Exercises in Algebraic Topology (master)

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Exercise sheet no 6 due: 13th of May 2025, 13:45h in H3

1 (5-Lemma revisited) (1 + 1 points)
Consider the following commutative diagram of exact sequences

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5 \end{tikzcd}$$

Under which assumptions on f_1, f_2, f_4, f_5 can we deduce that the map f_3 is a monomorphism or an epimorphism?

2 (Trivial versus actual gluing) (2 + 2 points)

(1) Are the homology groups of $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{S}^2 \vee \mathbb{S}^1 \vee \mathbb{S}^1$ isomorphic?
(2) What about the homology groups of the Klein bottle versus the homology groups of $\mathbb{S}^2 \vee \mathbb{S}^1 \vee \mathbb{S}^1$?

3 (More linear algebra) (2 points)
Let $A \in O(n+1)$. Then multiplication by A induces a continuous self-map on $\mathbb{S}^n$. (Why?) What is its degree?

4 (Degrees) (3 + 3 points)

(1) Prove the Brouwer fixed-point theorem: Let X be a closed ball $B_R(x) \subset \mathbb{R}^n$ for $n \geqslant 1$, $r > 0$, $x \in \mathbb{R}^n$, and let f be a continuous map $f\colon B_R(x) \to B_R(x)$. Show that f has a fixed point.
(2) Use this to show that every $(a_{ij}) = A \in M(n \times n; \mathbb{R})$ with non-negative a_{ij} must have an eigenvector with non-negative coordinates. Hint: Consider a suitable standard simplex instead of $B_R(x)$.