnothing$.*

Proof. If $X \neq \varnothing$, then we define $\varepsilon(\alpha) = 1$ for any $\alpha\colon \Delta^0 \to X$, thus $\varepsilon(\sum_{i\in I}\lambda_i\alpha_i) = \sum_{i\in I}\lambda_i$ on $S_0(X)$. As only finitely many λ_i are non-trivial, this is in fact a finite sum.

We have to show that this map is well-defined on homology, *i.e.*, that it vanishes on boundaries. One possibility is to see that ε can be interpreted as the map on singular chains that is induced by the projection map of X to a one-point space.

One can also show the claim directly: Let $S_0(X) \ni c = \partial b$ be a boundary and write $b = \sum_{i\in I} \nu_i\beta_i$ with $\beta_i\colon \Delta^1 \to X$. Then we get

$$\partial b = \partial \sum_{i\in I} \nu_i\beta_i = \sum_{i\in I} \nu_i(\beta_i \circ d_0 - \beta_i \circ d_1) = \sum_{i\in I} \nu_i\beta_i \circ d_0 - \sum_{i\in I} \nu_i\beta_i \circ d_1$$

and hence

$$\varepsilon(c) = \varepsilon(\partial b) = \sum_{i\in I} \nu_i - \sum_{i\in I} \nu_i = 0.$$

$\square$

We said that $S_0(\varnothing)$ is zero, so $H_0(\varnothing) = 0$ and in this case we define ε to be the zero map.

If $X \neq \varnothing$, then any $\alpha\colon \Delta^0 \to X$ can be identified with its image point, so the map ε on $S_0(X)$ counts points in X with multiplicities.

Proposition 3.2. *If X is a path-connected, non-empty space, then $\varepsilon\colon H_0(X) \cong \mathbb{Z}$.*

Proof. As X is non-empty, there is a point $x \in X$ and the constant map κ_x with value x is an element in $S_0(X)$ with $\varepsilon(\kappa_x) = 1$. Therefore ε is surjective. For any other point $y \in X$ there is a continuous path $\omega\colon [0, 1] \to X$ with $\omega(0) = x$ and $\omega(1) = y$. We define $\alpha_\omega\colon \Delta^1 \to X$ as

$$\alpha_\omega(t_0, t_1) = \omega(1 - t_0).$$

Then

$$\partial(\alpha_\omega) = \partial_0(\alpha_\omega) - \partial_1(\alpha_\omega) = \alpha_\omega(e_1) - \alpha_\omega(e_0) = \alpha_\omega(0, 1) - \alpha_\omega(1, 0) = \kappa_y - \kappa_x,$$

and the two generators κ_x, κ_y are homologous. This shows that ε is injective. $\square$

From now on we will identify paths w and their associated 1-simplices α_w.

Corollary 3.3. *If X is of the form $X = \bigsqcup_{i\in I} X_i$ such that the X_i are non-empty and path-connected, then*

$$H_0(X) \cong \bigoplus_{i\in I} \mathbb{Z}.$$

In this case, the zeroth homology group of X is the free abelian group generated by the path-components.

Proof. The singular chain complex of X splits as the direct sum of chain complexes of the X_i:

$$S_n(X) \cong \bigoplus_{i\in I} S_n(X_i)$$

for all n. Boundary summands ∂_i stay in a component, in particular,

$$\partial\colon S_1(X) \cong \bigoplus_{i\in I} S_1(X_i) \to \bigoplus_{i\in I} S_0(X_i) \cong S_0(X)$$

is the direct sum of the boundary operators $\partial\colon S_1(X_i) \to S_0(X_i)$ and the claim follows. $\square$

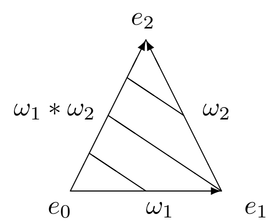
Next, we want to relate H_1 to the fundamental group. Let X be path-connected and $x \in X$.

Lemma 3.4. *Let $\omega_1, \omega_2, \omega$ be paths in X.*

(a) *Constant paths are null-homologous.*
(b) *If $\omega_1(1) = \omega_2(0)$, then $\omega_1 * \omega_2 - \omega_1 - \omega_2$ is a boundary. Here $\omega_1 * \omega_2$ is the concatenation of ω_1 followed by ω_2.*
(c) *If $\omega_1(0) = \omega_2(0), \omega_1(1) = \omega_2(1)$ and if ω_1 is homotopic to ω_2 relative to $\{0, 1\}$, then ω_1 and ω_2 are homologous as singular 1-chains.*
(d) *Any 1-chain of the form $\bar{\omega} * \omega$ is a boundary. Here, $\bar{\omega}(t) := \omega(1 - t)$.*

Proof. For a), consider the constant singular 2-simplex $\alpha(t_0, t_1, t_2) = x$ and c_x, the constant path on x. Then $\partial\alpha = c_x - c_x + c_x = c_x$.

For b), we define a singular 2-simplex $\beta\colon \Delta^2 \to X$ as follows.



We define β on the boundary components of Δ^2 as indicated and prolong it constantly along the sloped inner lines. Then

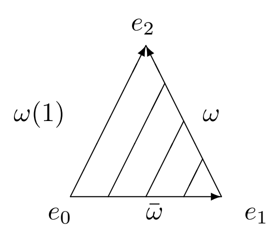
$$\partial\beta = \beta \circ d_0 - \beta \circ d_1 + \beta \circ d_2 = \omega_2 - \omega_1 * \omega_2 + \omega_1.$$

For c): Let $H\colon [0,1] \times [0,1] \to X$ a homotopy from ω_1 to ω_2. As we have that $H(0,t) = \omega_1(0) = \omega_2(0)$, we can factor H through the quotient $[0,1] \times [0,1]/\{0\} \times [0,1] \cong \Delta^2$ with induced map $h\colon \Delta^2 \to X$. Then

$$\partial h = h \circ d_0 - h \circ d_1 + h \circ d_2.$$

The first summand is null-homologous, because it's constant (with value $\omega_1(1) = \omega_2(1)$), the second one is ω_2 and the last is ω_1, thus $\omega_1 - \omega_2$ is null-homologous.

For d): Consider $\gamma\colon \Delta^2 \to X$ as indicated below.



□

Definition 3.5. Let $h\colon \pi_1(X,x) \to H_1(X)$ be the map, that sends the homotopy class of a closed path ω, $[\omega]_{\pi_1}$, to its homology class $[\omega] = [\omega]_{H_1}$. This map is called the *Hurewicz-homomorphism*.

Witold Hurewicz: 1904–1956 `https://en.wikipedia.org/wiki/Witold_Hurewicz` (Mayan pyramids are dangerous, at least for mathematicians.)

Lemma 3.4 ensures that h is well-defined and

$$h([\omega_1][\omega_2]) = h([\omega_1 * \omega_2]) = [\omega_1] + [\omega_2] = h([\omega_1]) + h([\omega_2]);$$

thus h is a homomorphism.

Note that for a closed path ω we have that $[\bar{\omega}] = -[\omega]$ in $H_1(X)$.

Definition 3.6. Let G be an arbitrary group, then its *abelianization*, G_{ab}, is $G/[G,G]$.

Recall that $[G,G]$ is the commutator subgroup of G. That is the smallest subgroup of G containing all commutators $ghg^{-1}h^{-1}, g, h \in G$. It is a normal subgroup of G: If $c \in [G,G]$, then for any $g \in G$ the element $gcg^{-1}c^{-1}$ is a commutator and also by the closure property of subgroups the element $gcg^{-1}c^{-1}c = gcg^{-1}$ is in the commutator subgroup.

Proposition 3.7. *The Hurewicz homomorphism factors through the abelianization of* $\pi_1(X, x)$ *and induces an isomorphism*

$$\pi_1(X, x)_{\text{ab}} \cong H_1(X)$$

for all path-connected X.

$$\begin{array}{ccc} \pi_1(X,x) & \xrightarrow{\quad h \quad} & H_1(X) \\ \big\downarrow p & \nearrow_{h_{\text{ab}}}^{\cong} & \\ \pi_1(X,x)_{\text{ab}} = \pi_1(X,x)/[\pi_1(X,x), \pi_1(X,x)] & & \end{array}$$

Proof. We will construct an inverse to h_{ab}. For any $y \in X$ we choose a path u_y from x to y. For $y = x$ we take u_x to be the constant path on x. Let α be an arbitrary singular 1-simplex and $y_i = \alpha(e_i)$. Define $\phi\colon S_1(X) \to \pi_1(X, x)_{\text{ab}}$ on generators as $\phi(\alpha) = [u_{y_0} * \alpha * \bar{u}_{y_1}]$ and extend ϕ linearly to all of $S_1(X)$, keeping in mind that the composition in π_1 is written multiplicatively.

We have to show that ϕ is trivial on boundaries, so let $\beta\colon \Delta^2 \to X$. Then

$$\phi(\partial\beta) = \phi(\beta \circ d_0 - \beta \circ d_1 + \beta \circ d_2) = \phi(\beta \circ d_0)\phi(\beta \circ d_1)^{-1}\phi(\beta \circ d_2).$$

Abbreviating $\beta \circ d_i$ with α_i we get as a result

$$[u_{y_1} * \alpha_0 * \bar{u}_{y_2}][u_{y_0} * \alpha_1 * \bar{u}_{y_2}]^{-1}[u_{y_0} * \alpha_2 * \bar{u}_{y_1}] = [u_{y_0} * \alpha_2 * \bar{u}_{y_1} * u_{y_1} * \alpha_0 * \bar{u}_{y_2} * u_{y_2} * \bar{\alpha_1} * \bar{u}_{y_0}].$$

Here, we've used that the image of ϕ is abelian. We can reduce $\bar{u}_{y_1} * u_{y_1}$ and $\bar{u}_{y_2} * u_{y_2}$ and are left with $[u_{y_0} * \alpha_2 * \alpha_0 * \bar{\alpha_1} * \bar{u}_{y_0}]$ but $\alpha_2 * \alpha_0 * \bar{\alpha_1}$ is the closed path tracing the boundary of β and therefore it is null-homotopic in X. Thus $\phi(\partial\beta) = 0$ and ϕ passes to a map

$$\phi\colon H_1(X) \to \pi_1(X, x)_{\text{ab}}.$$

The composition $\phi \circ h_{\text{ab}}$ evaluated on the class of a closed path ω gives

$$\phi \circ h_{\text{ab}}[\omega]_{\pi_1} = \phi[\omega]_{H_1} = [u_x * \omega * \bar{u}_x]_{\pi_1}.$$

But we chose u_x to be constant, thus $\phi \circ h_{\text{ab}} = \text{id}$.

If $c = \sum \lambda_i \alpha_i$ is a cycle, then $h_{\text{ab}} \circ \phi(c)$ is of the form $[c + D_{\partial c}]$ where the $D_{\partial c}$-part comes from the contributions of the u_{y_i}. The fact that $\partial(c) = 0$ implies that the summands in $D_{\partial c}$ cancel off and thus $h_{\text{ab}} \circ \phi = \text{id}_{H_1(X)}$. □

Note, that abelianization doesn't change anything for abelian groups, *i.e.*, whenever we have an abelian fundamental group, we know that $H_1(X) \cong \pi_1(X, x)$.

Corollary 3.8. *Knowledge of* π_1 *gives*

$$\begin{aligned} H_1(\mathbb{S}^n) &= 0, \ \textit{for } n > 1, \\ H_1(\mathbb{S}^1) &\cong \mathbb{Z}, \\ H_1(\underbrace{\mathbb{S}^1 \times \ldots \times \mathbb{S}^1}_{n}) &\cong \mathbb{Z}^n, \\ H_1(\mathbb{S}^1 \vee \mathbb{S}^1) &\cong (\mathbb{Z} * \mathbb{Z})_{\text{ab}} \cong \mathbb{Z} \oplus \mathbb{Z}, \\ H_1(\mathbb{R}P^n) &\cong \begin{cases} \mathbb{Z}, & n = 1, \\ \mathbb{Z}/2\mathbb{Z}, & n > 1, \end{cases} \\ H_1(F_g) &\cong \mathbb{Z}^{2g}, \ \textit{for } g \geqslant 1, \\ H_1(K) &\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

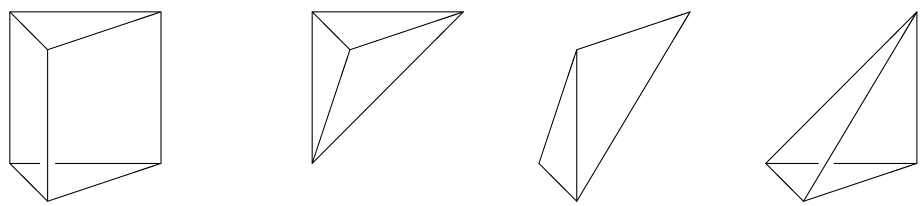
In the last case, K *denotes the Klein bottle.*

4. Homotopy invariance

We want to show that two continuous maps that are homotopic induce identical maps on the level of homology groups.

Heuristics: If $\alpha\colon \Delta^n \to X$ is a singular n-simplex and if f, g are homotopic maps from X to Y, then the homotopy from $f \circ \alpha$ to $g \circ \alpha$ starts on $\Delta^n \times [0,1]$. We want to translate this geometric homotopy into a chain homotopy on the singular chain complex. To that end we have to cut the prism $\Delta^n \times [0,1]$ into $(n+1)$-simplices. In low dimensions this is easy:

$\Delta^0 \times [0,1]$ is homeomorphic to Δ^1, $\Delta^1 \times [0,1] \cong [0,1]^2$ and this can be cut into two copies of Δ^2 and $\Delta^2 \times [0,1]$ is a 3-dimensional prism and that can be glued together from three tetrahedrons, *e.g.*, like



As you might guess now, we use $n+1$ copies of Δ^{n+1} to build $\Delta^n \times [0,1]$.

Definition 4.1. For $i = 0, \ldots, n$ define $p_i\colon \Delta^{n+1} \to \Delta^n \times [0,1]$ as

$$p_i(t_0, \ldots, t_{n+1}) = ((t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}), t_{i+1} + \ldots + t_{n+1}) \in \Delta^n \times [0,1].$$

On the standard basis vectors e_k we obtain

$$p_i(e_k) = \begin{cases} (e_k, 0), & \text{for } 0 \leqslant k \leqslant i, \\ (e_{k-1}, 1), & \text{for } k > i. \end{cases}$$

We obtain maps $P_i\colon S_n(X) \to S_{n+1}(X \times [0,1])$ via $P_i(\alpha) = (\alpha \times \mathrm{id}) \circ p_i$:

$$\Delta^{n+1} \xrightarrow{p_i} \Delta^n \times [0,1] \xrightarrow{\alpha \times \mathrm{id}} X \times [0,1].$$

For $k = 0, 1$ let $j_k\colon X \to X \times [0,1]$ be the inclusion $x \mapsto (x, k)$.

Lemma 4.2. *The maps P_i satisfy the following relations*

(a) $\partial_0 \circ P_0 = S_n(j_1)$,
(b) $\partial_{n+1} \circ P_n = S_n(j_0)$,
(c) $\partial_i \circ P_i = \partial_i \circ P_{i-1}$ *for* $1 \leqslant i \leqslant n$.
(d)

$$\partial_j \circ P_i = \begin{cases} P_i \circ \partial_{j-1}, & \text{for } i \leqslant j-2 \\ P_{i-1} \circ \partial_j, & \text{for } i \geqslant j+1. \end{cases}$$

Proof. Note that it suffices to check the corresponding claims for the p_i's and d_j's.

For the first two points, we note that on Δ^n we have

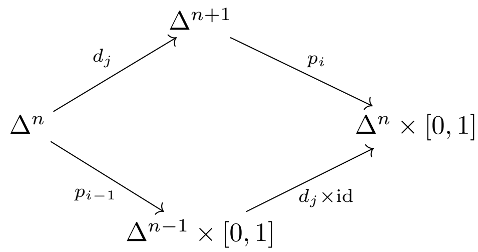
$$p_0 \circ d_0(t_0, \ldots, t_n) = p_0(0, t_0, \ldots, t_n) = ((t_0, \ldots, t_n), \sum t_i) = ((t_0, \ldots, t_n), 1) = j_1(t_0, \ldots, t_n)$$

and

$$p_n \circ d_{n+1}(t_0, \ldots, t_n) = p_n(t_0, \ldots, t_n, 0) = ((t_0, \ldots, t_n), 0) = j_0(t_0, \ldots, t_n).$$

For c), one checks that $p_i \circ d_i = p_{i-1} \circ d_i$ on Δ^n: both give $((t_0, \ldots, t_n), \sum_{j=i}^n t_j)$ on $(t_0, \ldots, t_n)$.

For d) in the case $i \geqslant j+1$, consider the following diagram



Checking coordinates one sees that this diagram commutes. The remaining case follows from a similar observation. □

Definition 4.3. We define $P\colon S_n(X) \to S_{n+1}(X \times [0,1])$ as $P = \sum_{i=0}^{n}(-1)^i P_i$.

Lemma 4.4. *The map P is a chain homotopy between $(S_n(j_0))_n$ and $(S_n(j_1))_n$*, i.e., $\partial \circ P + P \circ \partial = S_n(j_1) - S_n(j_0)$.

Proof. We take an $\alpha\colon \Delta^n \to X$ and calculate

$$\partial P\alpha + P\partial\alpha = \sum_{i=0}^{n}\sum_{j=0}^{n+1}(-1)^{i+j}\partial_j P_i \alpha + \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j} P_i \partial_j \alpha.$$

If we single out the terms involving the pairs of indices $(0,0)$ and $(n,n+1)$ in the first sum, we are left with

$$S_n(j_1)(\alpha) - S_n(j_0)(\alpha) + \sum_{(i,j)\neq(0,0),(n,n+1)} (-1)^{i+j}\partial_j P_i\alpha + \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j}P_i\partial_j\alpha.$$

Using Lemma 4.2 we see that only the first two summands survive. □

So, finally we can prove the main result of this section:

Theorem 4.5. *(Homotopy invariance)*
If $f, g\colon X \to Y$ are homotopic maps, then they induce the same map on homology.

Proof. Let $H\colon X \times [0,1] \to Y$ be a homotopy from f to g, *i.e.*, $H \circ j_0 = f$ and $H \circ j_1 = g$. Set $K_n := S_{n+1}(H) \circ P$. We claim that $(K_n)_n$ is a chain homotopy between $(S_n(f))_n$ and $(S_n(g))_n$. Note that H induces a chain map $(S_n(H))_n$. Therefore we get

$$\begin{aligned}\partial \circ S_{n+1}(H) \circ P + S_n(H) \circ P \circ \partial &= S_n(H) \circ \partial \circ P + S_n(H) \circ P \circ \partial \\ &= S_n(H) \circ (\partial \circ P + P \circ \partial) \\ &= S_n(H) \circ (S_n(j_1) - S_n(j_0)) = S_n(H \circ j_1) - S_n(H \circ j_0) \\ &= S_n(g) - S_n(f).\end{aligned}$$

Hence these two maps are chain homotopic and $H_n(g) = H_n(f)$ for all n. □

Corollary 4.6. *If two spaces X, Y are homotopy equivalent, then $H_*(X) \cong H_*(Y)$. In particular, if X is contractible, then*

$$H_*(X) \cong \begin{cases} \mathbb{Z}, & \text{for } * = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Examples. As $\mathbb{R}^n$ is contractible for all n, the above corollary gives that its homology is trivial but in degree zero where it consists of the integers.

As the Möbius strip is homotopy equivalent to $\mathbb{S}^1$, we know that their homology groups are isomorphic.

If you know about vector bundles: the zero section of a vector bundle induces a homotopy equivalence between the base and the total space, hence these two have isomorphic homology groups.

5. The long exact sequence in homology

A typical situation is that there is a subspace A of a topological space X and you might know something about A or X and want to calculate the homology of the other space using that partial information.

But before we can move on to topological applications we need some techniques about chain complexes. We need to know that a short exact sequence of chain complexes gives rise to a long exact sequence in homology.

Definition 5.1. Let A, B, C be abelian groups and

$$A \xrightarrow{f} B \xrightarrow{g} C$$

a sequence of homomorphisms. Then this sequence is *exact*, if the image of f is the kernel of g.

Definition 5.2. If

$$\ldots \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \xrightarrow{f_{i-1}} \ldots$$

is a sequence of homomorphisms of abelian groups (indexed over the integers), then this sequence is called *(long) exact*, if it is exact at every A_i, *i.e.*, the image of f_{i+1} is the kernel of f_i for all i.

An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a *short exact sequence.*

Examples. The sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is a short exact sequence.

If $\iota\colon U \to A$ is a monomorphism, then $0 \longrightarrow U \xrightarrow{\iota} A$ is exact. Similarly, an epimorphism $\varrho\colon B \to Q$ gives rise to an exact sequence $B \xrightarrow{\varrho} Q \longrightarrow 0$ and an isomorphism $\phi\colon A \cong A'$ sits in an exact sequence $0 \longrightarrow A \xrightarrow{\phi} A' \longrightarrow 0$.

A sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact iff f is injective, the image of f is the kernel of g and g is an epimorphism. Another equivalent description is to view a sequence as above as a chain complex with vanishing homology groups. Homology measures the deviation from exactness.

Definition 5.3. If A_*, B_*, C_* are chain complexes and $f_*\colon A_* \to B_*$, $g\colon B_* \to C_*$ are chain maps, then we call the sequence

$$A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_*$$

exact, if the image of f_n is the kernel of g_n for all $n \in \mathbb{Z}$.

Thus such an exact sequence of chain complexes is a commuting double ladder

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\
A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\
A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\
A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n+1} \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\
\vdots & & \vdots & & \vdots
\end{array}
$$

in which every row is exact.

Example. Let p be a prime, then

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} & \xrightarrow{0} & 0 \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p^2} & & \downarrow \\
\mathbb{Z} & \xrightarrow{p} & \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/p\mathbb{Z} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \text{id}} \\
\mathbb{Z}/p\mathbb{Z} & \xrightarrow{p} & \mathbb{Z}/p^2\mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/p\mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

has exact rows and columns, in particular it is an exact sequence of chain complexes. Here, π denotes varying canonical projection maps.

Proposition 5.4. *If* $0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ *is a short exact sequence of chain complexes, then there exists a homomorphism* $\delta\colon H_n(C_*) \to H_{n-1}(A_*)$ *for all* $n \in \mathbb{Z}$ *which is natural,* i.e., *if*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A_* & \xrightarrow{f} & B_* & \xrightarrow{g} & C_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A'_* & \xrightarrow{f'} & B'_* & \xrightarrow{g'} & C'_* & \longrightarrow & 0
\end{array}
$$

is a commutative diagram of chain maps in which the rows are exact then $H_{n-1}(\alpha) \circ \delta = \delta \circ H_n(\gamma)$*,*

$$
\begin{array}{ccc}
H_n(C_*) & \xrightarrow{\delta} & H_{n-1}(A_*) \\
{\scriptstyle H_n(\gamma)}\downarrow & & \downarrow{\scriptstyle H_{n-1}(\alpha)} \\
H_n(C'_*) & \xrightarrow{\delta} & H_{n-1}(A'_*)
\end{array}
$$

The method of proof is an instance of a *diagram chase*. The homomorphism δ is called *connecting homomorphism*. The implicit claim in the proposition above is that δ is not always the zero map.

Proof. We show the existence of a δ first and then prove that the constructed map satisfies the naturality condition.

a) Definition of δ:

Is $c \in C_n$ with $d(c) = 0$, then we choose a $b \in B_n$ with $g_n b = c$. This is possible because g_n is surjective. We know that $dg_n b = dc = 0 = g_{n-1} db$ thus db is in the kernel of g_{n-1}, hence it is in the image of f_{n-1}. Thus there is an $a \in A_{n-1}$ with $f_{n-1} a = db$. We have that $f_{n-2} da = df_{n-1} a = ddb = 0$ and as f_{n-2} is injective, this shows that a is a cycle.

We define $\delta[c] := [a]$.

$$\begin{array}{ccc} & & B_n \ni b \stackrel{g_n}{\longmapsto} c \in C_n \\ & & \\ A_{n-1} \ni a \stackrel{f_{n-1}}{\longmapsto} db \in B_{n-1} & & \end{array}$$

In order to check that δ is well-defined, we assume that there are b and b' with $g_n b = g_n b' = c$. Then $g_n(b - b') = 0$ and thus there is an $\tilde{a} \in A_n$ with $f_n \tilde{a} = b - b'$. Define a' as $a - d\tilde{a}$. Then

$$f_{n-1} a' = f_{n-1} a - f_{n-1} d\tilde{a} = db - db + db' = db'$$

because $f_{n-1} d\tilde{a} = db - db'$. As f_{n-1} is injective, we get that a' is uniquely determined with this property. As a is homologous to a' we get that $[a] = [a'] = \delta[c]$, thus the latter is independent of the choice of b.

In addition, we have to make sure that the value stays the same if we add a boundary term to c, *i.e.*, take $c' = c + d\tilde{c}$ for some $\tilde{c} \in C_{n+1}$. Choose preimages of $c, \tilde{c}$ under g_n and g_{n+1}, *i.e.*, b and $\tilde{b}$ with $g_n b = c$ and $g_{n+1}\tilde{b} = \tilde{c}$. Then the element $b' = b + d\tilde{b}$ has boundary $db' = db$ and thus both choices will result in the same a.

Therefore $\delta\colon H_n(C_*) \to H_{n-1}(A_*)$ is well-defined.

b) We have to show that δ is natural with respect to maps of short exact sequences.

Let $c \in Z_n(C_*)$, then $\delta[c] = [a]$ for a $b \in B_n$ with $g_n b = c$ and an $a \in A_{n-1}$ with $f_{n-1} a = db$. Therefore, $H_{n-1}(\alpha)(\delta[c]) = [\alpha_{n-1}(a)]$.

On the other hand, we have

$$f'_{n-1}(\alpha_{n-1} a) = \beta_{n-1}(f_{n-1} a) = \beta_{n-1}(db) = d\beta_n b$$

and

$$g'_n(\beta_n b) = \gamma_n g_n b = \gamma_n c$$

and we can conclude that by the construction of δ

$$\delta[\gamma_n(c)] = [\alpha_{n-1}(a)]$$

and this shows $\delta \circ H_n(\gamma) = H_{n-1}(\alpha) \circ \delta$. □

With this auxiliary result at hand we can now prove the main result in this section:

Proposition 5.5. *For any short exact sequence*

$$0 \longrightarrow A_* \stackrel{f}{\longrightarrow} B_* \stackrel{g}{\longrightarrow} C_* \longrightarrow 0$$

of chain complexes we obtain a long exact sequence of homology groups

$$\ldots \stackrel{\delta}{\longrightarrow} H_n(A_*) \stackrel{H_n(f)}{\longrightarrow} H_n(B_*) \stackrel{H_n(g)}{\longrightarrow} H_n(C_*) \stackrel{\delta}{\longrightarrow} H_{n-1}(A_*) \stackrel{H_{n-1}(f)}{\longrightarrow} \ldots$$

Proof. a) Exactness at the spot $H_n(B_*)$:

We have $H_n(g) \circ H_n(f)[a] = [g_n(f_n(a))] = 0$ because the composition of g_n and f_n is zero. This proves that the image of $H_n(f)$ is contained in the kernel of $H_n(g)$.

For the converse, let $[b] \in H_n(B_*)$ with $[g_n b] = 0$. Then there is a $c \in C_{n+1}$ with $dc = g_n b$. As g_{n+1} is surjective, we find a $b' \in B_{n+1}$ with $g_{n+1} b' = c$. Hence

$$g_n(b - db') = g_n b - dg_{n+1} b' = dc - dc = 0.$$

Exactness gives an $a \in A_n$ with $f_n a = b - db'$ and $da = 0$ and therefore $f_n a$ is homologous to b and $H_n(f)[a] = [b]$ thus the kernel of $H_n(g)$ is contained in the image of $H_n(f)$.

b) Exactness at the spot $H_n(C_*)$:

Let $b \in H_n(B_*)$, then $\delta[g_n b] = 0$ because b is a cycle, so 0 is the only preimage under f_{n-1} of $db = 0$. Therefore the image of $H_n(g)$ is contained in the kernel of δ.

Now assume that $\delta[c] = 0$, thus in the construction of δ, the a is a boundary, $a = da'$. Then for a preimage of c under g_n, b, we have by the definition of a

$$d(b - f_n a') = db - df_n a' = db - f_{n-1}a = 0.$$

Thus $b - f_n a'$ is a cycle and $g_n(b - f_n a') = g_n b - g_n f_n a' = g_n b - 0 = g_n b = c$, so we found a preimage for $[c]$ and the kernel of δ is contained in the image of $H_n(g)$.

c) Exactness at $H_{n-1}(A_*)$:

Let c be a cycle in $Z_n(C_*)$. Again, we choose a preimage b of c under g_n and an a with $f_{n-1}(a) = db$. Then $H_{n-1}(f)\delta[c] = [f_{n-1}(a)] = [db] = 0$. Thus the image of δ is contained in the kernel of $H_{n-1}(f)$.

If $a \in Z_{n-1}(A_*)$ with $H_{n-1}(f)[a] = 0$. Then $f_{n-1}a = db$ for some $b \in B_n$. Take $c = g_n b$. Then by definition $\delta[c] = [a]$. □

6. The long exact sequence of a pair of spaces

Let X be a topological space and $A \subset X$ a subspace of X. Consider the inclusion map $i\colon A \to X$, $i(a) = a$. We obtain an induced map $S_n(i)\colon S_n(A) \to S_n(X)$, but we know that the inclusion of spaces doesn't have to yield a monomorphism on homology groups. For instance, we can include $A = \mathbb{S}^1$ into $X = \mathbb{D}^2$.

We consider pairs of spaces (X, A).

Definition 6.1. The *relative chain complex of* (X, A) is

$$S_*(X, A) := S_*(X)/S_*(A).$$

Alternatively, $S_n(X, A)$ is isomorphic to the free abelian group generated by all n-simplices $\beta\colon \Delta^n \to X$ whose image is not completely contained in A, *i.e.*, $\beta(\Delta^n) \cap (X \setminus A) \neq \varnothing$.

Definition 6.2.

- Elements in $S_n(X, A)$ are called *relative chains in* (X, A)
- Cycles in $S_n(X, A)$ are chains c with $\partial^X(c)$ whose generators have image in A. These are *relative cycles.*
- Boundaries in $S_n(X, A)$ are chains c in X such that $c = \partial^X b + a$ where a is a chain in A.

A continuous map $f\colon X \to Y$ with $f(A) \subset B$ is denoted by $f\colon (X, A) \to (Y, B)$. Such maps induce chain maps $S_*(f)\colon S_*(X, A) \to S_*(Y, B)$.

The following facts are immediate from the definition:

(a) $S_n(X, \varnothing) \cong S_n(X)$.
(b) $S_n(X, X) = 0$.
(c) $S_n(X \sqcup X', X') \cong S_n(X)$.

Definition 6.3. The *relative homology groups of* (X, A) are

$$H_n(X, A) := H_n(S_*(X, A)).$$

Theorem 6.4. *For any pair of topological spaces $A \subset X$ we obtain a long exact sequence*

$$\ldots \xrightarrow{\delta} H_n(A) \xrightarrow{H_n(i)} H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} \ldots$$

For a map $f\colon (X, A) \to (Y, B)$ we get an induced map of long exact sequences

$$\begin{array}{ccccccccc}
\ldots \xrightarrow{\delta} & H_n(A) & \xrightarrow{H_n(i)} & H_n(X) & \longrightarrow & H_n(X, A) & \xrightarrow{\delta} & H_{n-1}(A) & \xrightarrow{H_{n-1}(i)} \ldots \\
& \big\downarrow {\scriptstyle H_n(f|_A)} & & \big\downarrow {\scriptstyle H_n(f)} & & \big\downarrow {\scriptstyle H_n(f)} & & \big\downarrow {\scriptstyle H_{n-1}(f|_A)} & \\
\ldots \xrightarrow[\delta]{} & H_n(B) & \xrightarrow[H_n(i)]{} & H_n(Y) & \longrightarrow & H_n(Y, B) & \xrightarrow[\delta]{} & H_{n-1}(B) & \xrightarrow[H_{n-1}(i)]{} \ldots
\end{array}$$

Proof. By definition of $S_*(X,A)$ the sequence

$$0 \longrightarrow S_*(A) \xrightarrow{S_*(i)} S_*(X) \xrightarrow{\pi} S_*(X,A) \longrightarrow 0$$

is an exact sequence of chain complexes and by Proposition 5.5 we obtain the first claim. For a map f as above the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_n(A) & \xrightarrow{S_n(i)} & S_n(X) & \xrightarrow{\pi} & S_n(X,A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle S_n(f|_A)} & & \downarrow{\scriptstyle S_n(f)} & & \downarrow{\scriptstyle S_n(f)/S_n(f|_A)} & & \\ 0 & \longrightarrow & S_n(B) & \xrightarrow{S_n(i)} & S_n(Y) & \xrightarrow{\pi} & S_n(Y,B) & \longrightarrow & 0 \end{array}$$

commutes. □

Example. Let $A = \mathbb{S}^{n-1}$ and $X = \mathbb{D}^n$, then we know that $H_j(i)$ is trivial for $j > 0$. From the long exact sequence we get that $\delta\colon H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong H_{j-1}(\mathbb{S}^{n-1})$ for $j > 1$ and $n \geqslant 1$.

Proposition 6.5. *If $i\colon A \hookrightarrow X$ is a weak retract,* i.e., *if there is an $r\colon X \to A$ with $r \circ i \simeq \mathrm{id}_A$, then*

$$H_n(X) \cong H_n(A) \oplus H_n(X,A), \quad 0 \leqslant n.$$

Proof. From the assumption we get that $H_n(r) \circ H_n(i) = H_n(\mathrm{id}_A) = \mathrm{id}_{H_n(A)}$ for all n and hence $H_n(i)$ is injective for all n. This implies that $0 \longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(X)$ is exact. Injectivity of $H_{n-1}(i)$ yields that the image of $\delta\colon H_n(X,A) \to H_{n-1}(A)$ is trivial. Therefore we get short exact sequences

$$0 \longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{\pi_*} H_n(X,A) \longrightarrow 0$$

for all n. As $H_n(r)$ is a left-inverse for $H_n(i)$ we obtain a splitting

$$H_n(X) \cong H_n(A) \oplus H_n(X,A)$$

because we map $[c] \in H_n(X)$ to $([rc], \pi_*[c])$ with inverse

$$H_n(A) \oplus H_n(X,A) \ni ([a],[b]) \mapsto H_n(i)[a] + [a'] - H_n(i \circ r)[a'] \in H_n(X)$$

for any $[a'] \in H_n(X)$ with $\pi_*[a'] = [b]$. The second map is well-defined: if $[a'']$ is another element with $\pi_*[a''] = [b]$, then $[a' - a'']$ is of the form $H_n(i)[\tilde{a}]$ because this element is in the kernel of π_* and hence $[a' - a''] - H_n(ir)[a' - a'']$ is trivial. □

Proposition 6.6. *For any $\varnothing \neq A \subset X$ such that $A \subset X$ is a deformation retract we get*

$$H_n(i)\colon H_n(A) \cong H_n(X), \quad H_n(X,A) \cong 0, \quad 0 \leqslant n.$$

Proof. Recall, that $i\colon A \hookrightarrow X$ is a *deformation retract*, if there is a homotopy $R\colon X \times [0,1] \to X$ such that

(a) $R(x,0) = x$ for all $x \in X$,
(b) $R(x,1) \in A$ for all $x \in X$, and
(c) $R(a,1) = a$ for all $a \in A$.

In particular, R is a homotopy from id_X to $i \circ r$ where $r = R(-,1)\colon X \to A$. Condition (c) can be rewritten as $r \circ i = \mathrm{id}_A$, *i.e.*, r is a retraction, and thus A and X are homotopically equivalent and $H_n(i)$ is an isomorphism for all $n \geqslant 0$. □

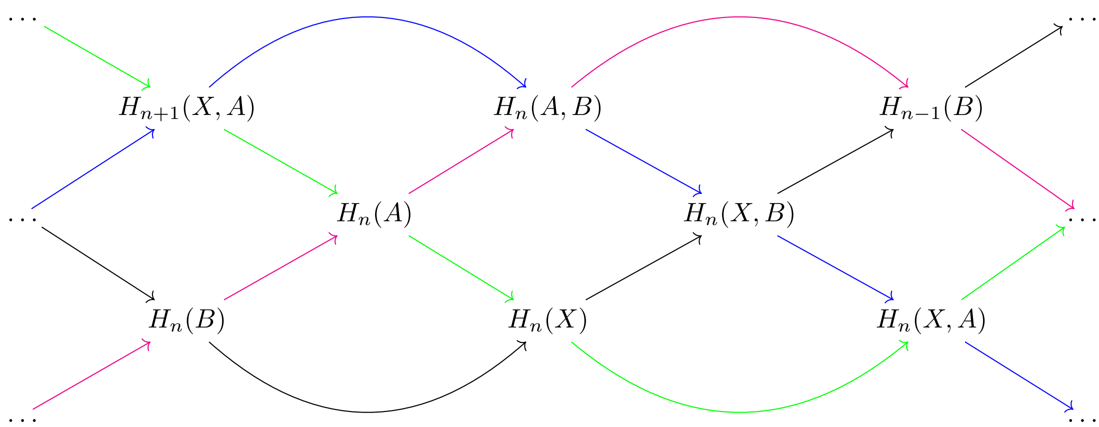
Definition 6.7. If X has two subspaces $A, B \subset X$, then (X,A,B) is called a *triple*, if $B \subset A \subset X$.

Any triple gives rise to three pairs of spaces (X,A), (X,B) and (A,B) and accordingly we have three long exact sequences in homology. But there is another one.

Proposition 6.8. *For any triple (X, A, B) there is a natural long exact sequence*

$$\ldots \longrightarrow H_n(A,B) \longrightarrow H_n(X,B) \longrightarrow H_n(X,A) \xrightarrow{\delta} H_{n-1}(A,B) \longrightarrow \ldots$$

This sequence is part of the following braided commutative diagram displaying four long exact sequences



In particular, the connecting homomorphism $\delta\colon H_n(X,A) \to H_{n-1}(A,B)$ is the composite $\delta = \pi_*^{(A,B)} \circ \delta^{(X,A)}$.

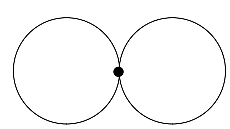
Proof. Consider the sequence

$$0 \longrightarrow S_n(A)/S_n(B) \longrightarrow S_n(X)/S_n(B) \longrightarrow S_n(X)/S_n(A) \longrightarrow 0.$$

This sequence is exact, because $S_n(B) \subset S_n(A) \subset S_n(X)$. □

7. Excision

The aim is to simplify relative homology groups. Let $A \subset X$ be a subspace. Then it is easy to see that $H_*(X, A)$ is not isomorphic to $H_*(X \setminus A)$: Consider the figure eight as X and A as the point connecting the two copies of $\mathbb{S}^1$, then $H_0(X, A)$ is trivial, but $H_0(X \setminus A) \cong \mathbb{Z} \oplus \mathbb{Z}$.



So if we want to simplify $H_*(X, A)$ by excising something, then we have to be more careful. The first step towards that is to make singular simplices 'smaller'. The technique is called barycentric subdivision and that is a tool that's frequently used.

First, we construct cones. Let $v \in \Delta^p$ and let $\alpha\colon \Delta^n \to \Delta^p$ be a singular n-simplex in Δ^p.

Definition 7.1. The *cone of α with respect to v* is $K_v(\alpha)\colon \Delta^{n+1} \to \Delta^p$,

$$(t_0, \ldots, t_{n+1}) \mapsto \begin{cases} (1 - t_{n+1})\alpha(\frac{t_0}{1-t_{n+1}}, \ldots, \frac{t_n}{1-t_{n+1}}) + t_{n+1}v, & t_{n+1} < 1, \\ v, & t_{n+1} = 1. \end{cases}$$

This map is well-defined and continuous. On the standard basis vectors K_v gives $K_v(e_i) = \alpha(e_i)$ for $0 \leqslant i \leqslant n$ but $K_v(e_{n+1}) = v$. Extending K_v linearly gives a map

$$K_v\colon S_n(\Delta^p) \to S_{n+1}(\Delta^p).$$

Lemma 7.2. *The map K_v satisfies*

- *$\partial K_v(c) = \varepsilon(c).\kappa_v - c$ for $c \in S_0(\Delta^p)$, $\kappa_v(e_0) = v$ and ε the augmentation.*
- *For $n > 0$ we have that $\partial \circ K_v - K_v \circ \partial = (-1)^{n+1}\mathrm{id}$.*

Proof. For a singular 0-simplex $\alpha\colon \Delta^0 \to \Delta^p$ we know that $\varepsilon(\alpha) = 1$ and we calculate

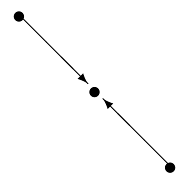
$$\partial K_v(\alpha)(e_0) = (K_v(\alpha) \circ d_0)(e_0) - (K_v(\alpha) \circ d_1)(e_0) = K_v(\alpha)(e_1) - K_v(\alpha)(e_0) = v - \alpha(e_0).$$

For $n > 0$ we have to calculate $\partial_i K_v(\alpha)$ and it is straightforward to see that $\partial_{n+1} K_v(\alpha) = \alpha$ and $\partial_i(K_v(\alpha)) = K_v(\partial_i \alpha)$ for all $i < n+1$. □

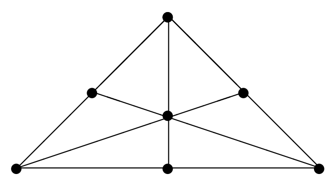
Definition 7.3. For $\alpha\colon \Delta^n \to \Delta^p$ let $v(\alpha) = v := \frac{1}{n+1}\sum_{i=0}^n \alpha(e_i)$. The *barycentric subdivision* $B\colon S_n(\Delta_p) \to S_n(\Delta_p)$ is defined inductively as $B(\alpha) = \alpha$ for $\alpha \in S_0(\Delta_p)$ and $B(\alpha) = (-1)^n K_v(B(\partial\alpha))$ for $n > 0$.

For $n \geqslant 1$ this yields $B(\alpha) = \sum_{i=0}^n (-1)^{n+i} K_v(B(\partial_i \alpha))$.

If we take $n = p$ and $\alpha = \mathrm{id}_{\Delta^n}$, then for small n this looks as follows: You cannot subdivide a point any further. For $n = 1$ we get



And for $n = 2$ we get (up to tilting)



Lemma 7.4. *The barycentric subdivision is a chain map.*

Proof. We have to show that $\partial B = B\partial$. If α is a singular zero chain, then $\partial B\alpha = \partial\alpha = 0$ and $B\partial\alpha = B(0) = 0$.

Let $n = 1$. Then

$$\partial B\alpha = -\partial K_v B(\partial_0\alpha) + \partial K_v B(\partial_1 \alpha).$$

But the boundary terms are zero chains and there B is the identity so we get

$$-\partial K_v(\partial_0\alpha) + \partial K_v(\partial_1\alpha) = -\kappa_v + \partial_0\alpha + \kappa_v - \partial_1\alpha = \partial\alpha = B\partial\alpha.$$

(Note, that the v is $v(\alpha)$, not a $v(\partial_i\alpha)$.)

We prove the claim inductively, so let $\alpha \in S_n(\Delta^p)$. Then

$$\begin{aligned}\partial B\alpha =& (-1)^n \partial K_v(B\partial\alpha)\\ =& (-1)^n((-1)^n B\partial\alpha + K_v \partial B\partial\alpha)\\ =& B\partial\alpha + (-1)^n K_v B\partial\partial\alpha = B\partial\alpha.\end{aligned}$$

Here, the first equality is by definition, the second one follows by Lemma 7.2 and then we use the induction hypothesis and the fact that $\partial\partial = 0$. □

Our aim is to show that B doesn't change anything on the level of homology groups and to that end we prove that it is chain homotopic to the identity.

We construct $\psi_n\colon S_n(\Delta^p) \to S_{n+1}(\Delta^p)$ again inductively as

$$\psi_0(\alpha) := 0, \quad \psi_n(\alpha) := (-1)^{n+1} K_v(B\alpha - \alpha - \psi_{n-1}\partial\alpha)$$

with $v = \frac{1}{n+1}\sum_{i=0}^n \alpha(e_i)$.

Lemma 7.5. *The sequence $(\psi_n)_n$ is a chain homotopy from B to the identity.*

Proof. For $n = 0$ we have $\partial\psi_0 = 0$ and this agrees with $B - \mathrm{id}$ in that degree.

For $n = 1$, we get

$$\partial\psi_1 + \psi_0\partial = \partial\psi_1 = \partial(K_vB - K_v - K_v\psi_0\partial) = \partial K_vB - \partial K_v.$$

With Lemma 7.2 we can transform the latter to $B + K_v\partial B - \partial K_v$ and as B is a chain map, this is $B + K_vB\partial - \partial K_v$. In chain degree one $B\partial$ agrees with ∂, thus this reduces to

$$B + K_v\partial - \partial K_v = B - (\partial K_v - K_v\partial) = B - \mathrm{id}.$$

So, finally we can do the inductive step:

$$\begin{aligned}\partial\psi_n =&(-1)^{n+1}\partial K_v(B - \mathrm{id} - \psi_{n-1}\partial)\\ =&(-1)^{n+1}\partial K_vB - (-1)^{n+1}\partial K_v - (-1)^{n+1}\partial K_v\psi_{n-1}\partial\\ =&(-1)^{n+1}((-1)^{n+1}B + K_v\partial B)\\ &\qquad - (-1)^{n+1}((-1)^{n+1}\mathrm{id} + K_v\partial)\\ &\qquad - (-1)^{n+1}((-1)^{n+1}\psi_{n-1}\partial + K_v\partial\psi_{n-1}\partial)\\ =&B - \mathrm{id} - \psi_{n-1}\partial + \text{remaining terms}\end{aligned}$$

The equation

$$K_v\partial\psi_{n-1}\partial + K_v\psi_{n-2}\partial^2 = K_vB\partial - K_v\partial$$

from the inductive assumption ensures that these remaining terms give zero. □

Definition 7.6. A singular n-simplex $\alpha\colon \Delta^n \to \Delta^p$ is called *affine*, if

$$\alpha(\sum_{i=0}^{n} t_ie_i) = \sum_{i=0}^{n} t_i\alpha(e_i).$$

We abbreviate $\alpha(e_i)$ with v_i, so $\alpha(\sum_{i=0}^n t_ie_i) = \sum_{i=0}^n t_iv_i$ and we call the v_i's the *vertices of* α.

Definition 7.7. Let A be a subset of a metric space (X, d). The *diameter of* A is

$$\sup\{d(x, y)|x, y \in A\}$$

and we denote it by $\mathrm{diam}(A)$.

Accordingly, the *diameter of an affine n-simplex α in Δ^p* is the diameter of its image, and we abbreviate that with $\mathrm{diam}(\alpha)$.

Lemma 7.8. *For any affine α every simplex in the chain $B\alpha$ has diameter $\leqslant \frac{n}{n+1}\mathrm{diam}(\alpha)$.*

Either you believe this lemma, or you prove it, or you check Bredon, Proof of Lemma 13.7 (p. 226).

Each simplex in $B\alpha$ is again affine; this allows us to iterate the application of B and get smaller and smaller diameter. Thus, the k-fold iteration, $B^k(\alpha)$, has diameter at most $\left(\frac{n}{n+1}\right)^k \mathrm{diam}(\alpha)$.

In the following we use the easy but powerful trick to express α as

$$\alpha = \alpha \circ \mathrm{id}_{\Delta^n} = S_n(\alpha)(\mathrm{id}_{\Delta^n}).$$

This allows us to use the barycentric subdivision for general spaces.

Definition 7.9.

(a) We define $B_n^X : S_n(X) \to S_n(X)$ as

$$B_n^X(\alpha) := S_n(\alpha) \circ B(\mathrm{id}_{\Delta^n}).$$

(b) Similarly, $\psi_n^X : S_n(X) \to S_{n+1}(X)$ is

$$\psi_n^X(\alpha) := S_{n+1}(\alpha) \circ \psi_n(\mathrm{id}_{\Delta^n}).$$

Lemma 7.10. *The maps B^X are natural in X and are homotopic to the identity on $S_n(X)$.*

Proof. Let $f\colon X \to Y$ be a continuous map. We have

$$\begin{aligned} S_n(f)B_n^X(\alpha) =& S_n(f) \circ S_n(\alpha) \circ B(\mathrm{id}_{\Delta^n}) \\ =& S_n(f \circ \alpha) \circ B(\mathrm{id}_{\Delta^n}) \\ =& B_n^Y(f \circ \alpha). \end{aligned}$$

The calculation for $\partial\psi_n^X + \psi_{n-1}^X\partial = B_n^X - \mathrm{id}_{S_n(X)}$ uses that α induces a chain map and thus we get

$$\partial\psi_n^X(\alpha) = \partial \circ S_{n+1}(\alpha) \circ \psi_n(\mathrm{id}_{\Delta^n}) = S_n(\alpha) \circ \partial \circ \psi_n(\mathrm{id}_{\Delta^n}).$$

Hence

$$\partial\psi_n^X + \psi_{n-1}^X\partial = S_n(\alpha) \circ (\partial \circ \psi_n(\mathrm{id}_{\Delta^n}) + \psi_{n-1} \circ \partial(\mathrm{id}_{\Delta^n})) = S_n(\alpha) \circ (B - \mathrm{id})(\mathrm{id}_{\Delta^n}) = B_n^X(\alpha) - \alpha.$$

□

Now we consider singular n-chains that are spanned by 'small' singular n-simplices.

Definition 7.11. Let $\mathfrak{U} = \{U_i, i \in I\}$ be an open covering of X. Then $S_n^{\mathfrak{U}}(X)$ is the free abelian group generated by all $\alpha\colon \Delta^n \to X$ such that the image of Δ^n under α is contained in one of the $U_i \in \mathfrak{U}$.

Note that $S_n^{\mathfrak{U}}(X)$ is an abelian subgroup of $S_n(X)$. As we will see now, these chains suffice to detect everything in singular homology.

Lemma 7.12. *Every chain in $S_n(X)$ is homologous to a chain in $S_n^{\mathfrak{U}}(X)$.*

Proof. Let $\alpha = \sum_{j=1}^m \lambda_j\alpha_j \in S_n(X)$ and let L_j for $1 \leqslant j \leqslant m$ be the Lebesgue numbers for the coverings $\{\alpha_j^{-1}(U_i), i \in I\}$ of Δ^n. Choose a k, such that $\left(\frac{n}{n+1}\right)^k \leqslant L_1, \dots, L_m$. Then $B^k\alpha_1$ up to $B^k\alpha_m$ are all in $S_n^{\mathfrak{U}}(X)$. Therefore

$$B^k(\alpha) = \sum_{j=1}^m \lambda_j B^k(\alpha_j) =: \alpha' \in S_n^{\mathfrak{U}}(X).$$

As B is homotopic to the identity we have

$$\alpha \sim B\alpha \sim \ldots \sim B^k\alpha = \alpha'.$$

□

With this we get the main result of this section:

Theorem 7.13. *Let $W \subset A \subset X$ such that $\bar{W} \subset \mathring{A}$. Then the inclusion $i\colon (X \setminus W, A \setminus W) \hookrightarrow (X, A)$ induces an isomorphism*

$$H_n(i)\colon H_n(X \setminus W, A \setminus W) \cong H_n(X, A)$$

for all $n \geqslant 0$.

Proof. We first prove that $H_n(i)$ is surjective, so let $c \in S_n(X, A)$ be a relative cycle, *i.e.*, let $\partial c \in S_{n-1}(A)$. There is a k such that $c' := B^k c$ is a chain in $S_n^{\mathfrak{U}}(X)$ for the open covering $\mathfrak{U} = \{\mathring{A}, X \setminus \bar{W}\} =: \{U, V\}$. We decompose c' as $c' = c^U + c^V$ with c^U and c^V being elements in the corresponding chain complex. (This decomposition is not unique.)

We know that the boundary of c' is $\partial c' = \partial B^k c = B^k \partial c$ and by assumption this is a chain in $S_{n-1}(A)$. But $\partial c' = \partial c^U + \partial c^V$ with $\partial c^U \in S_{n-1}(U) \subset S_{n-1}(A)$. Thus, $\partial c^V \in S_{n-1}(A)$, in fact, $\partial c^V \in S_{n-1}(A \setminus W)$ and therefore c^V is a relative cycle in $S_n(X \setminus W, A \setminus W)$. This shows that $H_n(i)[c^V] = [c] \in H_n(X, A)$ because $[c] = [c^U + c^V] = [c^V]$ in $H_n(X, A)$.

The injectivity of $H_n(i)$ is shown as follows. Assume that there is a $c \in S_n(X \setminus W)$ with $\partial c \in S_{n-1}(A \setminus W)$ and assume $H_n(i)[c] = 0$, *i.e.*, c is of the form $c = \partial b + a'$ with $b \in S_{n+1}(X)$ and $a' \in S_n(A)$ and write b as $b^U + b^V$ with $b^U \in S_{n+1}(U) \subset S_{n+1}(A)$ and $b^V \in S_{n+1}(V) \subset S_{n+1}(X \setminus W)$. Then

$$c = \partial b^U + \partial b^V + a'.$$

But ∂b^U and a' are elements in $S_n(A \setminus W)$ and hence $c = \partial b^V \in S_n(X \setminus W, A \setminus W)$. □

8. Mayer-Vietoris sequence

We consider the following situation: Assume that there are subspaces $X_1, X_2 \subset X$ such that X_1 and X_2 are open in X and such that $X = X_1 \cup X_2$. We consider the open covering $\mathfrak{U} = \{X_1, X_2\}$. We need the following maps:

$$\begin{array}{ccccc} & & X_1 & & \\ & \overset{i_1}{\nearrow} & & \overset{\kappa_1}{\searrow} & \\ X_1 \cap X_2 & & & & X. \\ & \underset{i_2}{\searrow} & & \underset{\kappa_2}{\nearrow} & \\ & & X_2 & & \end{array}$$

Note that by definition, the sequence

$$0 \longrightarrow S_*(X_1 \cap X_2) \xrightarrow{(i_1, i_2)} S_*(X_1) \oplus S_*(X_2) \longrightarrow S_*^{\mathfrak{U}}(X) \longrightarrow 0 \tag{8.1}$$

is exact. Here, the second map is

$$(\alpha_1, \alpha_2) \mapsto \kappa_1(\alpha_1) - \kappa_2(\alpha_2).$$

THEOREM 8.1. *(The Mayer-Vietoris sequence)*
There is a long exact sequence

$$\ldots \xrightarrow{\delta} H_n(X_1 \cap X_2) \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X) \xrightarrow{\delta} H_{n-1}(X_1 \cap X_2) \longrightarrow \ldots$$

Walther Mayer: 1887–1948 `https://en.wikipedia.org/wiki/Walther_Mayer`
Leopold Vietoris: 1891–2002 (!) `https://en.wikipedia.org/wiki/Leopold_Vietoris`

PROOF. The proof follows from Lemma 7.12, because $H_n^{\mathfrak{U}}(X) \cong H_n(X)$. □

As an application, we calculate the homology groups of spheres. Let $X = \mathbb{S}^m$ and let $X^{\pm} := \mathbb{S}^m \setminus \{\mp e_{m+1}\}$. The subspaces X^+ and X^- are contractible and therefore $H_*(X^{\pm}) = 0$ for all positive $*$.

The Mayer-Vietoris sequence is as follows

$$\ldots \xrightarrow{\delta} H_n(X^+ \cap X^-) \longrightarrow H_n(X^+) \oplus H_n(X^-) \longrightarrow H_n(\mathbb{S}^m) \xrightarrow{\delta} H_{n-1}(X^+ \cap X^-) \longrightarrow \ldots$$

For $n > 1$ we can deduce

$$H_n(\mathbb{S}^m) \cong H_{n-1}(X^+ \cap X^-) \cong H_{n-1}(\mathbb{S}^{m-1}).$$

The first map is the connecting homomorphism and the second map is $H_{n-1}(i)\colon H_{n-1}(\mathbb{S}^{m-1}) \to H_{n-1}(X^+ \cap X^-)$ where i is the inclusion of $\mathbb{S}^{m-1}$ into $X^+ \cap X^-$ and this inclusion is a homotopy equivalence. Thus define $D := H_{n-1}(i)^{-1} \circ \delta$. This D is an isomorphism for all $n \geqslant 2$.

We have to controll what is going on in small degrees and dimensions.

We know from the Hurewicz isomorphism that $H_1(\mathbb{S}^m)$ is trivial for $m > 1$. If we want to see that via the Mayer-Vietoris sequence, we have to understand the map

$$\mathbb{Z} \cong H_0(X^+ \cap X^-) \to H_0(X^+) \oplus H_0(X^-) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Let 1 be a base point of $X^+ \cap X^-$. Then the map on H_0 is

$$[1] \mapsto ([1], [1]).$$

This map is injective and therefore the connecting homomorphism $\delta\colon H_1(\mathbb{S}^m) \to H_0(X^+ \cap X^-)$ is trivial and we obtain that

$$H_1(\mathbb{S}^m) \cong 0, \quad m > 1.$$

Next, we consider the case of $n = 1 = m$. In this case the intersection $X^+ \cap X^-$ splits into two components. We choose a $P_+ \in X^+$ and a $P_- \in X^-$ such that $P_+, P_- \in X^+ \cap X^-$ lie in different path components. Then,

$$H_0(i_1)([P_+]) = [e_2] = H_0(i_1)([P_-]) \text{ and } H_0(i_2)([P_+]) = [-e_2] = H_0(i_2)([P_-])$$

and hence

$$(H_0(i_1)([P_+], -[P_-]) = 0 = H_0(i_2)([P_+], -[P_-]).$$

Therefore the kernel of $(H_0(i_1), H_0(i_2))$ is spanned by $([P_+], -[P_-])$ and is isomorphic to $\mathbb{Z}$. Considering the exact sequence

$$0 \longrightarrow H_1\mathbb{S}^1 \xrightarrow{\delta} H_0(X^+ \cap X^-) \xrightarrow{(H_0(i_1),H_0(i_2))} H_0(X^+) \oplus H_0(X^-) \longrightarrow H_0\mathbb{S}^1$$

therefore yields $H_1(\mathbb{S}^1) \cong \mathbb{Z}$. (We already knew this from the Hurewicz isomorphism.)

For $0 < n < m$ we get

$$H_n\mathbb{S}^m \xrightarrow{\cong} H_{n-1}\mathbb{S}^{m-1} \xrightarrow{\cong} \ldots \xrightarrow{\cong} H_1(\mathbb{S}^{m-n+1}) \cong \pi_1(\mathbb{S}^{m-n+1}).$$

and the latter is trivial.

Similarly, for $0 < m < n$ we have

$$H_n\mathbb{S}^m \xrightarrow{\cong} H_{n-1}\mathbb{S}^{m-1} \xrightarrow{\cong} \ldots \xrightarrow{\cong} H_{n-m+1}(\mathbb{S}^1) \cong 0.$$

The last claim follows directly by another simple Mayer-Vietoris argument.

The remaining case $0 < m = n$ gives something non-trivial

$$H_n\mathbb{S}^n \xrightarrow{\cong} H_{n-1}\mathbb{S}^{n-1} \xrightarrow{\cong} \ldots \xrightarrow{\cong} H_1(\mathbb{S}^1) \cong \mathbb{Z}.$$

We can summarize the result as follows.

Proposition 8.2.

$$H_n(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = m = 0, \\ \mathbb{Z}, & n = 0, m > 0, \\ \mathbb{Z}, & n = m > 0, \\ 0, & \textit{otherwise}. \end{cases}$$

Definition 8.3. Let $\mu_0 := [P_+] - [P_-] \in H_0(X^+ \cap X^-) \cong H_0(\mathbb{S}^0)$ and let $\mu_1 \in H_1(\mathbb{S}^1) \cong \pi_1(\mathbb{S}^1)$ be given by the degree one map (aka the class of the identity on $\mathbb{S}^1$, aka the class of the loop $t \mapsto e^{2\pi i t}$).

Define the higher μ_ns via $D\mu_n = \mu_{n-1}$. Then μ_n is called *the fundamental class in* $H_n(\mathbb{S}^n)$.

In order to obtain a relative version of the Mayer-Vietoris sequence, we need a tool from homological algebra.

Lemma 8.4. *(The five-lemma)*
Let

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 \end{array}$$

be a commutative diagram of exact sequences. If f_1, f_2, f_4, f_5 are isomorphisms, then so is f_3.

Proof. Again, we are chasing diagrams.

In order to prove that f_3 is injective, assume that there is an $a \in A_3$ with $f_3 a = 0$. Then $\beta_3 f_3 a = f_4 \alpha_3 a = 0$, as well. But f_4 is injective, thus $\alpha_3 a = 0$. Exactness of the top row gives, that there is an $a' \in A_2$ with $\alpha_2 a' = a$. This implies

$$f_3 \alpha_2 a' = f_3 a = 0 = \beta_2 f_2 a'.$$

Exactness of the bottom row gives us a $b \in B_1$ with $\beta_1 b = f_2 a'$, but f_1 is an isomorphism so we can lift b to $a_1 \in A_1$ with $f_1 a_1 = b$.

Thus $f_2 \alpha_1 a_1 = \beta_1 b = f_2 a'$ and as f_2 is injective, this implies that $\alpha_1 a_1 = a'$. So finally we get that $a = \alpha_2 a' = \alpha_2 \alpha_1 a_1$, but the latter is zero, thus $a = 0$.

For the surjectivity of f_3 assume $b \in B_3$ is given. Move b over to B_4 via β_3 and set $a := f_4^{-1} \beta_3 b$.

Consider $f_5\alpha_4 a$. This is equal to $\beta_4\beta_3 b$ and hence trivial. Therefore $\alpha_4 a = 0$ and thus there is an $a' \in A_3$ with $\alpha_3 a' = a$. Then $b - f_3 a'$ is in the kernel of β_3 because

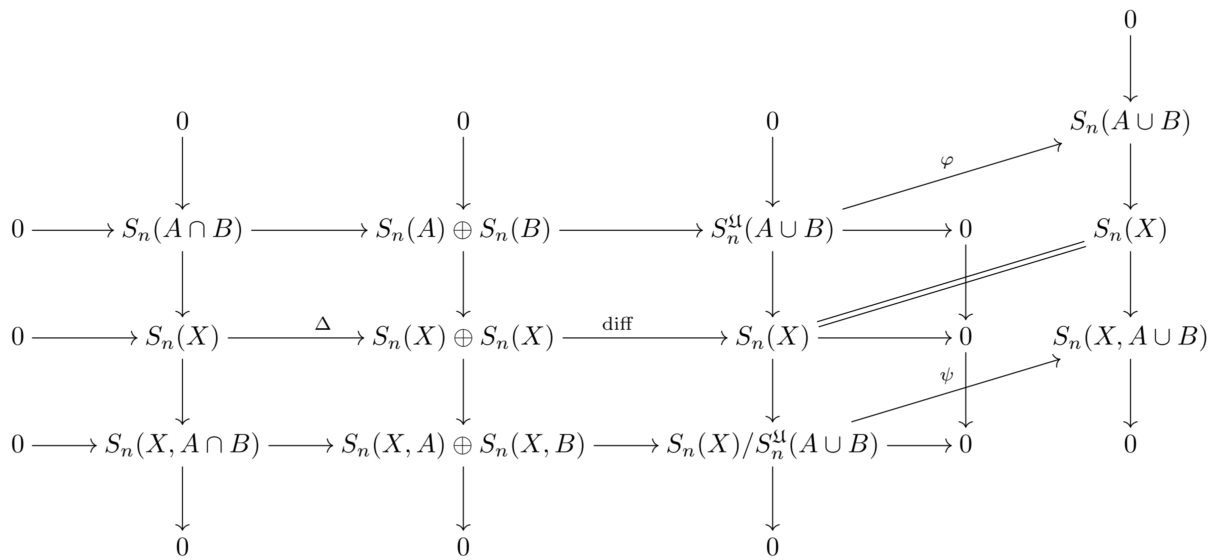
$$\beta_3(b - f_3 a') = \beta_3 b - f_4\alpha_3 a' = \beta_3 b - f_4 a = 0.$$

Hence we get a $b_2 \in B_2$ with $\beta_2 b_2 = b - f_3 a'$. Define a_2 as $f_2^{-1}(b_2)$, so $a' + \alpha_2 a_2$ is in A_3 and

$$f_3(a' + \alpha_2 a_2) = f_3 a' + \beta_2 f_2 a_2 = f_3 a' + \beta_2 b_2 = f_3 a' + b - f_3 a' = b.$$

□

We now consider a relative situation, so let X be a topological space with $A, B \subset X$ open in $A \cup B$ and set $\mathfrak{U} := \{A, B\}$. This is an open covering of $A \cup B$. The following diagram of exact sequences combines absolute chains with relative ones:



Here, ψ is induced by the inclusion $\varphi\colon S_n^{\mathfrak{U}}(A \cup B) \to S_n(A \cup B)$, Δ denotes the diagonal map and diff the difference map. It is clear that the first two rows are exact. That the third row is exact follows by the nine-lemma or a direct diagram chase.

Consider the two right-most non-trivial columns in this diagram. Each gives a long exact sequence in homology and we focus on five terms.

$$\begin{array}{ccccccccc}
H_n(S_*^{\mathfrak{U}}(A\cup B)) & \longrightarrow & H_n(X) & \longrightarrow & H_n(S_*(X)/S_*^{\mathfrak{U}}(A\cup B)) & \xrightarrow{\delta} & H_{n-1}(S_*^{\mathfrak{U}}(A\cup B)) & \longrightarrow & H_{n-1}(X) \\
\downarrow{\scriptstyle H_n(\varphi)} & & \| & & \downarrow{\scriptstyle H_n(\psi)} & & \downarrow{\scriptstyle H_{n-1}(\varphi)} & & \| \\
H_n(A\cup B) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A\cup B) & \xrightarrow{\delta} & H_{n-1}(A\cup B) & \longrightarrow & H_{n-1}(X)
\end{array}$$

Then by the five-lemma, as $H_n(\varphi)$ and $H_{n-1}(\varphi)$ are isomorphisms, so is $H_n(\psi)$. This observation together with the bottom non-trivial exact row proves the following.

THEOREM 8.5. *(Relative Mayer-Vietoris sequence)*
If $A, B \subset X$ are open in $A \cup B$, then the following sequence is exact:

$$\ldots \xrightarrow{\delta} H_n(X, A\cap B) \longrightarrow H_n(X,A) \oplus H_n(X,B) \longrightarrow H_n(X, A\cup B) \xrightarrow{\delta} \quad \ldots$$

9. Reduced homology and suspension

For any path-connected space we have that the zeroth homology group is isomorphic to the integers, so somehow this copy of $\mathbb{Z}$ is superfluous information and we want to get rid of it in a civilized manner. Let P denote the one-point topological space. Then for any space X there is a continuous map $\varepsilon\colon X \to P$.

Definition 9.1. We define $\widetilde{H}_n(X) := \ker(H_n(\varepsilon)\colon H_n(X) \to H_n(P))$ and call it the *reduced nth homology group of the space X*.

- Note that $\widetilde{H}_n(X) \cong H_n(X)$ for all positive n.
- If X is path-connected, then $\widetilde{H}_0(X) = 0$.
- For any choice of a base point $x \in X$ we get
$$\widetilde{H}_n(X) \oplus H_n(\{x\}) \cong H_n(X)$$
because $H_n(P) \cong H_n(\{x\})$ and the composition
$$\{x\} \hookrightarrow X \to \{x\}$$
is the identity. Therefore, $\widetilde{H}_n(X) \cong H_n(X, \{x\})$ because the retraction $r\colon X \to \{x\}$ splits the exact sequence
$$\ldots H_n(\{x\}) \to H_n(X) \to H_n(X, \{x\}) \to \ldots$$
- We can prolong the singular chain complex $S_*(X)$ and consider $\widetilde{S}_*(X)$:
$$\ldots \longrightarrow S_1(X) \longrightarrow S_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$
where $\varepsilon(\alpha) = 1$ for every singular 0-simplex α. This is precisely the augmentation we considered before. Then for all $n \geqslant 0$,
$$\widetilde{H}_*(X) \cong H_*(\widetilde{S}_*(X)).$$

As every continuous map $f\colon X \to Y$ induces a chain map $S_*(f)\colon S_*(X) \to S_*(Y)$ and as $\varepsilon^Y \circ S_0(f) = \varepsilon^X$ we obtain the following result.

Lemma 9.2. *The assignment $X \mapsto H_*(\widetilde{S}_*(X))$ is a functor,* i.e., *for a continuous $f\colon X \to Y$ we get an induced map $H_*(\widetilde{S}_*(f))\colon H_*(\widetilde{S}_*(X)) \to H_*(\widetilde{S}_*(Y))$ such that the identity on X induces the identity and composition of maps is respected.*

Similarly, $\widetilde{H}_*(-)$ is a functor.

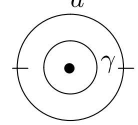
Definition 9.3. For $\varnothing \neq A \subset X$ we define
$$\widetilde{H}_n(X, A) := H_n(X, A).$$

As we identified reduced homology groups with relative homology groups we obtain a reduced version of the Mayer-Vietoris sequence. A similar remark applies to the long exact sequence for a pair of spaces.

Proposition 9.4. *For each pair of spaces, there is a long exact sequence*
$$\ldots \longrightarrow \widetilde{H}_n(A) \longrightarrow \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(X, A) \longrightarrow \widetilde{H}_{n-1}(A) \longrightarrow \ldots$$
and a reduced Mayer-Vietoris sequence.

Examples.

1) Recall that we can express $\mathbb{R}P^2$ as the quotient space of $\mathbb{S}^2$ modulo antipodal points or as a quotient of $\mathbb{D}^2$:
$$\mathbb{R}P^2 \cong \mathbb{S}^2/\pm\,\mathrm{id} \cong \mathbb{D}^2/z \sim -z \text{ for } z \in \mathbb{S}^1.$$
We use the latter definition and set $X = \mathbb{R}P^2$, $X_1 = X \setminus \{[0,0]\}$ (which is an open Möbius strip and hence homotopically equivalent to $\mathbb{S}^1$) and $X_2 = \mathring{\mathbb{D}}^2$. Then
$$X_1 \cap X_2 = \mathring{\mathbb{D}}^2 \setminus \{[0,0]\} \simeq \mathbb{S}^1.$$
Thus we know that $H_1(X_1) \cong \mathbb{Z}$, $H_1(X_2) \cong 0$ and $H_2X_1 = H_2X_2 = 0$. We choose generators for $H_1(X_1)$ and $H_1(X_1 \cap X_2)$ as follows.



Let a be the path that runs along the outer circle in mathematical positive direction half around starting from the point $(1,0)$. Let γ be the loop that runs along the inner circle in mathematical positive direction. Then the inclusion $i\colon X_1 \cap X_2 \to X_1$ induces

$$H_1(i)[\gamma] = 2[a].$$

This suffices to compute $H_*(\mathbb{R}P^2)$ up to degree two because the long exact sequence is

$$\widetilde{H}_2(X_1) \oplus \widetilde{H}_2(X_2) = 0 \to \widetilde{H}_2(X) \to \widetilde{H}_1(X_1 \cap X_2) \cong \mathbb{Z} \to \widetilde{H}_1(X_1) \cong \mathbb{Z} \to \widetilde{H}_1(X) \to \widetilde{H}_0(X_1 \cap X_2) = 0.$$

On the two copies of the integers, the map is given as above and thus we obtain:

$$\begin{aligned} H_2(\mathbb{R}P^2) &\cong \ker(2\cdot\colon \mathbb{Z} \to \mathbb{Z}) = 0, \\ H_1(\mathbb{R}P^2) &\cong \operatorname{coker}(2\cdot\colon \mathbb{Z} \to \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \\ H_0(\mathbb{R}P^2) &\cong \mathbb{Z}. \end{aligned}$$

The higher homology groups are trivial, because there $H_n(\mathbb{R}P^2)$ is located in a long exact sequence between trivial groups.

2) We can now calculate the homology groups of bouquets of spaces in terms of the homology groups of the single spaces, at least in good cases. Let $(X_i)_{i \in I}$ be a family of topological spaces with chosen basepoints $x_i \in X_i$. Consider

$$X = \bigvee_{i \in I} X_i.$$

If the inclusion of x_i into X_i is pathological, then we cannot apply the Mayer-Vietoris sequence. However, we get the following:

Proposition 9.5. *If there are open neighbourhoods U_i of $x_i \in X_i$ together with a deformation of U_i to $\{x_i\}$, then we have for any finite $E \subset I$*

$$\widetilde{H}_n(\bigvee_{i \in E} X_i) \cong \bigoplus_{i \in E} \widetilde{H}_n(X_i).$$

In the situation above we say that the X_i are *well-pointed* with respect to x_i.

Proof. First we consider the case of two bouquet summands. We have $X_1 \vee U_2 \cup U_1 \vee X_2$ as an open covering of $X_1 \vee X_2$. The Mayer-Vietoris sequence then gives that $H_n(X) \cong H_n(X_1 \vee U_2) \oplus H_n(U_1 \vee X_2)$ for $n > 0$. For H_0 we get the exact sequence

$$0 \to \widetilde{H}_0(X_1 \vee U_2) \oplus \widetilde{H}_0(U_1 \vee X_2) \to \widetilde{H}_0(X) \to 0.$$

By induction we obtain the case of finitely many bouquet summands. □

We also get

$$\widetilde{H}_n(\bigvee_{i \in I} X_i) \cong \bigoplus_{i \in I} \widetilde{H}_n(X_i)$$

but for this one needs a colimit argument. We postpone that for a while.

We can extend such results to the full relative case. Let $A \subset X$ be a closed subspace and assume that A is a deformation retract of an open neighbourhood $A \subset U$. Let $\pi\colon X \to X/A$ be the canonical projection and $b = \{A\}$ the image of A. Then X/A is well-pointed with respect to b.

Proposition 9.6. *In the situation above*

$$H_n(X, A) \cong \widetilde{H}_n(X/A), \quad 0 \leqslant n.$$

Proof. The canonical projection, π, induces a homeomorphism $(X \setminus A, U \setminus A) \cong (X/A \setminus \{b\}, \pi(U) \setminus \{b\})$. Consider the following diagram:

$$\begin{array}{ccccc} H_n(X,A) & \xrightarrow{\cong} & H_n(X,U) & \xleftarrow{\cong} & H_n(X \setminus A, U \setminus A) \\ \downarrow{\scriptstyle H_n(\pi)} & & & & \cong \downarrow{\scriptstyle H_n(\pi)} \\ H_n(X/A, b) & \xrightarrow{\cong} & H_n(X/A, \pi(U)) & \xleftarrow{\cong} & H_n(X/A \setminus \{b\}, \pi(U) \setminus \{b\}) \end{array}$$

The upper and lower left arrows are isomorphisms because A is a deformation retract of U, the isomorphism in the upper right is a consequence of excision, because $A = \bar{A} \subset U$ and the lower right one follows from excision as well. $\square$

Theorem 9.7. *(Suspension isomorphism) If $A \subset X$ is as above, then*

$$H_n(\Sigma X, \Sigma A) \cong \tilde{H}_{n-1}(X, A), \qquad \textit{for all } n > 0.$$

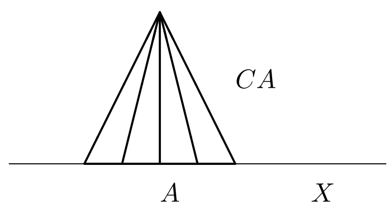
Proof. Consider the inclusion of pairs $(X, A) \subset (CX, CA) \subset (\Sigma X, \Sigma A)$ and the triple $(CX, X \cup CA, CA)$. We obtain the corresponding long exact sequence on homology groups

$$\ldots \longrightarrow H_n(CX, CA) \longrightarrow H_n(CX, CA \cup X) \xrightarrow{\delta} \tilde{H}_{n-1}(X \cup CA, CA) \longrightarrow \ldots$$

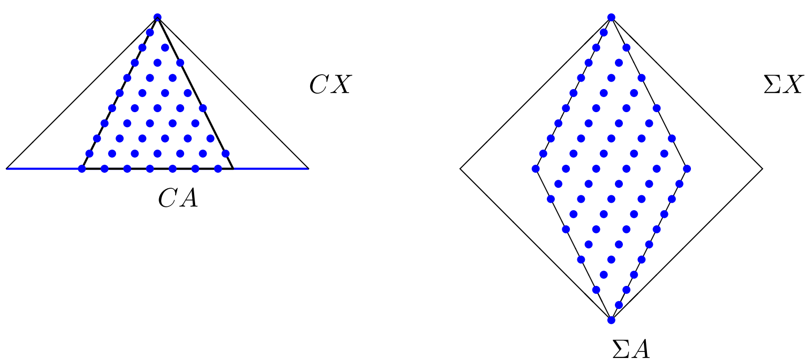
By Proposition 9.6 we get that $\tilde{H}_n(CX, CA \cup X) \cong \tilde{H}_n(CX/CA \cup X)$ and $\tilde{H}_{n-1}(X \cup CA, CA) \cong \tilde{H}_{n-1}(X \cup CA/CA)$ and the latter is isomorphic to $\tilde{H}_{n-1}(X/A) \cong \tilde{H}_{n-1}(X, A)$. Similarly, as $CX/CA \cup X \simeq \Sigma X/\Sigma A$, we get

$$\tilde{H}_n(CX, CA \cup X) \cong \tilde{H}_n(CX/CA \cup X) \cong \tilde{H}_n(\Sigma X/\Sigma A) \cong H_n(\Sigma X, \Sigma A).$$

$X \cup CA/CA \cong X/A$:



$CX/CA \cup X \cong \Sigma X/\Sigma A$:



$\square$

Note, that the corresponding statement is terribly wrong for homotopy groups. We have $\Sigma \mathbb{S}^2 \cong \mathbb{S}^3$, but $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, whereas $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$, so homotopy groups (unlike homology groups) don't satisfy such an easy form of a suspension isomorphism. There is a Freundenthal suspension theorem for homotopy groups, but that's more complicated (`https://en.wikipedia.org/wiki/Freudenthal_suspension_theorem`). For the above case it yields:

$$\mathbb{Z}/2\mathbb{Z} \cong \pi_{1+3}(\mathbb{S}^3) \cong \pi_{1+4}(\mathbb{S}^4) \cong \ldots =: \pi_1^s$$

where π_1^s denotes the first stable homotopy group.

Freudenthal: 1905–1990 `https://en.wikipedia.org/wiki/Hans_Freudenthal`

10. Mapping degree

Recall that we defined fundamental classes $\mu_n \in \tilde{H}_n(\mathbb{S}^n)$ for all $n \geqslant 0$. Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be any continuous map.

DEFINITION 10.1. The map f induces a homomorphism

$$\tilde{H}_n(f)\colon \tilde{H}_n(\mathbb{S}^n) \to \tilde{H}_n(\mathbb{S}^n)$$

and therefore we get

$$\tilde{H}_n(f)\mu_n = \deg(f)\mu_n$$

with $\deg(f) \in \mathbb{Z}$. We call this integer *the degree of f*.

In the case $n = 1$ we can relate this notion of a mapping degree to the one defined via the fundamental group of the 1-sphere: if we represent the generator of $\pi_1(\mathbb{S}^1, 1)$ as the class given by the loop

$$\omega\colon [0,1] \to \mathbb{S}^1, \quad t \mapsto e^{2\pi i t},$$

then the abelianized Hurewicz, $h_{\text{ab}}\colon \pi_1(\mathbb{S}^1, 1) \to H_1(\mathbb{S}^1)$, sends the class of ω precisely to μ_1 and therefore the naturality of h_{ab}

$$\begin{array}{ccc} \pi_1(\mathbb{S}^1,1) & \xrightarrow{\pi_1(f)} & \pi_1(\mathbb{S}^1,1) \\ \downarrow{\scriptstyle h_{\text{ab}}} & & \downarrow{\scriptstyle h_{\text{ab}}} \\ H_1(\mathbb{S}^1) & \xrightarrow{H_1(f)} & H_1(\mathbb{S}^1) \end{array}$$

shows that

$$\deg(f)\mu_1 = H_1(f)\mu_1 = h_{\text{ab}}(\pi_1(f)[w]) = h_{\text{ab}}(k[w]) = k\mu_1.$$

where k is the degree of f defined via the fundamental group. Thus both notions coincide for $n = 1$.

As we know that the connecting homomorphism induces an isomorphism between $H_n(\mathbb{D}^n, \mathbb{S}^{n-1})$ and $\tilde{H}_{n-1}(\mathbb{S}^{n-1})$, we can consider degrees of maps $f\colon (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (\mathbb{D}^n, \mathbb{S}^{n-1})$ by defining $\bar{\mu}_n := \delta^{-1}\mu_n$. Then $H_n(f)(\bar{\mu}_n) := \deg(f)\bar{\mu}_n$ gives a well-defined integer $\deg(f) \in \mathbb{Z}$.

The degree of self-maps of $\mathbb{S}^n$ satisfies the following properties:

PROPOSITION 10.2.
(a) *If f is homotopic to g, then* $\deg(f) = \deg(g)$.
(b) *The degree of the identity on $\mathbb{S}^n$ is one.*
(c) *The degree is multiplicative,* i.e., $\deg(g \circ f) = \deg(g)\deg(f)$.
(d) *If f is not surjective, then* $\deg(f) = 0$.

PROOF. The first three properties follow directly from the definition of the degree. If f is not surjective, then it is homotopic to a constant map and this has degree zero. □

It is true that the group of (pointed) homotopy classes of self-maps of $\mathbb{S}^n$ is isomorphic to $\mathbb{Z}$ and thus the first property can be upgraded to an 'if and only if', but we won't prove that here.

Recall that $\Sigma\mathbb{S}^n \cong \mathbb{S}^{n+1}$. If $f\colon \mathbb{S}^n \to \mathbb{S}^n$ is continuous, then $\Sigma(f)\colon \Sigma\mathbb{S}^n \to \Sigma\mathbb{S}^n$ is given as $\Sigma\mathbb{S}^n \ni [x,t] \mapsto [f(x), t]$.

LEMMA 10.3. *Suspensions leave the degree invariant,* i.e., *for $f\colon \mathbb{S}^n \to \mathbb{S}^n$ we have*

$$\deg(\Sigma(f)) = \deg(f).$$

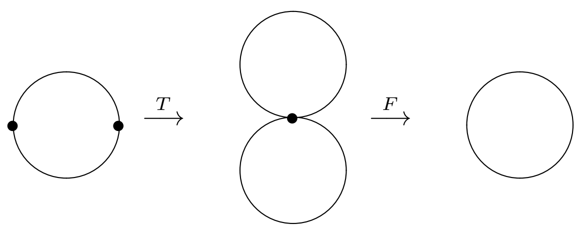
In particular, for every $k \in \mathbb{Z}$ there is an $f\colon \mathbb{S}^n \to \mathbb{S}^n$ with $\deg(f) = k$.

Proof. The suspension isomorphism of Theorem 9.7 is induced by a connecting homomorphism. Using the isomorphism $H_{n+1}(\mathbb{S}^{n+1}) \cong H_{n+1}(\Sigma\mathbb{S}^n)$, the connecting homomorphism sends $\mu_{n+1} \in H_{n+1}(\mathbb{S}^{n+1})$ to $\pm\mu_n \in \tilde{H}_n(\mathbb{S}^n)$. But then the commutativity of

$$\begin{array}{ccccccc} H_{n+1}(\mathbb{S}^{n+1}) & \xrightarrow{\cong} & H_{n+1}(\Sigma\mathbb{S}^n) & \xrightarrow{H_{n+1}(\Sigma f)} & H_{n+1}(\Sigma\mathbb{S}^n) & \xleftarrow{\cong} & H_{n+1}(\mathbb{S}^{n+1}) \\ & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\ & & \tilde{H}_n(\mathbb{S}^n) & \xrightarrow{H_n(f)} & \tilde{H}_n(\mathbb{S}^n) & & \end{array}$$

ensures that $\pm\deg(f)\mu_n = \pm\deg(\Sigma f)\mu_n$ with the same sign. □

For the degree of a self-map of $\mathbb{S}^1$ one has an additivity relation. We can generalize this to higher dimensions. Consider the *pinch map* $T\colon \mathbb{S}^n \to \mathbb{S}^n/\mathbb{S}^{n-1} \simeq \mathbb{S}^n \vee \mathbb{S}^n$ and the *fold map* $F\colon \mathbb{S}^n \vee \mathbb{S}^n \to \mathbb{S}^n$. Here, F is induced by the identity of $\mathbb{S}^n$.



Note that we can replace every continuous $f\colon \mathbb{S}^n \to \mathbb{S}^n$ by a basepoint-preserving map by composing with a rotation. That doesn't change the degree.

Proposition 10.4. *For $f, g\colon \mathbb{S}^n \to \mathbb{S}^n$ we have*

$$\deg(F \circ (f \vee g) \circ T) = \deg(f) + \deg(g).$$

Proof. The map $H_n(T)$ sends μ_n to $(\mu_n, \mu_n) \in \tilde{H}_n\mathbb{S}^n \oplus \tilde{H}_n\mathbb{S}^n \cong \tilde{H}_n(\mathbb{S}^n \vee \mathbb{S}^n)$. Under this isomorphism, the map $H_n(f \vee g)$ corresponds to $(\mu_n, \mu_n) \mapsto (\tilde{H}_n(f)\mu_n, \tilde{H}_n(g)\mu_n)$ and this yields $(\deg(f)\mu_n, \deg(g)\mu_n)$ which under the fold map is sent to the sum. □

We use the mapping degree to show some geometric properties of self-maps of spheres.

Proposition 10.5. *Let $f^{(n)}\colon \mathbb{S}^n \to \mathbb{S}^n$ be the map*

$$(x_0, x_1, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n).$$

Then $f^{(n)}$ has degree -1.

Proof. We prove the claim by induction. μ_0 was the difference class $[+1] - [-1]$, and

$$f^{(0)}([+1] - [-1]) = [-1] - [+1] = -\mu_0.$$

We defined μ_n in such a way that $D\mu_n = \mu_{n-1}$. Therefore, as D is natural,

$$H_n(f^{(n)})\mu_n = H_n(f^{(n)})D^{-1}\mu_{n-1} = D^{-1}H_{n-1}(f^{(n-1)})\mu_{n-1} = D^{-1}(-\mu_{n-1}) = -\mu_n.$$

□

Corollary 10.6. *The antipodal map $A\colon \mathbb{S}^n \to \mathbb{S}^n$, $A(x) = -x$, has degree $(-1)^{n+1}$.*

Proof. Let $f_i^{(n)}\colon \mathbb{S}^n \to \mathbb{S}^n$ be the map $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$. As in Proposition 10.5 one shows that the degree of $f_i^{(n)}$ is -1. As $A = f_n^{(n)} \circ \ldots \circ f_0^{(n)}$, the claim follows. □

In particular, the antipodal map cannot be homotopic to the identity as long as n is even!

Proposition 10.7. *Let $f, g\colon \mathbb{S}^n \to \mathbb{S}^n$ with $f(x) \neq g(x)$ for all $x \in \mathbb{S}^n$, then f is homotopic to $A \circ g$. In particular,*

$$\deg(f) = (-1)^{n+1}\deg(g).$$

Proof. By assumption the segment $t \mapsto (1-t)f(x) - tg(x)$ doesn't pass through the origin for $0 \leqslant t \leqslant 1$. Thus the homotopy

$$H(x,t) = \frac{(1-t)f(x) - tg(x)}{||(1-t)f(x) - tg(x)||}$$

connects f to $-g = A \circ g$. □

Corollary 10.8. *For any $f\colon \mathbb{S}^n \to \mathbb{S}^n$ with $\deg(f) = 0$ there is an $x_+ \in \mathbb{S}^n$ with $f(x_+) = x_+$ and an x_- with $f(x_-) = -x_-$.*

Proof. If $f(x) \neq x$ for all x, then $\deg(f) = \deg(A) \neq 0$. If $f(x) \neq -x$ for all x, then $\deg(f) = (-1)^{n+1}\deg(A) \neq 0$. □

Corollary 10.9. *Assume that n is even and let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be any continuous map. Then there is an $x \in \mathbb{S}^n$ with $f(x) = x$ or $f(x) = -x$.*

Finally, we can say the following about hairstyles of hedgehogs of arbitrary even dimension:

Proposition 10.10. *Any tangential vector field on $\mathbb{S}^{2k}$ is trivial in at least one point.*

Proof. Recall that we can describe the tangent space at a point $x \in \mathbb{S}^{2k}$ as

$$T_x(\mathbb{S}^{2k}) = \{y \in \mathbb{R}^{2k+1} | \langle x, y\rangle = 0\}.$$

Assume that $V\colon \mathbb{S}^{2k} \to T(\mathbb{S}^{2k})$ with $V(x) \in T_x(\mathbb{S}^{2k})$ for all x is a tangential vector field which does not vanish, *i.e.*, $V(x) \neq 0$ for all $x \in \mathbb{S}^{2k}$.

Define $f(x) := \frac{V(x)}{||V(x)||}$. If $f(x) = x$, then $V(x) = ||V(x)||x$. But this means that $V(x)$ points into the direction of x and thus it cannot be tangential. Similarly, $f(x) = -x$ yields the same contradiction. Thus such a V cannot exist. □