Algebraic Topology, summer term 2024

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CHAPTER 1

Homology theory

1. Chain complexes

Definition 1.1. A *chain complex* is a sequence of abelian groups, $(C_n)_{n\in\mathbb{Z}}$, together with homomorphisms $d_n\colon C_n \to C_{n-1}$ for $n \in \mathbb{Z}$, such that $d_{n-1} \circ d_n = 0$.

Let R be an associative ring with unit 1_R. A *chain complex of R-modules* can analoguously be defined as a sequence of R-modules $(C_n)_{n\in\mathbb{Z}}$ with R-linear maps $d_n\colon C_n \to C_{n-1}$ with $d_{n-1} \circ d_n = 0$.

Definition 1.2.

- The d_n are *differentials* or *boundary operators.*
- The $x \in C_n$ are called *n-chains.*
- Is $x \in C_n$ and $d_n x = 0$, then x is an *n-cycle.*

$$Z_n(C) := \{x \in C_n | d_n x = 0\}.$$

- If $x \in C_n$ is of the form $x = d_{n+1}y$ for some $y \in C_{n+1}$, then x is an *n-boundary.*

$$B_n(C) := Im(d_{n+1}) = \{d_{n+1}y, y \in C_{n+1}\}.$$

Note that the cycles and boundaries form subgroups of the chains. As $d_n \circ d_{n+1} = 0$, we know that the image of d_{n+1} is a subgroup of the kernel of d_n and thus

$$B_n(C) \subset Z_n(C).$$

We'll often drop the subscript n from the boundary maps and we'll just write C_* for the chain complex.

Definition 1.3. The abelian group $H_n(C) := Z_n(C)/B_n(C)$ is the *nth homology group of the complex* C_*.

Notation: We denote by $[c]$ the equivalence class of a $c \in Z_n(C)$.
If $c, c' \in C_n$ satisfy that $c - c'$ is a boundary, then c is *homologous* to c'. That's an equivalence relation.

Examples:

1) Consider

$$C_n = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

and let d_1 be the multiplication with $N \in \mathbb{N}$, then

$$H_n(C) = \begin{cases} \mathbb{Z}/N\mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2) Take $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}$ and

$$d_n = \begin{cases} \mathrm{id}_{\mathbb{Z}} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

What is the homology of this chain complex?

2') Consider $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}$ again, but let all boundary maps be trivial. What is the homology of this chain complex?

Definition 1.4. Let C_* and D_* be two chain complexes. A *chain map* $f\colon C_* \to D_*$ is a sequence of homomorphisms $f_n\colon C_n \to D_n$ such that $d_n^D \circ f_n = f_{n-1} \circ d_n^C$ for all n, *i.e.*, the diagram

$$\begin{tikzcd}
C_n \arrow[r, "d_n^C"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_n \arrow[r, "d_n^D"] & D_{n-1}
\end{tikzcd}$$

commutes for all n.

Such an f sends cycles to cycles and boundaries to boundaries. We therefore obtain an induced map

$$H_n(f)\colon H_n(C) \to H_n(D)$$

via $H_n(f)_*[c] = [f_n c]$.

There is a chain map from the chain complex mentioned in Example 1) to the chain complex D_* that is concentrated in degree zero and has $D_0 = \mathbb{Z}/N\mathbb{Z}$. Note, that $H_0(f)$ is an isomorphism on zeroth homology groups.

Are there chain maps between the complexes from Examples 2) and 2')?

Lemma 1.5. *If $f\colon C_* \to D_*$ and $g\colon D_* \to E_*$ are two chain maps, then $H_n(g) \circ H_n(f) = H_n(g \circ f)$ for all n.*

When do two chain maps induce the same map on homology?

Definition 1.6. A *chain homotopy* H between two chain maps $f, g\colon C_* \to D_*$ is a sequence of homomorphisms $(H_n)_{n\in\mathbb{Z}}$ with $H_n\colon C_n \to D_{n+1}$ such that for all n

$$d_{n+1}^D \circ H_n + H_{n-1} \circ d_n^C = f_n - g_n.$$

$$\begin{tikzcd}
\dots \arrow[r, "d_{n+2}^C"] & C_{n+1} \arrow[r, "d_{n+1}^C"] \arrow[dl, "H_{n+1}"'] \arrow[d, shift right, bend right, "f_{n+1}"'] \arrow[d, shift left, bend left, "g_{n+1}"] & C_n \arrow[r, "d_n^C"] \arrow[dl, "H_n"'] \arrow[d, shift right, bend right, "f_n"'] \arrow[d, shift left, bend left, "g_n"] & C_{n-1} \arrow[r, "d_{n-1}^C"] \arrow[dl, "H_{n-1}"'] \arrow[d, shift right, bend right, "f_{n-1}"'] \arrow[d, shift left, bend left, "g_{n-1}"] & \dots \\
\dots \arrow[r, "d_{n+2}^D"] & D_{n+1} \arrow[r, "d_{n+1}^D"] & D_n \arrow[r, "d_n^D"] & D_{n-1} \arrow[r, "d_{n-1}^D"] & \dots
\end{tikzcd}$$

If such an H exists, then f and g are *(chain) homotopic*: $f \simeq g$.

We will later see geometrically defined examples of chain homotopies.

Proposition 1.7.

(a) *Being chain homotopic is an equivalence relation.*
(b) *If f and g are homotopic, then $H_n(f) = H_n(g)$ for all n.*

Proof. (a) If H is a homotopy from f to g, then $-H$ is a homotopy from g to f. Each f is homotopic to itself with $H = 0$. If f is homotopic to g via H and g is homotopic to h via K, then f is homotopic to h via $H + K$.

(b) We have for every cycle $c \in Z_n(C_*)$:

$$H_n(f)[c] - H_n(g)[c] = [f_n c - g_n c] = [d_{n+1}^D \circ H_n(c)] + [H_{n-1} \circ d_n^C(c)] = 0.$$

□

Definition 1.8. Let $f\colon C_* \to D_*$ be a chain map. We call f a *chain homotopy equivalence*, if there is a chain map $g\colon D_* \to C_*$ such that $g \circ f \simeq \mathrm{id}_{C_*}$ and $f \circ g \simeq \mathrm{id}_{D_*}$. The chain complexes C_* and D_* are then *chain homotopically equivalent*.

Note, that such chain complexes have isomorphic homology. However, chain complexes with isomorphic homology do not have to be chain homotopically equivalent. (Can you find a counterexample?)

Definition 1.9. If C_* and C'_* are chain complexes, then their *direct sum*, $C_* \oplus C'_*$, is the chain complex with

$$(C_* \oplus C'_*)_n = C_n \oplus C'_n = C_n \times C'_n$$

with differential $d = d_\oplus$ given by

$$d_\oplus(c, c') = (dc, dc').$$

Similarly, if $(C_*^{(j)}, d^{(j)})_{j \in J}$ is a family of chain complexes, then we can define their direct sum as follows:

$$(\bigoplus_{j \in J} C_*^{(j)})_n := \bigoplus_{j \in J} C_n^{(j)}$$

as abelian groups and the differential $d_\oplus$ is defined via the property that its restriction to the jth summand is $d^{(j)}$.

2. Singular homology

Let $v_0, \ldots, v_n$ be $n+1$ points in $\mathbb{R}^{n+1}$. Consider the convex hull

$$K(v_0, \ldots, v_n) := \{\sum_{i=0}^n t_i v_i | \sum_{i=0}^n t_i = 1, t_i \geqslant 0\}.$$

Definition 2.1. If the vectors $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent, then $K(v_0, \ldots, v_n)$ is the *simplex generated by* $v_0, \ldots, v_n$. We denote such a simplex by $\mathrm{simp}(v_0, \ldots, v_n)$.

Example. The standard topological n-simplex is $\Delta^n := \mathrm{simp}(e_0, \ldots, e_n)$. Here, e_i is the vector in $\mathbb{R}^{n+1}$ that has a 1 in coordinate $i+1$ and is zero in all other coordinates. The first examples are: Δ^0 is the point e_0, Δ^1 is the line segment between e_0 and e_1, Δ^2 is a triangle in $\mathbb{R}^3$ and Δ^3 is homeomorphic to a tetrahedron.

The coordinate description of the n-simplex is

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | \sum t_i = 1, t_i \geqslant 0\}.$$

We consider Δ^n as $\Delta^n \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2} \subset \ldots$.

The boundary of Δ^1 consists of two copies of Δ^0, the boundary of Δ^2 consists of three copies of Δ^1. In general, the boundary of Δ^n consists of $n+1$ copies of Δ^{n-1}.

We need the following *face maps* for $0 \leqslant i \leqslant n$

$$d_i = d_i^{n-1} \colon \Delta^{n-1} \hookrightarrow \Delta^n; (t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

The image of d_i^{n-1} in Δ^n is the face that is opposite to e_i. It is the simplex generated by $e_0, \ldots, e_{i-1}$, $e_{i+1}, \ldots, e_n$.

Draw the examples of the faces in Δ^1 and Δ^2!

Lemma 2.2. *Concerning the composition of face maps, the following rule holds:*

$$d_i^{n-1} \circ d_j^{n-2} = d_j^{n-1} \circ d_{i-1}^{n-2}, \quad 0 \leqslant j < i \leqslant n.$$

Example: face maps for Δ^0 and composition into Δ^2: $d_2 \circ d_0 = d_0 \circ d_1$.

Proof. Both expressions yield

$$d_i^{n-1} \circ d_j^{n-2}(t_0, \ldots, t_{n-2}) = (t_0, \ldots, t_{j-1}, 0, \ldots, t_{i-2}, 0, \ldots, t_{n-2}) = d_j^{n-1} d_{i-1}^{n-2}(t_0, \ldots, t_{n-2}).$$

□

Let X be an arbitrary topological space, $X \neq \varnothing$.

Definition 2.3. A *singular n-simplex in X* is a continuous map $\alpha \colon \Delta^n \to X$.

Note, that α just has to be continuous, not smooth or anything!

Definition 2.4. Let $S_n(X)$ be the free abelian group generated by all singular n-simplices in X. We call $S_n(X)$ the *nth singular chain module of X*.

Elements of $S_n(X)$ are finite sums $\sum_{i \in I} \lambda_i \alpha_i$ with $\lambda_i = 0$ for almost all $i \in I$ and $\alpha_i \colon \Delta^n \to X$.

For all $n \geqslant 0$ there are non-trivial elements in $S_n(X)$, because we assumed that $X \neq \varnothing$: we can always take an $x_0 \in X$ and the constant map $\kappa_{x_0} \colon \Delta^n \to X$ as α. By convention, we define $S_n(\varnothing) = 0$ for all $n \geqslant 0$.

If we want to define maps from $S_n(X)$ to some abelian group then it suffices to define such a map on generators.

Example. What is $S_0(X)$? A continuous $\alpha \colon \Delta^0 \to X$ is determined by its value $\alpha(e_0) =: x_\alpha \in X$, which is a point in X. A singular 0-simplex $\sum_{i \in I} \lambda_i \alpha_i$ can thus be identified with the formal sum of points $\sum_{i \in I} \lambda_i x_{\alpha_i}$. For instance if you count the zeroes and poles of a meromorphic function with multiplicities then this gives an element in $S_0(X)$. In algebraic geometry a divisor is an element in $S_0(X)$.

Definition 2.5. We define $\partial_i \colon S_n(X) \to S_{n-1}(X)$ on generators

$$\partial_i(\alpha) = \alpha \circ d_i^{n-1}$$

and call it *the ith face of* α.

On $S_n(X)$ we therefore get $\partial_i(\sum_j \lambda_j \alpha_j) = \sum_j \lambda_j (\alpha_j \circ d_i^{n-1})$.

Lemma 2.6. *The face maps on $S_n(X)$ satisfy*

$$\partial_j \circ \partial_i = \partial_{i-1} \circ \partial_j, \quad 0 \leqslant j < i \leqslant n.$$

Proof. The proof follows from the one of Lemma 2.2. □

Definition 2.7. We define the boundary operator on singular chains as $\partial \colon S_n(X) \to S_{n-1}(X)$, $\partial = \sum_{i=0}^{n} (-1)^i \partial_i$.

Lemma 2.8. *The map ∂* is *a boundary operator,* i.e., $\partial \circ \partial = 0$.

Proof. We calculate

$$\begin{aligned}\partial \circ \partial = (\sum_{j=0}^{n-1} (-1)^j \partial_j) \circ (\sum_{i=0}^{n} (-1)^i \partial_i) &= \sum \sum (-1)^{i+j} \partial_j \circ \partial_i \\ &= \sum_{0 \leqslant j < i \leqslant n} (-1)^{i+j} \partial_j \circ \partial_i + \sum_{0 \leqslant i \leqslant j \leqslant n-1} (-1)^{i+j} \partial_j \circ \partial_i \\ &= \sum_{0 \leqslant j < i \leqslant n} (-1)^{i+j} \partial_{i-1} \circ \partial_j + \sum_{0 \leqslant i \leqslant j \leqslant n-1} (-1)^{i+j} \partial_j \circ \partial_i = 0.\end{aligned}$$

□

We therefore obtain the *singular chain complex*, $S_*(X)$,

$$\ldots \longrightarrow S_n(X) \xrightarrow{\partial} S_{n-1}(X) \xrightarrow{\partial} \ldots \xrightarrow{\partial} S_1(X) \xrightarrow{\partial} S_0(X) \longrightarrow 0.$$

We abbreviate $Z_n(S_*(X))$ by $Z_n(X)$, $B_n(S_*(X))$ by $B_n(X)$ and $H_n(S_*(X))$ by $H_n(X)$.

Definition 2.9. For a space X, $H_n(X)$ is the *nth singular homology group of* X.

Note that $Z_0(X) = S_0(X)$.

As an example of a 1-cycle consider a 1-chain $c = \alpha + \beta + \gamma$ where $\alpha, \beta, \gamma \colon \Delta^1 \to X$ such that $\alpha(e_1) = \beta(e_0)$, $\beta(e_1) = \gamma(e_0)$ and $\gamma(e_1) = \alpha(e_0)$ and calculate that $\partial c = 0$.

We need to understand how continuous maps of topological spaces interact with singular chains and singular homology.

Let $f \colon X \to Y$ be a continuous map.

Definition 2.10. The map $f_n = S_n(f) \colon S_n(X) \to S_n(Y)$ is defined on generators $\alpha \colon \Delta^n \to X$ as

$$f_n(\alpha) = f \circ \alpha : \Delta^n \xrightarrow{\alpha} X \xrightarrow{f} Y.$$

Lemma 2.11. *For any continuous $f\colon X \to Y$ we have*

$$\begin{tikzcd} S_n(X) \arrow[r, "f_n"] \arrow[d, "\partial^X"'] & S_n(Y) \arrow[d, "\partial^Y"] \\ S_{n-1}(X) \arrow[r, "f_{n-1}"] & S_{n-1}(Y), \end{tikzcd}$$

i.e., *$(f_n)_n$ is a chain map and hence induces a map $H_n(f)\colon H_n(X) \to H_n(Y)$.*

Proof. By definition

$$\partial^Y(f_n(\alpha)) = \sum_{i=0}^{n}(-1)^i(f\circ\alpha)\circ d_i = \sum_{i=0}^{n}(-1)^i f\circ(\alpha\circ d_i) = f_{n-1}(\partial^X\alpha).$$

□

Of course, the identity map on X induces the identity map on $H_n(X)$ for all $n \geqslant 0$ and if we have a composition of continuous maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

then $S_n(g\circ f) = S_n(g)\circ S_n(f)$ and $H_n(g\circ f) = H_n(g)\circ H_n(f)$. In categorical language, this says precisely that $S_n(-)$ and $H_n(-)$ are functors from the category of topological spaces and continuous maps into the category of abelian groups. Taking all $S_n(-)$ together turns $S_*(-)$ into a functor from topological spaces and continuous maps into the category of chain complexes with chain maps as morphisms.

One implication of Lemma 2.11 is that homeomorphic spaces have isomorphic homology groups:

$$X \cong Y \Rightarrow H_n(X) \cong H_n(Y) \text{ for all } n \geqslant 0.$$

Our first (not too exciting) calculation is the following:

Proposition 2.12. *The homology groups of a one-point space* pt *are trivial but in degree zero,*

$$H_n(\mathrm{pt}) \cong \begin{cases} 0, & \text{if } n > 0, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. For every $n \geqslant 0$ there is precisely one continuous map $\alpha\colon \Delta^n \to \mathrm{pt}$, namely the constant map. We denote this map by κ_n. Then the boundary of κ_n is

$$\partial\kappa_n = \sum_{i=0}^{n}(-1)^i\kappa_n\circ d_i = \sum_{i=0}^{n}(-1)^i\kappa_{n-1} = \begin{cases} \kappa_{n-1}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

For all n we have $S_n(\mathrm{pt}) \cong \mathbb{Z}$ generated by κ_n and therefore the singular chain complex looks as follows:

$$\ldots \xrightarrow{\partial=0} \mathbb{Z} \xrightarrow{\partial=\mathrm{id}_\mathbb{Z}} \mathbb{Z} \xrightarrow{\partial=0} \mathbb{Z}.$$

□

3. H_0 and H_1

Before we calculate anything, we define a map.

Proposition 3.1. *For any topological space X there is a homomorphism $\varepsilon\colon H_0(X) \to \mathbb{Z}$ with $\varepsilon \neq 0$ for $X \neq \varnothing$.*

Proof. If $X \neq \varnothing$, then we define $\varepsilon(\alpha) = 1$ for any $\alpha\colon \Delta^0 \to X$, thus $\varepsilon(\sum_{i\in I}\lambda_i\alpha_i) = \sum_{i\in I}\lambda_i$ on $S_0(X)$. As only finitely many λ_i are non-trivial, this is in fact a finite sum.

We have to show that this map is well-defined on homology, *i.e.*, that it vanishes on boundaries. One possibility is to see that ε can be interpreted as the map on singular chains that is induced by the projection map of X to a one-point space.

One can also show the claim directly: Let $S_0(X) \ni c = \partial b$ be a boundary and write $b = \sum_{i \in I} \nu_i \beta_i$ with $\beta_i \colon \Delta^1 \to X$. Then we get

$$\partial b = \partial \sum_{i \in I} \nu_i \beta_i = \sum_{i \in I} \nu_i (\beta_i \circ d_0 - \beta_i \circ d_1) = \sum_{i \in I} \nu_i \beta_i \circ d_0 - \sum_{i \in I} \nu_i \beta_i \circ d_1$$

and hence

$$\varepsilon(c) = \varepsilon(\partial b) = \sum_{i \in I} \nu_i - \sum_{i \in I} \nu_i = 0.$$

□

We said that $S_0(\varnothing)$ is zero, so $H_0(\varnothing) = 0$ and in this case we define ε to be the zero map.

If $X \neq \varnothing$, then any $\alpha \colon \Delta^0 \to X$ can be identified with its image point, so the map ε on $S_0(X)$ counts points in X with multiplicities.

Proposition 3.2. *If X is a path-connected, non-empty space, then $\varepsilon \colon H_0(X) \cong \mathbb{Z}$.*

Proof. As X is non-empty, there is a point $x \in X$ and the constant map κ_x with value x is an element in $S_0(X)$ with $\varepsilon(\kappa_x) = 1$. Therefore ε is surjective. For any other point $y \in X$ there is a continuous path $\omega \colon [0,1] \to X$ with $\omega(0) = x$ and $\omega(1) = y$. We define $\alpha_\omega \colon \Delta^1 \to X$ as

$$\alpha_\omega(t_0, t_1) = \omega(1 - t_0).$$

Then

$$\partial(\alpha_\omega) = \partial_0(\alpha_\omega) - \partial_1(\alpha_\omega) = \alpha_\omega(e_1) - \alpha_\omega(e_0) = \alpha_\omega(0,1) - \alpha_\omega(1,0) = \kappa_y - \kappa_x,$$

and the two generators κ_x, κ_y are homologous. This shows that ε is injective. □

From now on we will identify paths w and their associated 1-simplices α_w.

Corollary 3.3. *If X is of the form $X = \bigsqcup_{i \in I} X_i$ such that the X_i are non-empty and path-connected, then*

$$H_0(X) \cong \bigoplus_{i \in I} \mathbb{Z}.$$

In this case, the zeroth homology group of X is the free abelian group generated by the path-components.

Proof. The singular chain complex of X splits as the direct sum of chain complexes of the X_i:

$$S_n(X) \cong \bigoplus_{i \in I} S_n(X_i)$$

for all n. Boundary summands ∂_i stay in a component, in particular,

$$\partial \colon S_1(X) \cong \bigoplus_{i \in I} S_1(X_i) \to \bigoplus_{i \in I} S_0(X_i) \cong S_0(X)$$

is the direct sum of the boundary operators $\partial \colon S_1(X_i) \to S_0(X_i)$ and the claim follows. □

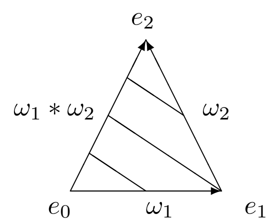
Next, we want to relate H_1 to the fundamental group. Let X be path-connected and $x \in X$.

Lemma 3.4. *Let $\omega_1, \omega_2, \omega$ be paths in X.*

(a) *Constant paths are null-homologous.*
(b) *If $\omega_1(1) = \omega_2(0)$, then $\omega_1 * \omega_2 - \omega_1 - \omega_2$ is a boundary. Here $\omega_1 * \omega_2$ is the concatenation of ω_1 followed by ω_2.*
(c) *If $\omega_1(0) = \omega_2(0), \omega_1(1) = \omega_2(1)$ and if ω_1 is homotopic to ω_2 relative to $\{0,1\}$, then ω_1 and ω_2 are homologous as singular 1-chains.*
(d) *Any 1-chain of the form $\bar{\omega} * \omega$ is a boundary. Here, $\bar{\omega}(t) := \omega(1-t)$.*

Proof. For a), consider the constant singular 2-simplex $\alpha(t_0, t_1, t_2) = x$ and c_x, the constant path on x. Then $\partial\alpha = c_x - c_x + c_x = c_x$.

For b), we define a singular 2-simplex $\beta\colon \Delta^2 \to X$ as follows.



We define β on the boundary components of Δ^2 as indicated and prolong it constantly along the sloped inner lines. Then

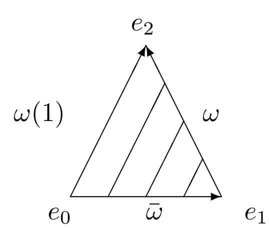
$$\partial\beta = \beta \circ d_0 - \beta \circ d_1 + \beta \circ d_2 = \omega_2 - \omega_1 * \omega_2 + \omega_1.$$

For c): Let $H\colon [0,1] \times [0,1] \to X$ a homotopy from ω_1 to ω_2. As we have that $H(0,t) = \omega_1(0) = \omega_2(0)$, we can factor H through the quotient $[0,1] \times [0,1]/\{0\} \times [0,1] \cong \Delta^2$ with induced map $h\colon \Delta^2 \to X$. Then

$$\partial h = h \circ d_0 - h \circ d_1 + h \circ d_2.$$

The first summand is null-homologous, because it's constant (with value $\omega_1(1) = \omega_2(1)$), the second one is ω_2 and the last is ω_1, thus $\omega_1 - \omega_2$ is null-homologous.

For d): Consider $\gamma\colon \Delta^2 \to X$ as indicated below.



□

Definition 3.5. Let $h\colon \pi_1(X,x) \to H_1(X)$ be the map, that sends the homotopy class of a closed path ω, $[\omega]_{\pi_1}$, to its homology class $[\omega] = [\omega]_{H_1}$. This map is called the *Hurewicz-homomorphism*.

Witold Hurewicz: 1904–1956 `https://en.wikipedia.org/wiki/Witold_Hurewicz` (Mayan pyramids are dangerous, at least for mathematicians.)

Lemma 3.4 ensures that h is well-defined and

$$h([\omega_1][\omega_2]) = h([\omega_1 * \omega_2]) = [\omega_1] + [\omega_2] = h([\omega_1]) + h([\omega_2]);$$

thus h is a homomorphism.

Note that for a closed path ω we have that $[\bar{\omega}] = -[\omega]$ in $H_1(X)$.

Definition 3.6. Let G be an arbitrary group, then its *abelianization*, G_{ab}, is $G/[G,G]$.

Recall that $[G,G]$ is the commutator subgroup of G. That is the smallest subgroup of G containing all commutators $ghg^{-1}h^{-1}, g,h \in G$. It is a normal subgroup of G: If $c \in [G,G]$, then for any $g \in G$ the element $gcg^{-1}c^{-1}$ is a commutator and also by the closure property of subgroups the element $gcg^{-1}c^{-1}c = gcg^{-1}$ is in the commutator subgroup.

Proposition 3.7. *The Hurewicz homomorphism factors through the abelianization of $\pi_1(X,x)$ and induces an isomorphism*

$$\pi_1(X,x)_{\mathrm{ab}} \cong H_1(X)$$

for all path-connected X.

$$\begin{tikzcd}
\pi_1(X,x) \arrow[rr, "h"] \arrow[d, "p"'] & & H_1(X) \\
\pi_1(X,x)_{\mathrm{ab}} = \pi_1(X,x)/[\pi_1(X,x),\pi_1(X,x)] \arrow[urr, "\cong", "h_{\mathrm{ab}}"'] & &
\end{tikzcd}$$

Proof. We will construct an inverse to h_{ab}. For any $y \in X$ we choose a path u_y from x to y. For $y = x$ we take u_x to be the constant path on x. Let α be an arbitrary singular 1-simplex and $y_i = \alpha(e_i)$. Define $\phi\colon S_1(X) \to \pi_1(X,x)_{\mathrm{ab}}$ on generators as $\phi(\alpha) = [u_{y_0} * \alpha * \bar{u}_{y_1}]$ and extend ϕ linearly to all of $S_1(X)$, keeping in mind that the composition in π_1 is written multiplicatively.

We have to show that ϕ is trivial on boundaries, so let $\beta\colon \Delta^2 \to X$. Then

$$\phi(\partial\beta) = \phi(\beta\circ d_0 - \beta\circ d_1 + \beta\circ d_2) = \phi(\beta\circ d_0)\phi(\beta\circ d_1)^{-1}\phi(\beta\circ d_2).$$

Abbreviating $\beta \circ d_i$ with α_i we get as a result

$$[u_{y_1} * \alpha_0 * \bar{u}_{y_2}][u_{y_0} * \alpha_1 * \bar{u}_{y_2}]^{-1}[u_{y_0} * \alpha_2 * \bar{u}_{y_1}] = [u_{y_0} * \alpha_2 * \bar{u}_{y_1} * u_{y_1} * \alpha_0 * \bar{u}_{y_2} * u_{y_2} * \bar{\alpha_1} * \bar{u}_{y_0}].$$

Here, we've used that the image of ϕ is abelian. We can reduce $\bar{u}_{y_1} * u_{y_1}$ and $\bar{u}_{y_2} * u_{y_2}$ and are left with $[u_{y_0} * \alpha_2 * \alpha_0 * \bar{\alpha_1} * \bar{u}_{y_0}]$ but $\alpha_2 * \alpha_0 * \bar{\alpha_1}$ is the closed path tracing the boundary of β and therefore it is null-homotopic in X. Thus $\phi(\partial\beta) = 0$ and ϕ passes to a map

$$\phi\colon H_1(X) \to \pi_1(X,x)_{\mathrm{ab}}.$$

The composition $\phi \circ h_{\mathrm{ab}}$ evaluated on the class of a closed path ω gives

$$\phi \circ h_{\mathrm{ab}}[\omega]_{\pi_1} = \phi[\omega]_{H_1} = [u_x * \omega * \bar{u}_x]_{\pi_1}.$$

But we chose u_x to be constant, thus $\phi \circ h_{\mathrm{ab}} = \mathrm{id}$.

If $c = \sum \lambda_i \alpha_i$ is a cycle, then $h_{\mathrm{ab}} \circ \phi(c)$ is of the form $[c + D_{\partial c}]$ where the $D_{\partial c}$-part comes from the contributions of the u_{y_i}. The fact that $\partial(c) = 0$ implies that the summands in $D_{\partial c}$ cancel off and thus $h_{\mathrm{ab}} \circ \phi = \mathrm{id}_{H_1(X)}$. □

Note, that abelianization doesn't change anything for abelian groups, *i.e.*, whenever we have an abelian fundamental group, we know that $H_1(X) \cong \pi_1(X,x)$.

Corollary 3.8. *Knowledge of π_1 gives*

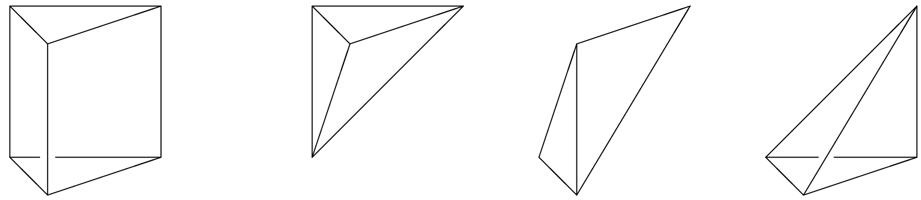
$$H_1(\mathbb{S}^n) = 0, \text{ for } n > 1, \quad H_1(\mathbb{S}^1) \cong \mathbb{Z},$$
$$H_1(\underbrace{\mathbb{S}^1 \times \ldots \times \mathbb{S}^1}_{n}) \cong \mathbb{Z}^n,$$
$$H_1(\mathbb{S}^1 \vee \mathbb{S}^1) \cong (\mathbb{Z} * \mathbb{Z})_{\mathrm{ab}} \cong \mathbb{Z} \oplus \mathbb{Z},$$
$$H_1(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z}, & n = 1, \\ \mathbb{Z}/2\mathbb{Z}, & n > 1. \end{cases}$$

4. Homotopy invariance

We want to show that two continuous maps that are homotopic induce identical maps on the level of homology groups.

Heuristics: If $\alpha\colon \Delta^n \to X$ is a singular n-simplex and if f, g are homotopic maps from X to Y, then the homotopy from $f \circ \alpha$ to $g \circ \alpha$ starts on $\Delta^n \times [0,1]$. We want to translate this geometric homotopy into a chain homotopy on the singular chain complex. To that end we have to cut the prism $\Delta^n \times [0,1]$ into $(n+1)$-simplices. In low dimensions this is easy:

$\Delta^0 \times [0,1]$ is homeomorphic to Δ^1, $\Delta^1 \times [0,1] \cong [0,1]^2$ and this can be cut into two copies of Δ^2 and $\Delta^2 \times [0,1]$ is a 3-dimensional prism and that can be glued together from three tetrahedrons, *e.g.*, like



As you might guess now, we use $n+1$ copies of Δ^{n+1} to build $\Delta^n \times [0,1]$.

Definition 4.1. For $i = 0, \dots, n$ define $p_i \colon \Delta^{n+1} \to \Delta^n \times [0,1]$ as

$$p_i(t_0, \dots, t_{n+1}) = ((t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1}), t_{i+1} + \ldots + t_{n+1}) \in \Delta^n \times [0,1].$$

On the standard basis vectors e_k we obtain

$$p_i(e_k) = \begin{cases} (e_k, 0), & \text{for } 0 \leqslant k \leqslant i, \\ (e_{k-1}, 1), & \text{for } k > i. \end{cases}$$

We obtain maps $P_i \colon S_n(X) \to S_{n+1}(X \times [0,1])$ via $P_i(\alpha) = (\alpha \times \mathrm{id}) \circ p_i$:

$$\Delta^{n+1} \xrightarrow{p_i} \Delta^n \times [0,1] \xrightarrow{\alpha \times \mathrm{id}} X \times [0,1].$$

For $k = 0, 1$ let $j_k \colon X \to X \times [0,1]$ be the inclusion $x \mapsto (x, k)$.

Lemma 4.2. *The maps P_i satisfy the following relations*

(a) $\partial_0 \circ P_0 = S_n(j_1)$,
(b) $\partial_{n+1} \circ P_n = S_n(j_0)$,
(c) $\partial_i \circ P_i = \partial_i \circ P_{i-1}$ *for* $1 \leqslant i \leqslant n$.
(d)

$$\partial_j \circ P_i = \begin{cases} P_i \circ \partial_{j-1}, & \textit{for } i \leqslant j-2 \\ P_{i-1} \circ \partial_j, & \textit{for } i \geqslant j+1. \end{cases}$$

Proof. Note that it suffices to check the corresponding claims for the p_i's and d_j's.
For the first two points, we note that on Δ^n we have

$$p_0 \circ d_0(t_0, \dots, t_n) = p_0(0, t_0, \dots, t_n) = ((t_0, \dots, t_n), \sum t_i) = ((t_0, \dots, t_n), 1) = j_1(t_0, \dots, t_n)$$

and

$$p_n \circ d_{n+1}(t_0, \dots, t_n) = p_n(t_0, \dots, t_n, 0) = ((t_0, \dots, t_n), 0) = j_0(t_0, \dots, t_n).$$

For c), one checks that $p_i \circ d_i = p_{i-1} \circ d_i$ on Δ^n: both give $((t_0, \dots, t_n), \sum_{j=i}^n t_j)$ on $(t_0, \dots, t_n)$.
For d) in the case $i \geqslant j+1$, consider the following diagram

$$\begin{tikzcd} & \Delta^{n+1} \arrow[dr, "p_i"] & \\ \Delta^n \arrow[ur, "d_j"] \arrow[dr, "p_{i-1}"'] & & \Delta^n \times [0,1] \\ & \Delta^{n-1} \times [0,1] \arrow[ur, "d_j \times \mathrm{id}"'] & \end{tikzcd}$$

Checking coordinates one sees that this diagram commutes. The remaining case follows from a similar observation. $\square$

Definition 4.3. We define $P \colon S_n(X) \to S_{n+1}(X \times [0,1])$ as $P = \sum_{i=0}^n (-1)^i P_i$.

Lemma 4.4. *The map P is a chain homotopy between $(S_n(j_0))_n$ and $(S_n(j_1))_n$,* i.e., $\partial \circ P + P \circ \partial = S_n(j_1) - S_n(j_0)$.

Proof. We take an $\alpha\colon \Delta^n \to X$ and calculate

$$\partial P\alpha + P\partial\alpha = \sum_{i=0}^{n}\sum_{j=0}^{n+1}(-1)^{i+j}\partial_j P_i\alpha + \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j}P_i\partial_j\alpha.$$

If we single out the terms involving the pairs of indices $(0,0)$ and $(n,n+1)$ in the first sum, we are left with

$$S_n(j_1)(\alpha) - S_n(j_0)(\alpha) + \sum_{(i,j)\neq(0,0),(n,n+1)}(-1)^{i+j}\partial_j P_i\alpha + \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j}P_i\partial_j\alpha.$$

Using Lemma 4.2 we see that only the first two summands survive. □

So, finally we can prove the main result of this section:

Theorem 4.5. *(Homotopy invariance)*
If $f,g\colon X\to Y$ are homotopic maps, then they induce the same map on homology.

Proof. Let $H\colon X\times[0,1]\to Y$ be a homotopy from f to g, *i.e.*, $H\circ j_0 = f$ and $H\circ j_1 = g$. Set $K_n := S_{n+1}(H)\circ P$. We claim that $(K_n)_n$ is a chain homotopy between $(S_n(f))_n$ and $(S_n(g))_n$. Note that H induces a chain map $(S_n(H))_n$. Therefore we get

$$\begin{aligned}\partial\circ S_{n+1}(H)\circ P + S_n(H)\circ P\circ\partial &= S_n(H)\circ\partial\circ P + S_n(H)\circ P\circ\partial\\ &= S_n(H)\circ(\partial\circ P + P\circ\partial)\\ &= S_n(H)\circ(S_n(j_1) - S_n(j_0)) = S_n(H\circ j_1) - S_n(H\circ j_0)\\ &= S_n(g) - S_n(f).\end{aligned}$$

Hence these two maps are chain homotopic and $H_n(g) = H_n(f)$ for all n. □

Corollary 4.6. *If two spaces X,Y are homotopy equivalent, then $H_*(X)\cong H_*(Y)$. In particular, if X is contractible, then*

$$H_*(X)\cong\begin{cases}\mathbb{Z}, & \text{for } *=0,\\ 0, & \text{otherwise.}\end{cases}$$

Examples. As $\mathbb{R}^n$ is contractible for all n, the above corollary gives that its homology is trivial but in degree zero where it consists of the integers.

As the Möbius strip is homotopy equivalent to $\mathbb{S}^1$, we know that their homology groups are isomorphic.

If you know about vector bundles: the zero section of a vector bundle induces a homotopy equivalence between the base and the total space, hence these two have isomorphic homology groups.

5. The long exact sequence in homology

A typical situation is that there is a subspace A of a topological space X and you might know something about A or X and want to calculate the homology of the other space using that partial information.

But before we can move on to topological applications we need some techniques about chain complexes. We need to know that a short exact sequence of chain complexes gives rise to a long exact sequence in homology.

Definition 5.1. Let A,B,C be abelian groups and

$$A\xrightarrow{f}B\xrightarrow{g}C$$

a sequence of homomorphisms. Then this sequence is *exact*, if the image of f is the kernel of g.

Definition 5.2. If

$$\ldots\xrightarrow{f_{i+1}}A_i\xrightarrow{f_i}A_{i-1}\xrightarrow{f_{i-1}}\ldots$$

is a sequence of homomorphisms of abelian groups (indexed over the integers), then this sequence is called *(long) exact*, if it is exact at every A_i, *i.e.*, the image of f_{i+1} is the kernel of f_i for all i.

An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a *short exact sequence.*

Examples. The sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is a short exact sequence.

If $\iota\colon U \to A$ is a monomorphism, then $0 \longrightarrow U \xrightarrow{\iota} A$ is exact. Similarly, an epimorphism $\varrho\colon B \to Q$ gives rise to an exact sequence $B \xrightarrow{\varrho} Q \longrightarrow 0$ and an isomorphism $\phi\colon A \cong A'$ sits in an exact sequence $0 \longrightarrow A \xrightarrow{\phi} A' \longrightarrow 0$.

A sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact iff f is injective, the image of f is the kernel of g and g is an epimorphism. Another equivalent description is to view a sequence as above as a chain complex with vanishing homology groups. Homology measures the deviation from exactness.

Definition 5.3. If A_*, B_*, C_* are chain complexes and $f_*\colon A_* \to B_*$, $g\colon B_* \to C_*$ are chain maps, then we call the sequence

$$A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_*$$

exact, if the image of f_n is the kernel of g_n for all $n \in \mathbb{Z}$.

Thus such an exact sequence of chain complexes is a commuting double ladder

$$\begin{tikzcd}
\vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] \\
A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d"] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d"] & C_{n+1} \arrow[d, "d"] \\
A_n \arrow[r, "f_n"] \arrow[d, "d"] & B_n \arrow[r, "g_n"] \arrow[d, "d"] & C_n \arrow[d, "d"] \\
A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "d"] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "d"] & C_{n+1} \arrow[d, "d"] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

in which every row is exact.

Example. Let p be a prime, then

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, "p"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "p^2"] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r, "p"] \arrow[d, "\pi"] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "\pi"] & \mathbb{Z}/p\mathbb{Z} \arrow[d, "\mathrm{id}"] \\
\mathbb{Z}/p\mathbb{Z} \arrow[r, "p"] \arrow[d] & \mathbb{Z}/p^2\mathbb{Z} \arrow[r, "\pi"] \arrow[d] & \mathbb{Z}/p\mathbb{Z} \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

has exact rows and columns, in particular it is an exact sequence of chain complexes. Here, π denotes varying canonical projection maps.

Proposition 5.4. *If* $0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ *is a short exact sequence of chain complexes, then there exists a homomorphism* $\delta\colon H_n(C_*) \to H_{n-1}(A_*)$ *for all* $n \in \mathbb{Z}$ *which is natural,* i.e., *if*

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "f"] \arrow[d, "\alpha"] & B_* \arrow[r, "g"] \arrow[d, "\beta"] & C_* \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A'_* \arrow[r, "f'"] & B'_* \arrow[r, "g'"] & C'_* \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of chain maps in which the rows are exact then $H_{n-1}(\alpha) \circ \delta = \delta \circ H_n(\gamma)$*,*

$$\begin{tikzcd}
H_n(C_*) \arrow[r, "\delta"] \arrow[d, "H_n(\gamma)"'] & H_{n-1}(A_*) \arrow[d, "H_{n-1}(\alpha)"] \\
H_n(C'_*) \arrow[r, "\delta"] & H_{n-1}(A'_*)
\end{tikzcd}$$

The method of proof is an instance of a *diagram chase*. The homomorphism δ is called *connecting homomorphism*. The implicit claim in the proposition above is that δ is not always the zero map.

Proof. We show the existence of a δ first and then prove that the constructed map satisfies the naturality condition.

a) Definition of δ:

Is $c \in C_n$ with $d(c) = 0$, then we choose a $b \in B_n$ with $g_n b = c$. This is possible because g_n is surjective. We know that $dg_n b = dc = 0 = g_{n-1} db$ thus db is in the kernel of g_{n-1}, hence it is in the image of f_{n-1}. Thus there is an $a \in A_{n-1}$ with $f_{n-1}a = db$. We have that $f_{n-2}da = df_{n-1}a = ddb = 0$ and as f_{n-2} is injective, this shows that a is a cycle.

We define $\delta[c] := [a]$.

$$B_n \ni b \xmapsto{g_n} c \in C_n$$

$$A_{n-1} \ni a \xmapsto{f_{n-1}} db \in B_{n-1}$$

In order to check that δ is well-defined, we assume that there are b and b' with $g_n b = g_n b' = c$. Then $g_n(b - b') = 0$ and thus there is an $\tilde{a} \in A_n$ with $f_n \tilde{a} = b - b'$. Define a' as $a - d\tilde{a}$. Then

$$f_{n-1}a' = f_{n-1}a - f_{n-1}d\tilde{a} = db - db + db' = db'$$

because $f_{n-1}d\tilde{a} = db - db'$. As f_{n-1} is injective, we get that a' is uniquely determined with this property. As a is homologous to a' we get that $[a] = [a'] = \delta[c]$, thus the latter is independent of the choice of b.

In addition, we have to make sure that the value stays the same if we add a boundary term to c, *i.e.*, take $c' = c + d\tilde{c}$ for some $\tilde{c} \in C_{n+1}$. Choose preimages of $c, \tilde{c}$ under g_n and g_{n+1}, *i.e.*, b and $\tilde{b}$ with $g_n b = c$ and $g_{n+1}\tilde{b} = \tilde{c}$. Then the element $b' = b + d\tilde{b}$ has boundary $db' = db$ and thus both choices will result in the same a.

Therefore $\delta\colon H_n(C_*) \to H_{n-1}(A_*)$ is well-defined.

b) We have to show that δ is natural with respect to maps of short exact sequences.

Let $c \in Z_n(C_*)$, then $\delta[c] = [a]$ for a $b \in B_n$ with $g_n b = c$ and an $a \in A_{n-1}$ with $f_{n-1}a = db$. Therefore, $H_{n-1}(\alpha)(\delta[c]) = [\alpha_{n-1}(a)]$.

On the other hand, we have

$$f'_{n-1}(\alpha_{n-1}a) = \beta_{n-1}(f_{n-1}a) = \beta_{n-1}(db) = d\beta_n b$$

and

$$g'_n(\beta_n b) = \gamma_n g_n b = \gamma_n c$$

and we can conclude that by the construction of δ

$$\delta[\gamma_n(c)] = [\alpha_{n-1}(a)]$$

and this shows $\delta \circ H_n(\gamma) = H_{n-1}(\alpha) \circ \delta$. □

With this auxiliary result at hand we can now prove the main result in this section:

Proposition 5.5. *For any short exact sequence*

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

of chain complexes we obtain a long exact sequence of homology groups

$$\ldots \xrightarrow{\delta} H_n(A_*) \xrightarrow{H_n(f)} H_n(B_*) \xrightarrow{H_n(g)} H_n(C_*) \xrightarrow{\delta} H_{n-1}(A_*) \xrightarrow{H_{n-1}(f)} \ldots$$

Proof. a) Exactness at the spot $H_n(B_*)$:

We have $H_n(g) \circ H_n(f)[a] = [g_n(f_n(a))] = 0$ because the composition of g_n and f_n is zero. This proves that the image of $H_n(f)$ is contained in the kernel of $H_n(g)$.

For the converse, let $[b] \in H_n(B_*)$ with $[g_n b] = 0$. Then there is a $c \in C_{n+1}$ with $dc = g_n b$. As g_{n+1} is surjective, we find a $b' \in B_{n+1}$ with $g_{n+1}b' = c$. Hence

$$g_n(b - db') = g_n b - dg_{n+1}b' = dc - dc = 0.$$

Exactness gives an $a \in A_n$ with $f_n a = b - db'$ and $da = 0$ and therefore $f_n a$ is homologous to b and $H_n(f)[a] = [b]$ thus the kernel of $H_n(g)$ is contained in the image of $H_n(f)$.

b) Exactness at the spot $H_n(C_*)$:

Let $b \in H_n(B_*)$, then $\delta[g_n b] = 0$ because b is a cycle, so 0 is the only preimage under f_{n-1} of $db = 0$. Therefore the image of $H_n(g)$ is contained in the kernel of δ.

Now assume that $\delta[c] = 0$, thus in the construction of δ, the a is a boundary, $a = da'$. Then for a preimage of c under g_n, b, we have by the definition of a

$$d(b - f_n a') = db - df_n a' = db - f_{n-1}a = 0.$$

Thus $b - f_n a'$ is a cycle and $g_n(b - f_n a') = g_n b - g_n f_n a' = g_n b - 0 = g_n b = c$, so we found a preimage for $[c]$ and the kernel of δ is contained in the image of $H_n(g)$.

c) Exactness at $H_{n-1}(A_*)$:

Let c be a cycle in $Z_n(C_*)$. Again, we choose a preimage b of c under g_n and an a with $f_{n-1}(a) = db$. Then $H_{n-1}(f)\delta[c] = [f_{n-1}(a)] = [db] = 0$. Thus the image of δ is contained in the kernel of $H_{n-1}(f)$.

If $a \in Z_{n-1}(A_*)$ with $H_{n-1}(f)[a] = 0$. Then $f_{n-1}a = db$ for some $b \in B_n$. Take $c = g_n b$. Then by definition $\delta[c] = [a]$. □

6. The long exact sequence of a pair of spaces

Let X be a topological space and $A \subset X$ a subspace of X. Consider the inclusion map $i\colon A \to X$, $i(a) = a$. We obtain an induced map $S_n(i)\colon S_n(A) \to S_n(X)$, but we know that the inclusion of spaces doesn't have to yield a monomorphism on homology groups. For instance, we can include $A = \mathbb{S}^1$ into $X = \mathbb{D}^2$.

We consider pairs of spaces (X, A).

Definition 6.1. The *relative chain complex of* (X, A) is

$$S_*(X, A) := S_*(X)/S_*(A).$$

Alternatively, $S_n(X, A)$ is isomorphic to the free abelian group generated by all n-simplices $\beta\colon \Delta^n \to X$ whose image is not completely contained in A, *i.e.*, $\beta(\Delta^n) \cap (X \setminus A) \neq \varnothing$.

Definition 6.2.

- Elements in $S_n(X, A)$ are called *relative chains in* (X, A)
- Cycles in $S_n(X, A)$ are chains c with $\partial^X(c)$ whose generators have image in A. These are *relative cycles*.
- Boundaries in $S_n(X, A)$ are chains c in X such that $c = \partial^X b + a$ where a is a chain in A.

A continuous map $f\colon X \to Y$ with $f(A) \subset B$ is denoted by $f\colon (X, A) \to (Y, B)$. Such maps induce chain maps $S_*(f)\colon S_*(X, A) \to S_*(Y, B)$.

The following facts are immediate from the definition:

(a) $S_n(X, \varnothing) \cong S_n(X)$.
(b) $S_n(X, X) = 0$.
(c) $S_n(X \sqcup X', X') \cong S_n(X)$.

Definition 6.3. The *relative homology groups of* (X, A) are

$$H_n(X, A) := H_n(S_*(X, A)).$$

Theorem 6.4. *For any pair of topological spaces $A \subset X$ we obtain a long exact sequence*

$$\ldots \xrightarrow{\delta} H_n(A) \xrightarrow{H_n(i)} H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} \ldots$$

For a map $f\colon (X, A) \to (Y, B)$ we get an induced map of long exact sequences

$$\begin{tikzcd}
\ldots \arrow[r, "\delta"] & H_n(A) \arrow[r, "H_n(i)"] \arrow[d, "H_n(f|_A)"] & H_n(X) \arrow[r] \arrow[d, "H_n(f)"] & H_n(X, A) \arrow[r, "\delta"] \arrow[d, "H_n(f)"] & H_{n-1}(A) \arrow[r, "H_{n-1}(i)"] \arrow[d, "H_{n-1}(f|_A)"] & \ldots \\
\ldots \arrow[r, "\delta"'] & H_n(B) \arrow[r, "H_n(i)"'] & H_n(Y) \arrow[r] & H_n(Y, B) \arrow[r, "\delta"'] & H_{n-1}(B) \arrow[r, "H_{n-1}(i)"'] & \ldots
\end{tikzcd}$$

Proof. By definition of $S_*(X, A)$ the sequence

$$0 \longrightarrow S_*(A) \xrightarrow{S_*(i)} S_*(X) \xrightarrow{\pi} S_*(X, A) \longrightarrow 0$$

is an exact sequence of chain complexes and by Proposition 5.5 we obtain the first claim. For a map f as above the following diagram

$$\begin{tikzcd}
0 \arrow[r] & S_n(A) \arrow[r, "S_n(i)"] \arrow[d, "S_n(f|_A)"] & S_n(X) \arrow[r, "\pi"] \arrow[d, "S_n(f)"] & S_n(X, A) \arrow[r] \arrow[d, "S_n(f)/S_n(f|_A)"] & 0 \\
0 \arrow[r] & S_n(B) \arrow[r, "S_n(i)"] & S_n(Y) \arrow[r, "\pi"] & S_n(Y, B) \arrow[r] & 0
\end{tikzcd}$$

commutes. □

Example. Let $A = \mathbb{S}^{n-1}$ and $X = \mathbb{D}^n$, then we know that $H_j(i)$ is trivial for $j > 0$. From the long exact sequence we get that $\delta\colon H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong H_{j-1}(\mathbb{S}^{n-1})$ for $j > 1$ and $n \geqslant 1$.

Proposition 6.5. *If $i\colon A \hookrightarrow X$ is a weak retract,* i.e., *if there is an $r\colon X \to A$ with $r \circ i \simeq \mathrm{id}_A$, then*

$$H_n(X) \cong H_n(A) \oplus H_n(X,A), \quad 0 \leqslant n.$$

Proof. From the assumption we get that $H_n(r) \circ H_n(i) = H_n(\mathrm{id}_A) = \mathrm{id}_{H_n(A)}$ for all n and hence $H_n(i)$ is injective for all n. This implies that $0 \longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(X)$ is exact. Injectivity of $H_{n-1}(i)$ yields that the image of $\delta\colon H_n(X,A) \to H_{n-1}(A)$ is trivial. Therefore we get short exact sequences

$$0 \longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{\pi_*} H_n(X,A) \longrightarrow 0$$

for all n. As $H_n(r)$ is a left-inverse for $H_n(i)$ we obtain a splitting

$$H_n(X) \cong H_n(A) \oplus H_n(X,A)$$

because we map $[c] \in H_n(X)$ to $([rc], \pi_*[c])$ with inverse

$$H_n(A) \oplus H_n(X,A) \ni ([a],[b]) \mapsto H_n(i)[a] + [a'] - H_n(i \circ r)[a'] \in H_n(X)$$

for any $[a'] \in H_n(X)$ with $\pi_*[a'] = [b]$. The second map is well-defined: if $[a'']$ is another element with $\pi_*[a''] = [b]$, then $[a' - a'']$ is of the form $H_n(i)[\tilde{a}]$ because this element is in the kernel of π_* and hence $[a' - a''] - H_n(ir)[a' - a'']$ is trivial. $\square$

Proposition 6.6. *For any $\varnothing \neq A \subset X$ such that $A \subset X$ is a deformation retract we get*

$$H_n(i)\colon H_n(A) \cong H_n(X), \quad H_n(X,A) \cong 0, \quad 0 \leqslant n.$$

Proof. Recall, that $i\colon A \hookrightarrow X$ is a *deformation retract*, if there is a homotopy $R\colon X \times [0,1] \to X$ such that

(a) $R(x,0) = x$ for all $x \in X$,
(b) $R(x,1) \in A$ for all $x \in X$, and
(c) $R(a,1) = a$ for all $a \in A$.

In particular, R is a homotopy from id_X to $i \circ r$ where $r = R(-,1)\colon X \to A$. Condition (c) can be rewritten as $r \circ i = \mathrm{id}_A$, *i.e.*, r is a retraction, and thus A and X are homotopically equivalent and $H_n(i)$ is an isomorphism for all $n \geqslant 0$. $\square$

Definition 6.7. If X has two subspaces $A, B \subset X$, then (X,A,B) is called a *triple*, if $B \subset A \subset X$.

Any triple gives rise to three pairs of spaces (X,A), (X,B) and (A,B) and accordingly we have three long exact sequences in homology. But there is another one.

Proposition 6.8. *For any triple (X,A,B) there is a natural long exact sequence*

$$\ldots \longrightarrow H_n(A,B) \longrightarrow H_n(X,B) \longrightarrow H_n(X,A) \xrightarrow{\delta} H_{n-1}(A,B) \longrightarrow \ldots$$

This sequence is part of the following braided commutative diagram displaying four long exact sequences

$$\begin{tikzcd}
\ldots \arrow[dr] & & & & & & \ldots \\
& H_{n+1}(X,A) \arrow[dr] \arrow[rr, bend left] & & H_n(A,B) \arrow[dr] \arrow[rr, bend left] & & H_{n-1}(B) \arrow[ur] \arrow[dr] & \\
\ldots \arrow[ur] \arrow[dr] & & H_n(A) \arrow[ur] \arrow[dr] & & H_n(X,B) \arrow[ur] \arrow[dr] & & \ldots \\
& H_n(B) \arrow[ur] \arrow[rr, bend right] & & H_n(X) \arrow[ur] \arrow[rr, bend right] & & H_n(X,A) \arrow[ur] \arrow[dr] & \\
\ldots \arrow[ur] & & & & & & \ldots
\end{tikzcd}$$

In particular, the connecting homomorphism $\delta\colon H_n(X,A) \to H_{n-1}(A,B)$ *is the composite* $\delta = \pi_*^{(A,B)} \circ \delta^{(X,A)}$.

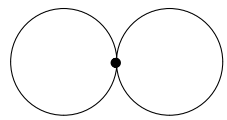
PROOF. Consider the sequence

$$0 \longrightarrow S_n(A)/S_n(B) \longrightarrow S_n(X)/S_n(B) \longrightarrow S_n(X)/S_n(A) \longrightarrow 0.$$

This sequence is exact, because $S_n(B) \subset S_n(A) \subset S_n(X)$. □

7. Excision

The aim is to simplify relative homology groups. Let $A \subset X$ be a subspace. Then it is easy to see that $H_*(X,A)$ is not isomorphic to $H_*(X \setminus A)$: Consider the figure eight as X and A as the point connecting the two copies of $\mathbb{S}^1$, then $H_0(X,A)$ is trivial, but $H_0(X \setminus A) \cong \mathbb{Z} \oplus \mathbb{Z}$.



So if we want to simplify $H_*(X,A)$ by excising something, then we have to be more careful. The first step towards that is to make singular simplices 'smaller'. The technique is called barycentric subdivision and that is a tool that's frequently used.

First, we construct cones. Let $v \in \Delta^p$ and let $\alpha\colon \Delta^n \to \Delta^p$ be a singular n-simplex in Δ^p.

DEFINITION 7.1. The *cone of* α *with respect to* v is $K_v(\alpha)\colon \Delta^{n+1} \to \Delta^p$,

$$(t_0,\dots,t_{n+1}) \mapsto \begin{cases} (1-t_{n+1})\alpha(\frac{t_0}{1-t_{n+1}},\dots,\frac{t_n}{1-t_{n+1}}) + t_{n+1}v, & t_{n+1} < 1, \\ v, & t_{n+1} = 1. \end{cases}$$

This map is well-defined and continuous. On the standard basis vectors K_v gives $K_v(e_i) = \alpha(e_i)$ for $0 \leqslant i \leqslant n$ but $K_v(e_{n+1}) = v$. Extending K_v linearly gives a map

$$K_v\colon S_n(\Delta^p) \to S_{n+1}(\Delta^p).$$

LEMMA 7.2. *The map* K_v *satisfies*

- $\partial K_v(c) = \varepsilon(c).\kappa_v - c$ *for* $c \in S_0(\Delta^p)$, $\kappa_v(e_0) = v$ *and* ε *the augmentation.*
- *For* $n > 0$ *we have that* $\partial \circ K_v - K_v \circ \partial = (-1)^{n+1}\mathrm{id}$.

PROOF. For a singular 0-simplex $\alpha\colon \Delta^0 \to \Delta^p$ we know that $\varepsilon(\alpha) = 1$ and we calculate

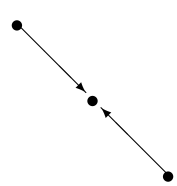
$$\partial K_v(\alpha)(e_0) = (K_v(\alpha) \circ d_0)(e_0) - (K_v(\alpha) \circ d_1)(e_0) = K_v(\alpha)(e_1) - K_v(\alpha)(e_0) = v - \alpha(e_0).$$

For $n > 0$ we have to calculate $\partial_i K_v(\alpha)$ and it is straightforward to see that $\partial_{n+1} K_v(\alpha) = \alpha$ and $\partial_i(K_v(\alpha)) = K_v(\partial_i \alpha)$ for all $i < n+1$. □

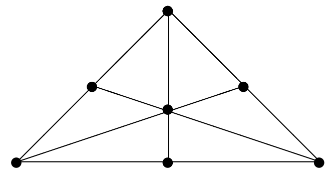
DEFINITION 7.3. For $\alpha\colon \Delta^n \to \Delta^p$ let $v(\alpha) = v := \frac{1}{n+1}\sum_{i=0}^n \alpha(e_i)$. The *barycentric subdivision* $B\colon S_n(\Delta_p) \to S_n(\Delta_p)$ is defined inductively as $B(\alpha) = \alpha$ for $\alpha \in S_0(\Delta_p)$ and $B(\alpha) = (-1)^n K_v(B(\partial\alpha))$ for $n > 0$.

For $n \geqslant 1$ this yields $B(\alpha) = \sum_{i=0}^n (-1)^{n+i} K_v(B(\partial_i \alpha))$.

If we take $n = p$ and $\alpha = \mathrm{id}_{\Delta^n}$, then for small n this looks as follows: You cannot subdivide a point any further. For $n = 1$ we get



And for $n = 2$ we get (up to tilting)



LEMMA 7.4. *The barycentric subdivision is a chain map.*

PROOF. We have to show that $\partial B = B\partial$. If α is a singular zero chain, then $\partial B\alpha = \partial\alpha = 0$ and $B\partial\alpha = B(0) = 0$.

Let $n = 1$. Then

$$\partial B\alpha = -\partial K_v B(\partial_0\alpha) + \partial K_v B(\partial_1\alpha).$$

But the boundary terms are zero chains and there B is the identity so we get

$$-\partial K_v(\partial_0\alpha) + \partial K_v(\partial_1\alpha) = -\kappa_v + \partial_0\alpha + \kappa_v - \partial_1\alpha = \partial\alpha = B\partial\alpha.$$

(Note, that the v is $v(\alpha)$, not a $v(\partial_i\alpha)$.)

We prove the claim inductively, so let $\alpha \in S_n(\Delta^p)$. Then

$$\begin{aligned}\partial B\alpha =&(-1)^n\partial K_v(B\partial\alpha)\\ =&(-1)^n((-1)^nB\partial\alpha + K_v\partial B\partial\alpha)\\ =&B\partial\alpha + (-1)^nK_vB\partial\partial\alpha = B\partial\alpha.\end{aligned}$$

Here, the first equality is by definition, the second one follows by Lemma 7.2 and then we use the induction hypothesis and the fact that $\partial\partial = 0$. $\square$

Our aim is to show that B doesn't change anything on the level of homology groups and to that end we prove that it is chain homotopic to the identity.

We construct $\psi_n\colon S_n(\Delta^p) \to S_{n+1}(\Delta^p)$ again inductively as

$$\psi_0(\alpha) := 0, \quad \psi_n(\alpha) := (-1)^{n+1}K_v(B\alpha - \alpha - \psi_{n-1}\partial\alpha)$$

with $v = \frac{1}{n+1}\sum_{i=0}^n \alpha(e_i)$.

LEMMA 7.5. *The sequence $(\psi_n)_n$ is a chain homotopy from B to the identity.*

PROOF. For $n = 0$ we have $\partial\psi_0 = 0$ and this agrees with $B - \mathrm{id}$ in that degree.

For $n = 1$, we get

$$\partial\psi_1 + \psi_0\partial = \partial\psi_1 = \partial(K_vB - K_v - K_v\psi_0\partial) = \partial K_vB - \partial K_v.$$

With Lemma 7.2 we can transform the latter to $B + K_v\partial B - \partial K_v$ and as B is a chain map, this is $B + K_vB\partial - \partial K_v$. In chain degree one $B\partial$ agrees with ∂, thus this reduces to

$$B + K_v\partial - \partial K_v = B - (\partial K_v - K_v\partial) = B - \mathrm{id}.$$

So, finally we can do the inductive step:

$$\begin{aligned}\partial\psi_n =&(-1)^{n+1}\partial K_v(B - \mathrm{id} - \psi_{n-1}\partial)\\ =&(-1)^{n+1}\partial K_vB - (-1)^{n+1}\partial K_v - (-1)^{n+1}\partial K_v\psi_{n-1}\partial\\ =&(-1)^{n+1}((-1)^{n+1}B + K_v\partial B)\\ &\qquad - (-1)^{n+1}((-1)^{n+1}\mathrm{id} + K_v\partial)\\ &\qquad - (-1)^{n+1}((-1)^{n+1}\psi_{n-1}\partial + K_v\partial\psi_{n-1}\partial)\\ =&B - \mathrm{id} - \psi_{n-1}\partial + \text{remaining terms}\end{aligned}$$

The equation

$$K_v\partial\psi_{n-1}\partial + K_v\psi_{n-2}\partial^2 = K_vB\partial - K_v\partial$$

from the inductive assumption ensures that these remaining terms give zero. $\square$

Definition 7.6. A singular n-simplex $\alpha\colon \Delta^n \to \Delta^p$ is called *affine*, if

$$\alpha(\sum_{i=0}^{n} t_i e_i) = \sum_{i=0}^{n} t_i \alpha(e_i).$$

We abbreviate $\alpha(e_i)$ with v_i, so $\alpha(\sum_{i=0}^n t_i e_i) = \sum_{i=0}^n t_i v_i$ and we call the v_i's the *vertices of* α.

Definition 7.7. Let A be a subset of a metric space (X, d). The *diameter of* A is

$$\sup\{d(x,y) | x, y \in A\}$$

and we denote it by $\mathrm{diam}(A)$.

Accordingly, the *diameter of an affine n-simplex α in Δ^p* is the diameter of its image, and we abbreviate that with $\mathrm{diam}(\alpha)$.

Lemma 7.8. *For any affine α every simplex in the chain $B\alpha$ has diameter $\leqslant \frac{n}{n+1}\mathrm{diam}(\alpha)$.*

Either you believe this lemma, or you prove it, or you check Bredon, Proof of Lemma 13.7 (p. 226).

Each simplex in $B\alpha$ is again affine; this allows us to iterate the application of B and get smaller and smaller diameter. Thus, the k-fold iteration, $B^k(\alpha)$, has diameter at most $\left(\frac{n}{n+1}\right)^k \mathrm{diam}(\alpha)$.

In the following we use the easy but powerful trick to express α as

$$\alpha = \alpha \circ \mathrm{id}_{\Delta^n} = S_n(\alpha)(\mathrm{id}_{\Delta^n}).$$

This allows us to use the barycentric subdivision for general spaces.

Definition 7.9.

(a) We define $B_n^X : S_n(X) \to S_n(X)$ as

$$B_n^X(\alpha) := S_n(\alpha) \circ B(\mathrm{id}_{\Delta^n}).$$

(b) Similarly, $\psi_n^X : S_n(X) \to S_{n+1}(X)$ is

$$\psi_n^X(\alpha) := S_{n+1}(\alpha) \circ \psi_n(\mathrm{id}_{\Delta^n}).$$

Lemma 7.10. *The maps B^X are natural in X and are homotopic to the identity on $S_n(X)$.*

Proof. Let $f\colon X \to Y$ be a continuous map. We have

$$\begin{aligned} S_n(f)B_n^X(\alpha) =& S_n(f) \circ S_n(\alpha) \circ B(\mathrm{id}_{\Delta^n}) \\ =& S_n(f \circ \alpha) \circ B(\mathrm{id}_{\Delta^n}) \\ =& B_n^Y(f \circ \alpha). \end{aligned}$$

The calculation for $\partial\psi_n^X + \psi_{n-1}^X\partial = B_n^X - \mathrm{id}_{S_n(X)}$ uses that α induces a chain map and thus we get

$$\partial\psi_n^X(\alpha) = \partial \circ S_{n+1}(\alpha) \circ \psi_n(\mathrm{id}_{\Delta^n}) = S_n(\alpha) \circ \partial \circ \psi_n(\mathrm{id}_{\Delta^n}).$$

Hence

$$\partial\psi_n^X + \psi_{n-1}^X\partial = S_n(\alpha) \circ (\partial \circ \psi_n(\mathrm{id}_{\Delta^n}) + \psi_{n-1} \circ \partial(\mathrm{id}_{\Delta^n})) = S_n(\alpha) \circ (B - \mathrm{id})(\mathrm{id}_{\Delta^n}) = B_n^X(\alpha) - \alpha.$$

□

Now we consider singular n-chains that are spanned by 'small' singular n-simplices.

Definition 7.11. Let $\mathfrak{U} = \{U_i, i \in I\}$ be an open covering of X. Then $S_n^{\mathfrak{U}}(X)$ is the free abelian group generated by all $\alpha\colon \Delta^n \to X$ such that the image of Δ^n under α is contained in one of the $U_i \in \mathfrak{U}$.

Note that $S_n^{\mathfrak{U}}(X)$ is an abelian subgroup of $S_n(X)$. As we will see now, these chains suffice to detect everything in singular homology.

Lemma 7.12. *Every chain in $S_n(X)$ is homologous to a chain in $S_n^{\mathfrak{U}}(X)$.*

Proof. Let $\alpha = \sum_{j=1}^{m} \lambda_j \alpha_j \in S_n(X)$ and let L_j for $1 \leqslant j \leqslant m$ be the Lebesgue numbers for the coverings $\{\alpha_j^{-1}(U_i), i \in I\}$ of Δ^n. Choose a k, such that $\left(\frac{n}{n+1}\right)^k \leqslant L_1, \ldots, L_m$. Then $B^k\alpha_1$ up to $B^k\alpha_m$ are all in $S_n^{\mathfrak{U}}(X)$. Therefore

$$B^k(\alpha) = \sum_{j=1}^{m} \lambda_j B^k(\alpha_j) =: \alpha' \in S_n^{\mathfrak{U}}(X).$$

As B is homotopic to the identity we have

$$\alpha \sim B\alpha \sim \ldots \sim B^k\alpha = \alpha'.$$

□

With this we get the main result of this section:

Theorem 7.13. *Let $W \subset A \subset X$ such that $\bar{W} \subset \mathring{A}$. Then the inclusion $i\colon (X \setminus W, A \setminus W) \hookrightarrow (X, A)$ induces an isomorphism*

$$H_n(i)\colon H_n(X \setminus W, A \setminus W) \cong H_n(X, A)$$

for all $n \geqslant 0$.

Proof. We first prove that $H_n(i)$ is surjective, so let $c \in S_n(X, A)$ be a relative cycle, *i.e.*, let $\partial c \in S_{n-1}(A)$. There is a k such that $c' := B^k c$ is a chain in $S_n^{\mathfrak{U}}(X)$ for the open covering $\mathfrak{U} = \{\mathring{A}, X \setminus \bar{W}\} =: \{U, V\}$. We decompose c' as $c' = c^U + c^V$ with c^U and c^V being elements in the corresponding chain complex. (This decomposition is not unique.)

We know that the boundary of c' is $\partial c' = \partial B^k c = B^k \partial c$ and by assumption this is a chain in $S_{n-1}(A)$. But $\partial c' = \partial c^U + \partial c^V$ with $\partial c^U \in S_{n-1}(U) \subset S_{n-1}(A)$. Thus, $\partial c^V \in S_{n-1}(A)$, in fact, $\partial c^V \in S_{n-1}(A \setminus W)$ and therefore c^V is a relative cycle in $S_n(X \setminus W, A \setminus W)$. This shows that $H_n(i)[c^V] = [c] \in H_n(X, A)$ because $[c] = [c^U + c^V] = [c^V]$ in $H_n(X, A)$.

The injectivity of $H_n(i)$ is shown as follows. Assume that there is a $c \in S_n(X \setminus W)$ with $\partial c \in S_{n-1}(A \setminus W)$ and assume $H_n(i)[c] = 0$, *i.e.*, c is of the form $c = \partial b + a'$ with $b \in S_{n+1}(X)$ and $a' \in S_n(A)$ and write b as $b^U + b^V$ with $b^U \in S_{n+1}(U) \subset S_{n+1}(A)$ and $b^V \in S_{n+1}(V) \subset S_{n+1}(X \setminus W)$. Then

$$c = \partial b^U + \partial b^V + a'.$$

But ∂b^U and a' are elements in $S_n(A \setminus W)$ and hence $c = \partial b^V \in S_n(X \setminus W, A \setminus W)$. □

8. Mayer-Vietoris sequence

We consider the following situation: Assume that there are subspaces $X_1, X_2 \subset X$ such that X_1 and X_2 are open in X and such that $X = X_1 \cup X_2$. We consider the open covering $\mathfrak{U} = \{X_1, X_2\}$. We need the following maps:

$$\begin{tikzcd} & X_1 \arrow[dr, "\kappa_1"] & \\ X_1 \cap X_2 \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & X. \\ & X_2 \arrow[ur, "\kappa_2"'] & \end{tikzcd}$$

Note that by definition, the sequence

(8.1) $$0 \longrightarrow S_*(X_1 \cap X_2) \xrightarrow{(i_1, i_2)} S_*(X_1) \oplus S_*(X_2) \longrightarrow S_*^{\mathfrak{U}}(X) \longrightarrow 0$$

is exact. Here, the second map is

$$(\alpha_1, \alpha_2) \mapsto \kappa_1(\alpha_1) - \kappa_2(\alpha_2).$$

Theorem 8.1. *(The Mayer-Vietoris sequence)*
There is a long exact sequence

$$\ldots \xrightarrow{\delta} H_n(X_1 \cap X_2) \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X) \xrightarrow{\delta} H_{n-1}(X_1 \cap X_2) \longrightarrow \ldots$$

Walther Mayer: 1887–1948 `https://en.wikipedia.org/wiki/Walther_Mayer`
Leopold Vietoris: 1891–2002 (!) `https://en.wikipedia.org/wiki/Leopold_Vietoris`

Proof. The proof follows from Lemma 7.12, because $H_n^{\mathfrak{U}}(X) \cong H_n(X)$. $\square$

As an application, we calculate the homology groups of spheres. Let $X = \mathbb{S}^m$ and let $X^{\pm} := \mathbb{S}^m \setminus \{\mp e_{m+1}\}$. The subspaces X^+ and X^- are contractible and therefore $H_*(X^{\pm}) = 0$ for all positive $*$.

The Mayer-Vietoris sequence is as follows

$$\ldots \xrightarrow{\delta} H_n(X^+ \cap X^-) \longrightarrow H_n(X^+) \oplus H_n(X^-) \longrightarrow H_n(\mathbb{S}^m) \xrightarrow{\delta} H_{n-1}(X^+ \cap X^-) \longrightarrow \ldots$$

For $n > 1$ we can deduce

$$H_n(\mathbb{S}^m) \cong H_{n-1}(X^+ \cap X^-) \cong H_{n-1}(\mathbb{S}^{m-1}).$$

The first map is the connecting homomorphism and the second map is $H_{n-1}(i)\colon H_{n-1}(\mathbb{S}^{m-1}) \to H_{n-1}(X^+ \cap X^-)$ where i is the inclusion of $\mathbb{S}^{m-1}$ into $X^+ \cap X^-$ and this inclusion is a homotopy equivalence. Thus define $D := H_{n-1}(i)^{-1} \circ \delta$. This D is an isomorphism for all $n \geqslant 2$.

We have to controll what is going on in small degrees and dimensions.

We know from the Hurewicz isomorphism that $H_1(\mathbb{S}^m)$ is trivial for $m > 1$. If we want to see that via the Mayer-Vietoris sequence, we have to understand the map

$$\mathbb{Z} \cong H_0(X^+ \cap X^-) \to H_0(X^+) \oplus H_0(X^-) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Let 1 be a base point of $X^+ \cap X^-$. Then the map on H_0 is

$$[1] \mapsto ([1], [1]).$$

This map is injective and therefore the connecting homomorphism $\delta\colon H_1(\mathbb{S}^m) \to H_0(X^+ \cap X^-)$ is trivial and we obtain that

$$H_1(\mathbb{S}^m) \cong 0, \quad m > 1.$$

Next, we consider the case of $n = 1 = m$. In this case the intersection $X^+ \cap X^-$ splits into two components. We choose a $P_+ \in X^+$ and a $P_- \in X^-$. Then, for $H_0(i_1, i_2)$ we have

$$H_0(X^+) \oplus H_0(X^-) \ni (H_0(i_1)[P_+], H_0(i_2)[P_-]) \sim ([e_2], [-e_2]).$$

Thus $[P_+] \mapsto ([e_2], 0)$ and $[P_-] \mapsto (0, [-e_2])$ and the difference $[P_+] - [P_-]$ generates the kernel of $H_0(\kappa_1) - H_0(\kappa_2)$:

$$(H_0(\kappa_1) - H_0(\kappa_2))([e_2], [-e_2]) = 0.$$

Consider the exact sequence

$$0 \longrightarrow H_1\mathbb{S}^1 \xrightarrow{\delta} H_0(X^+ \cap X^-) \xrightarrow{(H_0(i_1), H_0(i_2))} H_0(X^+) \oplus H_0(X^-) \longrightarrow H_0\mathbb{S}^1$$

which gives

$$0 \longrightarrow H_1\mathbb{S}^1 \xrightarrow{\delta} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}.$$

where $[P_+] - [P_-] \mapsto ([e_2], [e_2]) \mapsto 0$. The image of $(H_0(i_1), H_0(i_2))$ is isomorphic to the kernel of the difference of $H_0(\kappa_1)$ and $H_0(\kappa_2)$ and this is isomorphic to the free abelian group generated by $([e_2], [e_2])$ which is $\mathbb{Z}$. Therefore

$$0 \longrightarrow H_1\mathbb{S}^1 \xrightarrow{\delta} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

is short exact and $H_1\mathbb{S}^1 \cong \mathbb{Z}$. (We already knew this from the Hurewicz isomorphism.)

For $0 < n < m$ we get

$$H_n\mathbb{S}^m \xrightarrow{\cong} H_{n-1}\mathbb{S}^{m-1} \xrightarrow{\cong} \ldots \xrightarrow{\cong} H_1(\mathbb{S}^{m-n+1}) \cong \pi_1(\mathbb{S}^{m-n+1}).$$

and the latter is trivial.

Similarly, for $0 < m < n$ we have

$$H_n\mathbb{S}^m \xrightarrow{\cong} H_{n-1}\mathbb{S}^{m-1} \xrightarrow{\cong} \ldots \xrightarrow{\cong} H_{n-m+1}(\mathbb{S}^1) \cong 0.$$

The last claim follows directly by another simple Mayer-Vietoris argument.

The remaining case $0 < m = n$ gives something non-trivial

$$H_n\mathbb{S}^n \xrightarrow{\cong} H_{n-1}\mathbb{S}^{n-1} \xrightarrow{\cong} \ldots \xrightarrow{\cong} H_1(\mathbb{S}^1) \cong \mathbb{Z}.$$

We can summarize the result as follows.

Proposition 8.2.

$$H_n(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = m = 0, \\ \mathbb{Z}, & n = 0, m > 0, \\ \mathbb{Z}, & n = m > 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Definition 8.3. Let $\mu_0 := [P_+] - [P_-] \in H_0(X^+ \cap X^-) \cong H_0(\mathbb{S}^0)$ and let $\mu_1 \in H_1(\mathbb{S}^1) \cong \pi_1(\mathbb{S}^1)$ be given by the degree one map (aka the class of the identity on $\mathbb{S}^1$, aka the class of the loop $t \mapsto e^{2\pi i t}$).

Define the higher μ_ns via $D\mu_n = \mu_{n-1}$. Then μ_n is called *the fundamental class in $H_n(\mathbb{S}^n)$*.

In order to obtain a relative version of the Mayer-Vietoris sequence, we need a tool from homological algebra.

Lemma 8.4. *(The five-lemma)*
Let

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\ B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5 \end{tikzcd}$$

be a commutative diagram of exact sequences. If f_1, f_2, f_4, f_5 are isomorphisms, then so is f_3.

Proof. Again, we are chasing diagrams.

In order to prove that f_3 is injective, assume that there is an $a \in A_3$ with $f_3 a = 0$. Then $\beta_3 f_3 a = f_4 \alpha_3 a = 0$, as well. But f_4 is injective, thus $\alpha_3 a = 0$. Exactness of the top row gives, that there is an $a' \in A_2$ with $\alpha_2 a' = a$. This implies

$$f_3 \alpha_2 a' = f_3 a = 0 = \beta_2 f_2 a'.$$

Exactness of the bottom row gives us a $b \in B_1$ with $\beta_1 b = f_2 a'$, but f_1 is an isomorphism so we can lift b to $a_1 \in A_1$ with $f_1 a_1 = b$.

Thus $f_2 \alpha_1 a_1 = \beta_1 b = f_2 a'$ and as f_2 is injective, this implies that $\alpha_1 a_1 = a'$. So finally we get that $a = \alpha_2 a' = \alpha_2 \alpha_1 a_1$, but the latter is zero, thus $a = 0$.

For the surjectivity of f_3 assume $b \in B_3$ is given. Move b over to B_4 via β_3 and set $a := f_4^{-1} \beta_3 b$.

Consider $f_5 \alpha_4 a$. This is equal to $\beta_4 \beta_3 b$ and hence trivial. Therefore $\alpha_4 a = 0$ and thus there is an $a' \in A_3$ with $\alpha_3 a' = a$. Then $b - f_3 a'$ is in the kernel of β_3 because

$$\beta_3(b - f_3 a') = \beta_3 b - f_4 \alpha_3 a' = \beta_3 b - f_4 a = 0.$$

Hence we get a $b_2 \in B_2$ with $\beta_2 b_2 = b - f_3 a'$. Define a_2 as $f_2^{-1}(b_2)$, so $a' + \alpha_2 a_2$ is in A_3 and

$$f_3(a' + \alpha_2 a_2) = f_3 a' + \beta_2 f_2 a_2 = f_3 a' + \beta_2 b_2 = f_3 a' + b - f_3 a' = b.$$

□

We now consider a relative situation, so let X be a topological space with $A, B \subset X$ open in $A \cup B$ and set $\mathfrak{U} := \{A, B\}$. This is an open covering of $A \cup B$. The following diagram of exact sequences combines absolute chains with relative ones:

$$\begin{tikzcd}
 & & & & & 0 \arrow[d] \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & S_n(A\cup B) \arrow[d] \\
0 \arrow[r] & S_n(A\cap B) \arrow[r] \arrow[d] & S_n(A)\oplus S_n(B) \arrow[r] \arrow[d] & S_n^{\mathfrak{U}}(A\cup B) \arrow[r] \arrow[d] \arrow[urr, "\varphi"] & 0 \arrow[d] & S_n(X) \arrow[d] \\
0 \arrow[r] & S_n(X) \arrow[r, "\Delta"] \arrow[d] & S_n(X)\oplus S_n(X) \arrow[r, "\mathrm{diff}"] \arrow[d] & S_n(X) \arrow[r] \arrow[d] \arrow[urr, equal] & 0 \arrow[d] & S_n(X, A\cup B) \arrow[d] \\
0 \arrow[r] & S_n(X, A\cap B) \arrow[r] \arrow[d] & S_n(X,A)\oplus S_n(X,B) \arrow[r] \arrow[d] & S_n(X)/S_n^{\mathfrak{U}}(A\cup B) \arrow[r] \arrow[d] \arrow[urr, "\psi"] & 0 & 0 \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

Here, ψ is induced by the inclusion $\varphi\colon S_n^{\mathfrak{U}}(A\cup B) \to S_n(A\cup B)$, Δ denotes the diagonal map and diff the difference map. It is clear that the first two rows are exact. That the third row is exact follows by the nine-lemma or a direct diagram chase.

Consider the two right-most non-trivial columns in this diagram. Each gives a long exact sequence in homology and we focus on five terms.

$$\begin{tikzcd}
H_n(S_*^{\mathfrak{U}}(A\cup B)) \arrow[r] \arrow[d, "H_n(\varphi)"'] & H_n(X) \arrow[r] \arrow[d, equal] & H_n(S_*(X)/S_*^{\mathfrak{U}}(A\cup B)) \arrow[r, "\delta"] \arrow[d, "H_n(\psi)"'] & H_{n-1}(S_*^{\mathfrak{U}}(A\cup B)) \arrow[r] \arrow[d, "H_{n-1}(\varphi)"'] & H_{n-1}(X) \arrow[d, equal] \\
H_n(A\cup B) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A\cup B) \arrow[r, "\delta"] & H_{n-1}(A\cup B) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

Then by the five-lemma, as $H_n(\varphi)$ and $H_{n-1}(\varphi)$ are isomorphisms, so is $H_n(\psi)$. This observation together with the bottom non-trivial exact row proves the following.

Theorem 8.5. *(Relative Mayer-Vietoris sequence)*
If $A, B \subset X$ are open in $A\cup B$, then the following sequence is exact:

$$\ldots \xrightarrow{\delta} H_n(X, A\cap B) \longrightarrow H_n(X,A)\oplus H_n(X,B) \longrightarrow H_n(X, A\cup B) \xrightarrow{\delta} \quad \ldots$$

9. Reduced homology and suspension

For any path-connected space we have that the zeroth homology group is isomorphic to the integers, so somehow this copy of $\mathbb{Z}$ is superfluous information and we want to get rid of it in a civilized manner. Let P denote the one-point topological space. Then for any space X there is a continuous map $\varepsilon\colon X \to P$.

Definition 9.1. We define $\widetilde{H}_n(X) := \ker(H_n(\varepsilon)\colon H_n(X) \to H_n(P))$ and call it the *reduced nth homology group of the space X*.

- Note that $\widetilde{H}_n(X) \cong H_n(X)$ for all positive n.
- If X is path-connected, then $\widetilde{H}_0(X) = 0$.
- For any choice of a base point $x \in X$ we get

$$\widetilde{H}_n(X) \oplus H_n(\{x\}) \cong H_n(X)$$

  because $H_n(P) \cong H_n(\{x\})$ and the composition

$$\{x\} \hookrightarrow X \to \{x\}$$

is the identity. Therefore, $\widetilde{H}_n(X) \cong H_n(X, \{x\})$ because the retraction $r\colon X \to \{x\}$ splits the exact sequence

$$\dots H_n(\{x\}) \to H_n(X) \to H_n(X, \{x\}) \to \dots$$

- We can prolong the singular chain complex $S_*(X)$ and consider $\widetilde{S}_*(X)$:

$$\dots \longrightarrow S_1(X) \longrightarrow S_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

where $\varepsilon(\alpha) = 1$ for every singular 0-simplex α. This is precisely the augmentation we considered before. Then for all $n \geqslant 0$,

$$\widetilde{H}_*(X) \cong H_*(\widetilde{S}_*(X)).$$

As every continuous map $f\colon X \to Y$ induces a chain map $S_*(f)\colon S_*(X) \to S_*(Y)$ and as $\varepsilon^Y \circ S_0(f) = \varepsilon^X$ we obtain the following result.

Lemma 9.2. *The assignment $X \mapsto H_*(\widetilde{S}_*(X))$ is a functor,* i.e., *for a continuous $f\colon X \to Y$ we get an induced map $H_*(\widetilde{S}_*(f))\colon H_*(\widetilde{S}_*(X)) \to H_*(\widetilde{S}_*(Y))$ such that the identity on X induces the identity and composition of maps is respected.*

Similarly, $\widetilde{H}_*(-)$ is a functor.

Definition 9.3. For $\varnothing \neq A \subset X$ we define

$$\widetilde{H}_n(X, A) := H_n(X, A).$$

As we identified reduced homology groups with relative homology groups we obtain a reduced version of the Mayer-Vietoris sequence. A similar remark applies to the long exact sequence for a pair of spaces.

Proposition 9.4. *For each pair of spaces, there is a long exact sequence*

$$\dots \longrightarrow \widetilde{H}_n(A) \longrightarrow \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(X, A) \longrightarrow \widetilde{H}_{n-1}(A) \longrightarrow \dots$$

and a reduced Mayer-Vietoris sequence.

Examples.

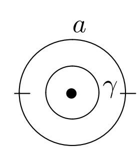
1) Recall that we can express $\mathbb{R}P^2$ as the quotient space of $\mathbb{S}^2$ modulo antipodal points or as a quotient of $\mathbb{D}^2$:

$$\mathbb{R}P^2 \cong \mathbb{S}^2/\pm\mathrm{id} \cong \mathbb{D}^2/z \sim -z \text{ for } z \in \mathbb{S}^1.$$

We use the latter definition and set $X = \mathbb{R}P^2$, $X_1 = X \setminus \{[0,0]\}$ (which is an open Möbius strip and hence homotopically equivalent to $\mathbb{S}^1$) and $X_2 = \mathring{\mathbb{D}}^2$. Then

$$X_1 \cap X_2 = \mathring{\mathbb{D}}^2 \setminus \{[0,0]\} \simeq \mathbb{S}^1.$$

Thus we know that $H_1(X_1) \cong \mathbb{Z}$, $H_1(X_2) \cong 0$ and $H_2X_1 = H_2X_2 = 0$. We choose generators for $H_1(X_1)$ and $H_1(X_1 \cap X_2)$ as follows.



Let a be the path that runs along the outer circle in mathematical positive direction half around starting from the point $(1,0)$. Let γ be the loop that runs along the inner circle in mathematical positive direction. Then the inclusion $i\colon X_1 \cap X_2 \to X_1$ induces

$$H_1(i)[\gamma] = 2[a].$$

This suffices to compute $H_*(\mathbb{R}P^2)$ up to degree two because the long exact sequence is

$$\widetilde{H}_2(X_1) \oplus \widetilde{H}_2(X_2) = 0 \to \widetilde{H}_2(X) \to \widetilde{H}_1(X_1 \cap X_2) \cong \mathbb{Z} \to \widetilde{H}_1(X_1) \cong \mathbb{Z} \to \widetilde{H}_1(X) \to \widetilde{H}_0(X_1 \cap X_2) = 0.$$

On the two copies of the integers, the map is given as above and thus we obtain:

$$\begin{aligned}
&H_2(\mathbb{R}P^2) \cong \ker(2\cdot\colon \mathbb{Z} \to \mathbb{Z}) = 0,\\
&H_1(\mathbb{R}P^2) \cong \operatorname{coker}(2\cdot\colon \mathbb{Z} \to \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},\\
&H_0(\mathbb{R}P^2) \cong \mathbb{Z}.
\end{aligned}$$

The higher homology groups are trivial, because there $H_n(\mathbb{R}P^2)$ is located in a long exact sequence between trivial groups.

2) We can now calculate the homology groups of bouquets of spaces in terms of the homology groups of the single spaces, at least in good cases. Let $(X_i)_{i\in I}$ be a family of topological spaces with chosen basepoints $x_i \in X_i$. Consider

$$X = \bigvee_{i\in I} X_i.$$

If the inclusion of x_i into X_i is pathological, then we cannot apply the Mayer-Vietoris sequence. However, we get the following:

Proposition 9.5. *If there are open neighbourhoods U_i of $x_i \in X_i$ together with a deformation of U_i to $\{x_i\}$, then we have for any finite $E \subset I$*

$$\widetilde{H}_n(\bigvee_{i\in E} X_i) \cong \bigoplus_{i\in E} \widetilde{H}_n(X_i).$$

In the situation above we say that the X_i are *well-pointed* with respect to x_i.

Proof. First we consider the case of two bouquet summands. We have $X_1 \vee U_2 \cup U_1 \vee X_2$ as an open covering of $X_1 \vee X_2$. The Mayer-Vietoris sequence then gives that $H_n(X) \cong H_n(X_1 \vee U_2) \oplus H_n(U_1 \vee X_2)$ for $n > 0$. For H_0 we get the exact sequence

$$0 \to \widetilde{H}_0(X_1 \vee U_2) \oplus \widetilde{H}_0(U_1 \vee X_2) \to \widetilde{H}_0(X) \to 0.$$

By induction we obtain the case of finitely many bouquet summands. □

We also get

$$\widetilde{H}_n(\bigvee_{i\in I} X_i) \cong \bigoplus_{i\in I} \widetilde{H}_n(X_i)$$

but for this one needs a colimit argument. We postpone that for a while.

We can extend such results to the full relative case. Let $A \subset X$ be a closed subspace and assume that A is a deformation retract of an open neighbourhood $A \subset U$. Let $\pi\colon X \to X/A$ be the canonical projection and $b = \{A\}$ the image of A. Then X/A is well-pointed with respect to b.

Proposition 9.6. *In the situation above*

$$H_n(X, A) \cong \widetilde{H}_n(X/A), \quad 0 \leqslant n.$$

Proof. The canonical projection, π, induces a homeomorphism $(X \setminus A, U \setminus A) \cong (X/A \setminus \{b\}, \pi(U) \setminus \{b\})$. Consider the following diagram:

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\cong"] \arrow[d, "H_n(\pi)"'] & H_n(X,U) & H_n(X \setminus A, U \setminus A) \arrow[l, "\cong"'] \arrow[d, "\cong"', "H_n(\pi)"] \\
H_n(X/A, b) \arrow[r, "\cong"] & H_n(X/A, \pi(U)) & H_n(X/A \setminus \{b\}, \pi(U) \setminus \{b\}) \arrow[l, "\cong"']
\end{tikzcd}$$

The upper and lower left arrows are isomorphisms because A is a deformation retract of U, the isomorphism in the upper right is a consequence of excision, because $A = \bar{A} \subset U$ and the lower right one follows from excision as well. □

Theorem 9.7. *(Suspension isomorphism) If $A \subset X$ is as above, then*

$$H_n(\Sigma X, \Sigma A) \cong \widetilde{H}_{n-1}(X, A), \quad \textit{for all } n > 0.$$

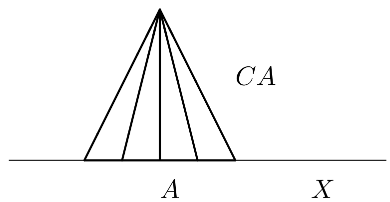
Proof. Consider the inclusion of pairs $(X, A) \subset (CX, CA) \subset (\Sigma X, \Sigma A)$ and the triple $(CX, X \cup CA, CA)$. We obtain the corresponding long exact sequence on homology groups

$$\ldots \longrightarrow H_n(CX, CA) \longrightarrow H_n(CX, CA \cup X) \xrightarrow{\delta} \tilde{H}_{n-1}(X \cup CA, CA) \longrightarrow \ldots$$

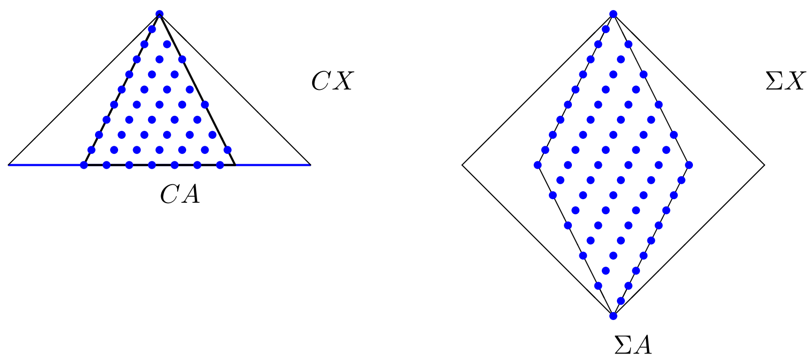
By Proposition 9.6 we get that $\tilde{H}_n(CX, CA \cup X) \cong \tilde{H}_n(CX/CA \cup X)$ and $\tilde{H}_{n-1}(X \cup CA, CA) \cong \tilde{H}_{n-1}(X \cup CA/CA)$ and the latter is isomorphic to $\tilde{H}_{n-1}(X/A) \cong \tilde{H}_{n-1}(X, A)$. Similarly, as $CX/CA \cup X \simeq \Sigma X/\Sigma A$, we get

$$\tilde{H}_n(CX, CA \cup X) \cong \tilde{H}_n(CX/CA \cup X) \cong \tilde{H}_n(\Sigma X/\Sigma A) \cong H_n(\Sigma X, \Sigma A).$$

$X \cup CA/CA \cong X/A$:



$CX/CA \cup X \cong \Sigma X/\Sigma A$:



□

Note, that the corresponding statement is terribly wrong for homotopy groups. We have $\Sigma \mathbb{S}^2 \cong \mathbb{S}^3$, but $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, whereas $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$, so homotopy groups (unlike homology groups) don't satisfy such an easy form of a suspension isomorphism. There is a Freundenthal suspension theorem for homotopy groups, but that's more complicated (`https://en.wikipedia.org/wiki/Freudenthal_suspension_theorem`). For the above case it yields:

$$\mathbb{Z}/2\mathbb{Z} \cong \pi_{1+3}(\mathbb{S}^3) \cong \pi_{1+4}(\mathbb{S}^4) \cong \ldots =: \pi_1^s$$

where π_1^s denotes the first stable homotopy group.

Freudenthal: 1905–1990 `https://en.wikipedia.org/wiki/Hans_Freudenthal`

10. Mapping degree

Recall that we defined fundamental classes $\mu_n \in \tilde{H}_n(\mathbb{S}^n)$ for all $n \geqslant 0$. Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be any continuous map.

Definition 10.1. The map f induces a homomorphism

$$\tilde{H}_n(f) : \tilde{H}_n(\mathbb{S}^n) \to \tilde{H}_n(\mathbb{S}^n)$$

and therefore we get

$$\tilde{H}_n(f)\mu_n = \deg(f)\mu_n$$

with $\deg(f) \in \mathbb{Z}$. We call this integer *the degree of* f.

In the case $n = 1$ we can relate this notion of a mapping degree to the one defined via the fundamental group of the 1-sphere: if we represent the generator of $\pi_1(\mathbb{S}^1, 1)$ as the class given by the loop

$$\omega\colon [0,1] \to \mathbb{S}^1, \quad t \mapsto e^{2\pi i t},$$

then the abelianized Hurewicz, $h_{\mathrm{ab}}\colon \pi_1(\mathbb{S}^1, 1) \to H_1(\mathbb{S}^1)$, sends the class of ω precisely to μ_1 and therefore the naturality of h_{ab}

$$\begin{tikzcd}
\pi_1(\mathbb{S}^1,1) \arrow[r, "\pi_1(f)"] \arrow[d, "h_{\mathrm{ab}}"'] & \pi_1(\mathbb{S}^1,1) \arrow[d, "h_{\mathrm{ab}}"] \\
H_1(\mathbb{S}^1) \arrow[r, "H_1(f)"] & H_1(\mathbb{S}^1)
\end{tikzcd}$$

shows that

$$\deg(f)\mu_1 = H_1(f)\mu_1 = h_{\mathrm{ab}}(\pi_1(f)[w]) = h_{\mathrm{ab}}(k[w]) = k\mu_1.$$

where k is the degree of f defined via the fundamental group. Thus both notions coincide for $n = 1$.

As we know that the connecting homomorphism induces an isomorphism between $H_n(\mathbb{D}^n, \mathbb{S}^{n-1})$ and $\tilde{H}_{n-1}(\mathbb{S}^{n-1})$, we can consider degrees of maps $f\colon (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (\mathbb{D}^n, \mathbb{S}^{n-1})$ by defining $\bar{\mu}_n := \delta^{-1}\mu_n$. Then $H_n(f)(\bar{\mu}_n) := \deg(f)\bar{\mu}_n$ gives a well-defined integer $\deg(f) \in \mathbb{Z}$.

The degree of self-maps of $\mathbb{S}^n$ satisfies the following properties:

PROPOSITION 10.2.
(a) *If f is homotopic to g, then* $\deg(f) = \deg(g)$.
(b) *The degree of the identity on $\mathbb{S}^n$ is one.*
(c) *The degree is multiplicative,* i.e., $\deg(g \circ f) = \deg(g)\deg(f)$.
(d) *If f is not surjective, then* $\deg(f) = 0$.

PROOF. The first three properties follow directly from the definition of the degree. If f is not surjective, then it is homotopic to a constant map and this has degree zero. □

It is true that the group of (pointed) homotopy classes of self-maps of $\mathbb{S}^n$ is isomorphic to $\mathbb{Z}$ and thus the first property can be upgraded to an 'if and only if', but we won't prove that here.

Recall that $\Sigma\mathbb{S}^n \cong \mathbb{S}^{n+1}$. If $f\colon \mathbb{S}^n \to \mathbb{S}^n$ is continuous, then $\Sigma(f)\colon \Sigma\mathbb{S}^n \to \Sigma\mathbb{S}^n$ is given as $\Sigma\mathbb{S}^n \ni [x,t] \mapsto [f(x), t]$.

LEMMA 10.3. *Suspensions leave the degree invariant,* i.e., *for $f\colon \mathbb{S}^n \to \mathbb{S}^n$ we have*

$$\deg(\Sigma(f)) = \deg(f).$$

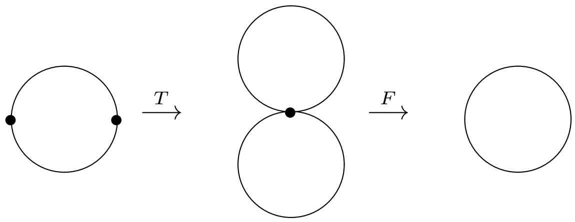
In particular, for every $k \in \mathbb{Z}$ there is an $f\colon \mathbb{S}^n \to \mathbb{S}^n$ with $\deg(f) = k$.

PROOF. The suspension isomorphism of Theorem 9.7 is induced by a connecting homomorphism. Using the isomorphism $H_{n+1}(\mathbb{S}^{n+1}) \cong H_{n+1}(\Sigma\mathbb{S}^n)$, the connecting homomorphism sends $\mu_{n+1} \in H_{n+1}(\mathbb{S}^{n+1})$ to $\pm\mu_n \in \tilde{H}_n(\mathbb{S}^n)$. But then the commutativity of

$$\begin{tikzcd}
H_{n+1}(\mathbb{S}^{n+1}) \arrow[r, "\cong"] & H_{n+1}(\Sigma\mathbb{S}^n) \arrow[r, "H_{n+1}(\Sigma f)"] \arrow[d, "\delta"'] & H_{n+1}(\Sigma\mathbb{S}^n) \arrow[d, "\delta"] & H_{n+1}(\mathbb{S}^{n+1}) \arrow[l, "\cong"'] \\
& \tilde{H}_n(\mathbb{S}^n) \arrow[r, "H_n(f)"] & \tilde{H}_n(\mathbb{S}^n) &
\end{tikzcd}$$

ensures that $\pm\deg(f)\mu_n = \pm\deg(\Sigma f)\mu_n$ with the same sign. □

For the degree of a self-map of $\mathbb{S}^1$ one has an additivity relation. We can generalize this to higher dimensions. Consider the *pinch map* $T\colon \mathbb{S}^n \to \mathbb{S}^n/\mathbb{S}^{n-1} \simeq \mathbb{S}^n \vee \mathbb{S}^n$ and the *fold map* $F\colon \mathbb{S}^n \vee \mathbb{S}^n \to \mathbb{S}^n$. Here, F is induced by the identity of $\mathbb{S}^n$.



Note that we can replace every continuous $f\colon \mathbb{S}^n \to \mathbb{S}^n$ by a basepoint-preserving map by composing with a rotation. That doesn't change the degree.

Proposition 10.4. *For $f, g\colon \mathbb{S}^n \to \mathbb{S}^n$ we have*

$$\deg(F \circ (f \vee g) \circ T) = \deg(f) + \deg(g).$$

Proof. The map $H_n(T)$ sends μ_n to $(\mu_n, \mu_n) \in \tilde{H}_n\mathbb{S}^n \oplus \tilde{H}_n\mathbb{S}^n \cong \tilde{H}_n(\mathbb{S}^n \vee \mathbb{S}^n)$. Under this isomorphism, the map $H_n(f \vee g)$ corresponds to $(\mu_n, \mu_n) \mapsto (\tilde{H}_n(f)\mu_n, \tilde{H}_n(g)\mu_n)$ and this yields $(\deg(f)\mu_n, \deg(g)\mu_n)$ which under the fold map is sent to the sum. □

We use the mapping degree to show some geometric properties of self-maps of spheres.

Proposition 10.5. *Let $f^{(n)}\colon \mathbb{S}^n \to \mathbb{S}^n$ be the map*

$$(x_0, x_1, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n).$$

Then $f^{(n)}$ has degree -1.

Proof. We prove the claim by induction. μ_0 was the difference class $[+1] - [-1]$, and

$$f^{(0)}([+1] - [-1]) = [-1] - [+1] = -\mu_0.$$

We defined μ_n in such a way that $D\mu_n = \mu_{n-1}$. Therefore, as D is natural,

$$H_n(f^{(n)})\mu_n = H_n(f^{(n)})D^{-1}\mu_{n-1} = D^{-1}H_{n-1}(f^{(n-1)})\mu_{n-1} = D^{-1}(-\mu_{n-1}) = -\mu_n.$$

□

Corollary 10.6. *The antipodal map $A\colon \mathbb{S}^n \to \mathbb{S}^n$, $A(x) = -x$, has degree $(-1)^{n+1}$.*

Proof. Let $f_i^{(n)}\colon \mathbb{S}^n \to \mathbb{S}^n$ be the map $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$. As in Proposition 10.5 one shows that the degree of $f_i^{(n)}$ is -1. As $A = f_n^{(n)} \circ \ldots \circ f_0^{(n)}$, the claim follows. □

In particular, the antipodal map cannot be homotopic to the identity as long as n is even!

Proposition 10.7. *Let $f, g\colon \mathbb{S}^n \to \mathbb{S}^n$ with $f(x) \neq g(x)$ for all $x \in \mathbb{S}^n$, then f is homotopic to $A \circ g$. In particular,*

$$\deg(f) = (-1)^{n+1}\deg(g).$$

Proof. By assumption the segment $t \mapsto (1-t)f(x) - tg(x)$ doesn't pass through the origin for $0 \leqslant t \leqslant 1$. Thus the homotopy

$$H(x,t) = \frac{(1-t)f(x) - tg(x)}{||(1-t)f(x) - tg(x)||}$$

connects f to $-g = A \circ g$. □

Corollary 10.8. *For any $f\colon \mathbb{S}^n \to \mathbb{S}^n$ with $\deg(f) = 0$ there is an $x_+ \in \mathbb{S}^n$ with $f(x_+) = x_+$ and an x_- with $f(x_-) = -x_-$.*

Proof. If $f(x) \neq x$ for all x, then $\deg(f) = \deg(A) \neq 0$. If $f(x) \neq -x$ for all x, then $\deg(f) = (-1)^{n+1}\deg(A) \neq 0$. □

Corollary 10.9. *Assume that n is even and let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be any continuous map. Then there is an $x \in \mathbb{S}^n$ with $f(x) = x$ or $f(x) = -x$.*

Finally, we can say the following about hairstyles of hedgehogs of arbitrary even dimension:

Proposition 10.10. *Any tangential vector field on $\mathbb{S}^{2k}$ is trivial in at least one point.*

Proof. Recall that we can describe the tangent space at a point $x \in \mathbb{S}^{2k}$ as

$$T_x(\mathbb{S}^{2k}) = \{y \in \mathbb{R}^{2k+1} | \langle x, y\rangle = 0\}.$$

Assume that $V : \mathbb{S}^{2k} \to T(\mathbb{S}^{2k})$ with $V(x) \in T_x(\mathbb{S}^{2k})$ for all x is a tangential vector field which does not vanish, *i.e.*, $V(x) \neq 0$ for all $x \in \mathbb{S}^{2k}$.

Define $f(x) := \frac{V(x)}{||V(x)||}$. If $f(x) = x$, then $V(x) = ||V(x)||x$. But this means that $V(x)$ points into the direction of x and thus it cannot be tangential. Similarly, $f(x) = -x$ yields the same contradiction. Thus such a V cannot exist. □

11. CW complexes

Definition 11.1. Let X be a topological space. Then X is called an *n-cell*, if X is homeomorphic to $\mathbb{R}^n$. The number n is then the *dimension of the cell*.

Examples. Every point is a zero cell and $\mathring{D}^n \cong \mathbb{R}^n \cong \mathbb{S}^n \setminus N$ are n-cells.
Note that an n-cell cannot be an m-cell for $n \neq m$, because $\mathbb{R}^n \not\cong \mathbb{R}^m$ for $n \neq m$.

Definition 11.2. A *cell decomposition of a space* X is a decomposition of X into subspaces, each of which is a cell of some dimension, *i.e.*,

$$X = \bigsqcup_{i \in I} X_i, \quad X_i \cong \mathbb{R}^{n_i}.$$

Here, this decomposition is meant as a set, not as a topological space.

Examples.

- A hollow 3-dimensional cube has a cell decomposition into 8 points, 12 open edges, and 6 open faces.
- The standard 3-simplex can be decomposed into 4 zero-cells, 6 1-cells, 4 2-cells, and a 3-cell.
- The n-dimensional sphere (for $n > 0$) has a cell decomposition into the north pole and its complement.

Definition 11.3. A topological Hausdorff space X together with a cell decomposition is called a *CW complex*, if it satisfies the following conditions:

(a) For every n-cell $\sigma \subset X$ there is a continuous map $\Phi_\sigma : \mathbb{D}^n \to X$ such that the restriction of Φ_σ to $\mathring{\mathbb{D}}^n$ is a homeomorphism
$$\Phi_\sigma|_{\mathring{\mathbb{D}}^n} : \mathring{\mathbb{D}}^n \cong \sigma$$
and Φ_σ maps $\mathbb{S}^{n-1}$ to the union of cells of dimension at most $n-1$.
(b) For every n-cell σ, the closure $\bar{\sigma} \subset X$ has a non-trivial intersection with only finitely many cells of X.
(c) A subset $A \subset X$ is closed if and only if $A \cap \bar{\sigma}$ is closed for all cells σ in X.

- The map Φ_σ as in (a) is called the *characteristic map of the cell* σ. Its restriction $\Phi_\sigma|_{\mathbb{S}^{n-1}}$ is called *attaching map*.
- Property (b) is the *closure finite* condition: the closure of every cell is contained in finitely many cells. That's the 'C' in CW.
- Property (c) tells us that X has the *weak topology*. That's the 'W'.
- If X is a CW complex with only finitely many cells, then we call X *finite*.

Definition 11.4. Assume that X is a CW complex.

- We set $X^n := \bigcup_{\sigma \subset X, \dim(\sigma) \leqslant n} \sigma$ and call it *the n-skeleton of* X.
- If we have $X = X^n$, but $X^{n-1} \subsetneq X$, then we say that X is n-dimensional, *i.e.*, $\dim(X) = n$.
- A subset $Y \subset X$ of a CW complex X is called a *subcomplex* (sub-CW complex), if it has a cell decomposition by cells in X and if for any cell $\sigma \subset Y$ we also have $\bar{\sigma} \subset Y$.

- For any subcomplex $Y \subset X$, (X,Y) is a *CW-pair*.

Note, that any subcomplex of a CW complex is again a CW complex: the characteristic maps Φ_σ for Y are the same as for X. We obtain that Y is closed in X because of the second requirement and this guarantees that Y has the weak topology. If $\bar{\sigma} \subset X$ and $\sigma \subset Y$, then $\bar{\sigma} \subset Y$. As Y is closed, this says that Y satisfies condition (b) of a CW complex.

Examples The unit interval $[0,1]$ has a CW structure with two zero cells and one 1-cell. But for instance the decomposition $\sigma_0^0 = \{0\}$, $\sigma_k^0 = \{\frac{1}{k}\}, k > 0$ and $\sigma_k^1 = (\frac{1}{k+1}, \frac{1}{k})$ does not give a CW structure on $[0,1]$. Consider the following $A \subset [0,1]$

$$A := \left\{ \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k+1}\right) | k \in \mathbb{N} \right\}.$$

Then $A \cap \bar{\sigma}_k^1$ is precisely the point $\frac{1}{2}(\frac{1}{k} + \frac{1}{k+1})$ and this is closed, but A isn't.

We want to understand the topology of CW complexes. Note that cells don't have to be open in X: if X is a CW complex and σ is an n-cell, then σ is open in the n-skeleton of X, X^n and X^n is closed in X.

Of course, as a set we have $X = \bigcup_{n \geqslant 0} X^n$. From the condition that A is closed in X if and only if the intersection of A with $\bar{\sigma}$ is closed for any cell σ we see that A is closed in X if and only if $A \cap X^n$ is closed for all $n \geqslant 0$. This is an instance of a *direct limit topology on X* and this is denoted by

$$X = \varinjlim X^n.$$

Such a direct limit has the following universal property: for any system of maps $(f_n \colon X^n \to Z)_{n \geqslant 0}$ such that $f_{n+1}|_{X^n} = f_n$ there is a uniquely determined continuous map $f \colon X \to Z$ such that $f|_{X^n} = f_n$.

Note that CW structures on a fixed topological space are not unique. For instance you can consider $\mathbb{S}^2$ with the CW structure coming from the cell decomposition $\mathbb{S}^2 = \mathbb{S}^2 \setminus N \sqcup N$. Then the zero skeleton of $\mathbb{S}^2$ only consists of the north pole N and this agrees with the 1-skeleton, but the 2-skeleton is equal to $\mathbb{S}^2$.

But of course there are many other CW structures. Take your favorite dice, *i.e.*, a tetrahedron, a cube, an octahedron, a dodecahedron, an icosahedron or something less regular like a rhombic dodecahedron. Imagine these dice are hollow and project them to $\mathbb{S}^2$. Then you get different CW structures on $\mathbb{S}^2$ that way.

Definition 11.5. Let X and Y be CW complexes. A continuous map $f \colon X \to Y$ is called *cellular*, if $f(X^n) \subset Y^n$ for all $n \geqslant 0$.

The category of CW complexes together with cellular maps is rather flexible. Most of the classical constructions don't lead out of it, but one has to be careful with respect to products:

Proposition 11.6. *If X and Y are CW complexes then $X \times Y$ is a CW complex if one of the factors is locally compact.*

Proof. As products of cells are cells, $X \times Y$ inherits a cell decomposition from its factors. We need to ensure that $X \times Y$ carries the weak topology. For this we prove a slightly more general auxiliary fact: if X, Y and Z are topological spaces satisfying the Hausdorff condition and if $\pi \colon X \to Y$ gives Y the quotient topology, and if Z is locally compact, then

$$\pi \times \mathrm{id} \colon X \times Z \to Y \times Z$$

gives $Y \times Z$ the quotient topology. For this we show that $Y \times Z$ has the universal property of a quotient space, so if $g \colon Y \times Z \to W$ is a map of sets and assume that the composition $g \circ (\pi \times \mathrm{id})$ is continuous. As Z is locally compact and as all spaces are Hausdorff, there is a homeomorphism

$$C(X \times Z, W) \cong C(X, C(Z, W))$$

of topological spaces. (Here for two spaces U, V, $C(U,V)$ is the set of all continuous maps from U to V and the topology of $C(U,V)$ is generated (under finite intersections and arbitrary unions) by the sets $V(K,O) := \{f \in C(U,V) | f(K) \subset O\}$ for compact $K \subset U$ and open $O \subset V$.)

Under this adjunction $g \circ (\pi \times \mathrm{id})$ corresponds to the composite

$$\tilde{g} \colon X \xrightarrow{\pi} Y \xrightarrow{\bar{g}} C(Z, W).$$

As $\tilde{g}$ is continuous and as Y carries the quotient topology we get that $\bar{g}$ is continuous and hence g is continuous, too.

With the help of this result we consider the characteristic maps of X and Y,

$$\Phi_\sigma \colon \mathring{\mathbb{D}}^n \to X, \sigma \text{ a cell in } X$$
$$\Psi_\tau \colon \mathring{\mathbb{D}}^m \to Y, \tau \text{ a cell in } Y.$$

Then we can use these maps to write $X \times Y$ as a target of a map

$$\Phi \times \Psi \colon (\bigsqcup_\sigma \mathring{\mathbb{D}}^n) \times (\bigsqcup_\tau \mathring{\mathbb{D}}^m) \to X \times Y.$$

Assume that Y is locally compact. We have to show that $X \times Y$ carries the quotient topology with respect to this map. We know that each $\mathring{\mathbb{D}}^n$ is locally compact, thus so is the disjoint union of open discs. The map $\mathrm{id}_{\bigsqcup \mathring{\mathbb{D}}^n} \times \Psi$ gives $(\bigsqcup \mathring{\mathbb{D}}^n) \times Y$ the quotient topology and by assumption Y is locally compact and therefore $\Phi \times \mathrm{id}_Y$ induces the quotient topology on $X \times Y$. □

Lemma 11.7. *If D is a subset of a CW complex X and D intersects each cell in at most one point, then D is discrete.*

Proof. Let S be an arbitrary subset of D. We show that S is closed. We know that $S \cap \bar{\sigma}$ is finite, because $\bar{\sigma}$ is covered by finitely many cells. Therefore $S \cap \bar{\sigma}$ is closed in $\bar{\sigma}$, because X is Hausdorff (and therefore T_1). But then the weak topology guarantees that S is closed. □

Corollary 11.8. *Let X be a CW complex.*

(a) *Every compact subset $K \subset X$ is contained in a finite union of cells.*
(b) *The space X is compact if and only if it is a finite CW complex.*
(c) *The space X is locally compact if and only if it is locally finite,* i.e., *every point has a neighborhood that is contained in finitely many cells.*

Proof. It is easy to see that (a) implies (b) and that (b) implies (c). Thus we only prove (a): consider the intersections $K \cap \sigma$ and choose a point p_σ in every non-empty intersection. Then $D := \{p_\sigma | \sigma \text{ a cell in } X\}$ is discrete. It is also compact and therefore finite. □

Corollary 11.9. *If $f \colon K \to X$ is a continuous map from a compact space K to a CW complex X, then the image of K under f is contained in a finite skeleton.*

For the proof just note that $f(K)$ is compact in X.

Proposition 11.10. *Let A be a subcomplex of a CW complex X. Then $X \times \{0\} \cup A \times [0,1]$ is a strong deformation retract of $X \times [0,1]$.*

Proof. For $r \colon \mathbb{D}^n \times [0,1] \to \mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times [0,1]$ we can choose the standard retraction of a cylinder onto its bottom and sides.

As $X^n \times [0,1]$ is built out of $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times [0,1]$ by gluing in copies of $\mathbb{D}^n \times [0,1]$ along $\mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times [0,1]$ we get that $X^n \times [0,1]$ is a deformation retract of $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times [0,1]$. We can parametrize the retracting homotopy in such a way that it takes place in the time interval $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$. Using the direct limit topology on X, we obtain a deformation of $X \times I$ to $X \times \{0\} \cup A \times [0,1]$. □

The property in Proposition 11.10 implies the so-called *homotopy extension property*, (HEP): If $g \colon X \to Y$ is a map and $H \colon A \times [0,1] \to Y$ is a homotopy such that $H|_{A \times \{0\}} = g$, then there is an extension of H to $X \times [0,1]$, compatible with g and H. This identifies $A \to X$ as a so-called *cofibration.*

In the following we just collect some facts about the topology of CW complexes, that I won't prove:

Lemma 11.11.

- *For any subcomplex $A \subset X$ there is an open neighborhood U of A in X together with a strong deformation retraction to A. In particular, for each skeleton X^n there is an open neighborhood U in X (and as well in X^{n+1}) of X^n such that X^n is a strong deformation retract of U.*
- *Every CW complex is paracompact, locally path-connected and locally contractible.*

- *Every CW complex is semi-locally* 1*-connected, hence possesses a universal covering space.*

LEMMA 11.12. *For any CW complex X we get for the skeleta:*

(a)
$$X^n \setminus X^{n-1} = \bigsqcup_{\sigma \text{ an } n\text{-cell}} \sigma \cong \bigsqcup_{\sigma \text{ an } n\text{-cell}} \mathring{\mathbb{D}}^n.$$

(b)
$$X^n / X^{n-1} \cong \bigvee_{\sigma \text{ an } n\text{-cell}} \mathbb{S}^n.$$

PROOF. The first claim follows directly from the definition of a CW complex. For the second claim note that the characteristic maps send the boundary $\partial\mathbb{D}^n$ to the $n-1$-skeleton and hence for every n-cell we get a copy of $\mathbb{S}^n$ in the quotient. □

Example Consider the hollow cube W^2. Then $W^2/W^1 \cong \bigvee_{i=1}^6 \mathbb{S}^2$.

12. Cellular homology

In the following, X will always be a CW complex.

LEMMA 12.1. *For all $q \neq n \geqslant 1$, $H_q(X^n, X^{n-1}) = 0$.*

PROOF. Using the identification of relative homology and reduced homology of the quotient gives
$$H_q(X^n, X^{n-1}) \cong \tilde{H}_q(X^n/X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \tilde{H}_q(\mathbb{S}^n).$$
□

LEMMA 12.2. *Consider the inclusion $i_n \colon X^n \to X$.*

(a) *The induced map $H_n(i_n) \colon H_n(X^n) \to H_n(X)$ is surjective.*
(b) *On the $(n+1)$-skeleton we get an isomorphism*
$$H_n(i_{n+1}) \colon H_n(X^{n+1}) \cong H_n(X).$$

PROOF. (a) We can factor i_n as

$$\begin{tikzcd}
X^n \arrow[rrrr, "i_n"] \arrow[d, "\alpha_1"] & & & & X \\
X^{n+1} \arrow[urrrr, "i_{n+1}"] \arrow[r, "\alpha_2"'] & X^{n+2} \arrow[urrr, "i_{n+2}"] \arrow[r, "\alpha_3"'] & X^{n+3} \arrow[urr, "i_{n+3}"] \arrow[r, "\alpha_4"'] & \ldots &
\end{tikzcd}$$

The map $H_n(\alpha_1) \colon H_n(X^n) \to H_n(X^{n+1})$ is surjective, because $H_n(X^{n+1}, X^n) = 0$. For $i > 1$ we have the following piece of the long exact sequence of the pair (X^{n+i}, X^{n+i-1})

$$0 \cong H_{n+1}(X^{n+i}, X^{n+i-1}) \longrightarrow H_n(X^{n+i-1}) \xrightarrow{H_n(\alpha_i)} H_n(X^{n+i}) \longrightarrow H_n(X^{n+i}, X^{n+i-1}) \cong 0.$$

Therefore $H_n(\alpha_i)$ is an isomorphism in this range. If X is finite-dimensional, this already proves the claim.

Every singular simplex in X has an image that is contained in one of the X^ms because the standard simplices are compact. If $a \in S_n(X)$ is a chain, $a = \sum_{i=1}^m \lambda_i \beta_i$ then we can find an M such that the images of all the β_i's are contained in X^M, say for $M = n+q$. Therefore every $[a] \in H_n(X)$ can be written as $i_M[b]$, but $\alpha_q \circ \ldots \circ \alpha_1$ is surjective, hence $[b]$ is of the form $\alpha_q \circ \ldots \circ \alpha_1[c]$ but then
$$[a] = i_M \circ \alpha_q \circ \ldots \circ \alpha_1[c] = i_n[c]$$
thus i_n is surjective.

(b) If $[a] = H_n(i_{n+1})[u] = 0$, then again we can write $[a]$ as $i_M \circ \alpha_q \circ \ldots \circ \alpha_2[u]$ where the α_i's are now isos and i_M indicates that it suffices to use the M-skeleton of X in order to define $[a]$, hence $[a]$ was trivial to start with. □

Corollary 12.3. *For CW complexes X, Y we have*

(a) *If the n-skeleta X^n and Y^n are homeomorphic, then $H_q(X) \cong H_q(Y)$, for all $q < n$.*
(b) *If X has no q-cells, then $H_q(X) \cong 0$.*
(c) *In particular, if q exceeds the dimension of X, then $H_q(X) \cong 0$.*

Proof. The first claim is a direct consequence of the lemma above.

By assumption in (b) $X^{q-1} = X^q$, therefore we have $H_q(X^{q-1}) \cong H_q(X^q)$ and the latter surjects onto $H_q(X)$. We show that $H_n(X^r) \cong 0$ for $n > r$. To that end we use the chain of isomorphisms

$$H_n(X^r) \cong H_n(X^{r-1}) \cong \ldots \cong H_n(X^0)$$

which holds because the adjacent relative groups $H_n(X^i, X^{i-1})$ are trivial for $i < n$. □

Again, X is a CW complex.

Definition 12.4. The *cellular chain complex* $C_*(X)$ consists of $C_n(X) := H_n(X^n, X^{n-1})$ with boundary operator

$$d\colon H_n(X^n, X^{n-1}) \xrightarrow{\delta} H_{n-1}(X^{n-1}) \xrightarrow{\varrho} H_{n-1}(X^{n-1}, X^{n-2})$$

where ϱ is the map induced by the projection map $S_{n-1}(X^{n-1}) \to S_{n-1}(X^{n-1}, X^{n-2})$.

Note that $C_n(X)$ is a free abelian group with

$$C_n(X) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \tilde{H}_n(\mathbb{S}^n) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \mathbb{Z}.$$

For $n < 0$, $C_n(X)$ is trivial. If X has only finitely many n-cells, then $C_n(X)$ is finitely generated. If X is a finite CW complex, then $C_*(X)$ is finitely generated as a chain complex, *i.e.*, $C_n(X)$ is only non-trivial in finitely many degrees n, and in these degrees, $C_n(X)$ is finitely generated. In this case, the boundary operator can be calculated using matrices over the integers.

Lemma 12.5. *The map d is a boundary operator.*

Proof. The composition d^2 is $\varrho \circ \delta \circ \varrho \circ \delta$, but $\delta \circ \varrho$ is a composition in an exact sequence. □

Theorem 12.6. *(Comparison of cellular and singular homology) For every CW complex X, there is an isomorphism $\Upsilon\colon H_*(C_*(X), d) \cong H_*(X)$.*

Proof. Consider the diagram

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, equal] \arrow[dd, "d"] & H_{n+1}(X^{n+1}, X^n) \arrow[d, "\lambda"] \arrow[dr, "\delta"] & & \\
& H_{n+1}(X, X^n) \arrow[r, "\delta'"] & H_n(X^n) \arrow[r, "H_n(i_n)"] \arrow[dl, "\varrho"] & H_n(X) \\
C_n(X) \arrow[r, equal] \arrow[dd, "d"] & H_n(X^n, X^{n-1}) \arrow[d, "\lambda"] \arrow[dr, "\delta"] & & \\
& H_n(X, X^{n-1}) \arrow[r, "\delta'"] & H_{n-1}(X^{n-1}) \arrow[r, "H_{n-1}(i_{n-1})"] \arrow[dl, "\varrho"] & H_{n-1}(X) \\
C_{n-1}(X) \arrow[r, equal] \arrow[dd, "d"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[d, "\lambda"] \arrow[dr, "\delta"] & & \\
& H_{n-1}(X, X^{n-1}) \arrow[r, "\delta'"] & H_{n-2}(X^{n-2}) \arrow[r, "H_{n-2}(i_{n-2})"] \arrow[dl, "\varrho"] & H_{n-2}(X) \\
\cdots & \cdots & &
\end{tikzcd}$$

- All occurring ϱ-maps are injective because $H_k(X^{k-1}) \cong 0$ for all k.
- For every $a \in H_n(X^n)$ $\varrho(a)$ is a cycle for d:

$$d\varrho(a) = \varrho\delta\varrho(a) = 0.$$

- Let $c \in C_n(X)$ be a d-cycle, thus $0 = dc = \varrho\delta c$ and as ϱ is injective we obtain $\delta c = 0$. Exactness yields that $c = \varrho(a)$ for an $a \in H_n(X^n)$. Hence,

$$H_n(X^n) \cong \ker(d\colon C_n(X) \to C_{n-1}(X)).$$

- We define $\Upsilon\colon \ker(d) \to H_n(X)$ as $\Upsilon(c) = H_n(i_n)(a)$ for $c = \varrho(a)$ and $H_n(i_n)\colon H_n(X^n) \to H_n(X)$.
- The map Υ is surjective because $H_n(i_n)$ is surjective.
- In the diagram, the triangles commute, *i.e.*, $\delta = \delta' \circ \lambda$.
- Consider the sequence

$$H_{n+1}(X^{n+1}) \twoheadrightarrow H_{n+1}(X) \longrightarrow H_{n+1}(X, X^{n+1}) \longrightarrow H_n(X^{n+1}) \xrightarrow{\cong} H_n(X)$$

which tells us that $H_{n+1}(X, X^{n+1}) = 0$ and this in turn implies that λ is surjective.
- Using this we obtain

$$\mathrm{im}(\delta) = \mathrm{im}(\delta') = \ker(H_n(i_n)).$$

As $d = \varrho \circ \delta$, the map ϱ induces an isomorphism between the image of d and the image of δ.
- Taking all facts into account we get that ϱ induces an isomorphism

$$\frac{\ker(d\colon C_n(X) \to C_{n-1}(X))}{\mathrm{im}(d\colon C_{n+1}(X) \to C_n(X))} \cong \frac{H_n(X^n)}{\ker(H_n(i_n))}$$

But the sequence

$$0 \longrightarrow \ker H_n(i_n) \longrightarrow H_n(X^n) \longrightarrow \mathrm{im}(H_n(i_n)) \longrightarrow 0$$

is exact and therefore

$$H_n(X^n)/\ker(H_n(i_n)) \cong \mathrm{im} H_n(i_n) \cong H_n(X).$$

$\square$

Examples Projective Spaces

Let K be $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ with $m := \dim_{\mathbb{R}}(K)$ and let $K^* = K \setminus \{0\}$. We let K^* act on K^{n+1} via

$$K^* \times K^{n+1} \setminus \{0\} \to K^{n+1} \setminus \{0\}, \quad (\lambda, v) \mapsto \lambda v.$$

We define $KP^n = (K^{n+1} \setminus \{0\})/K^*$ and we denote the equivalence class of $(x_0, \ldots, x_n)$ in KP^n by $[x_0 : \ldots : x_n]$.

We define

$$X_i := \{[x_0 : \ldots : x_n] | x_i \neq 0, x_{i+1} = \ldots = x_n = 0\}$$

and consider the map

$$\xi_i \colon X_i \to K^i, \quad \xi_i[x_0 : \ldots : x_n] = (\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}).$$

As ξ_i is a homeomorphism, we see that X_i is a cell of dimension $i \cdot m$. We can write KP^n as $X_0 \sqcup \ldots \sqcup X_n$ and we have characteristic maps $\Phi_i \colon \mathbb{D}^{mi} \to KP^n$ as

$$\Phi_i(y) = \Phi_i(y_0, \ldots, y_{i-1}) = [y_0 : \ldots : y_{i-1} : 1 - ||y|| : 0 : \ldots : 0]$$

with $X_i = \Phi_i(\mathring{\mathbb{D}}^{mi})$.

1) First we consider the case $K = \mathbb{C}$. Here, we have a cell in each even dimension $0, 2, 4, \ldots, 2n$ for $\mathbb{C}P^n$. Therefore the cellular chain complex is

$$C_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & k = 2i, 0 \leqslant i \leqslant n, \\ 0 & k = 2i - 1 \text{ or } k > 2n. \end{cases}$$

The boundary operator is zero in each degree and thus

$$H_*(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & * = 2i, 0 \leqslant * \leqslant 2n, \\ 0, & \text{otherwise.} \end{cases}$$

2) The case of the quaternions is similar. Here the cells are spread in degrees congruent to zero modulo four, thus

$$H_*(\mathbb{H}P^n) = \begin{cases} \mathbb{Z}, & * = 4i, 0 \leqslant * \leqslant 4n, \\ 0, & \text{otherwise.} \end{cases}$$

3) Non-trivial boundary operators occur in the case of the real numbers. Here, we have a cell in each dimension up to n and thus the homology of $\mathbb{R}P^n$ is the homology of the chain complex

$$0 \longrightarrow C_n \cong \mathbb{Z} \xrightarrow{d} C_{n-1} \cong \mathbb{Z} \xrightarrow{d} \ldots \xrightarrow{d} C_0 \cong \mathbb{Z}.$$

For the computation of d we consider the diagram

$$\begin{tikzcd}
H_i(\mathbb{R}P^i, \mathbb{R}P^{i-1}) \arrow[d, "\delta"] \arrow[dd, bend right, "d"'] \cong H_i(\mathbb{R}P^i/\mathbb{R}P^{i-1}) \cong H_i(\mathbb{S}^i) \cong \tilde{H}_{i-1}(\mathbb{S}^{i-1}) \\
H_{i-1}(\mathbb{R}P^{i-1}) \arrow[d, "\varrho"] \\
H_{i-1}(\mathbb{R}P^{i-1}, \mathbb{R}P^{i-2}) \cong \tilde{H}_{i-1}(\mathbb{R}P^{i-1}/\mathbb{R}P^{i-2}) \cong \tilde{H}_{i-1}(\mathbb{S}^{i-1})
\end{tikzcd}$$

Let $\varphi_i = \Phi_i|_{\mathbb{S}^{i-1}} \colon \mathbb{S}^{i-1} \to \mathbb{S}^{i-1}/\pm \mathrm{id}$. The preimage of a class $[x] \in \mathbb{S}^{i-1}/\pm \mathrm{id}$ is $\{\pm x\}$. We consider the composition $\bar{\varphi}_i$

$$\begin{tikzcd}
\mathbb{S}^{i-1} \arrow[r, "\varphi_i"] \arrow[dr, "\bar{\varphi}_i"'] & \mathbb{S}^{i-1}/\pm \mathrm{id} = \mathbb{R}P^{i-1} \arrow[d, "\pi"] \\
& \mathbb{R}P^{i-1}/\mathbb{R}P^{i-2} \cong \mathbb{S}^{i-1}
\end{tikzcd}$$

and have to determine its degree.

By construction $\bar{\varphi}_i \circ A = \bar{\varphi}_i$ and thus

$$\deg(\bar{\varphi}_i) = \deg(\bar{\varphi}_i \circ A) = (-1)^i \deg(\bar{\varphi}_i)$$

and hence the degree of $\bar{\varphi}_i$ is trivial for odd i. The complement $\mathbb{S}^{i-1} \setminus \mathbb{S}^{i-2}$ has two components X_+, X_- and A exchanges these two components. The map $\bar{\varphi}_i$ sends X_+ and X_- to $[X_+]$. Therefore the degree of $\bar{\varphi}_i$ is

$$\deg(\bar{\varphi}_i) = \deg(F \circ (\mathrm{id} \vee A) \circ T) = \deg(\mathrm{id}) + \deg(A) = 1 + (-1)^i.$$

and d is either zero or two. Thus, depending on n we get

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & k \leqslant n, k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

for n even.

For odd dimensions n we get

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ \mathbb{Z}/2\mathbb{Z} & 0 < k < n, k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathbb{R}P^1 \cong \mathbb{S}^1$ and $\mathbb{R}P^3 \cong SO(3)$.

13. Homology with coefficients

Let G be an arbitrary abelian group.

Definition 13.1. The *singular chain complex of a topological space X with coefficients in G*, $S_*(X;G)$, has as elements in $S_n(X;G)$ finite sums of the form $\sum_{i=1}^N g_i\alpha_i$ with g_i in G and $\alpha_i \colon \Delta^n \to X$. Addition in $S_n(X;G)$ is given by

$$\sum_{i=1}^N g_i\alpha_i + \sum_{i=1}^N h_i\alpha_i = \sum_{i=1}^N (g_i + h_i)\alpha_i.$$

The *nth (singular) homology group of X with coefficients in G* is

$$H_n(X;G) := H_n(S_*(X;G))$$

where the boundary operator $\partial \colon S_n(X;G) \to S_{n-1}(X;G)$ is given by

$$\partial(\sum_{i=1}^N g_i\alpha_i) = \sum_{j=0}^n (-1)^j (\sum_{i=1}^N g_i(\alpha_i \circ d_j)).$$

We use a similar definition for *cellular homology of a CW complex X with coefficients in G*. Recall, that $C_n(X) = H_n(X^n, X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \mathbb{Z}$.

Definition 13.2. We denote a $c \in C_n(X;G)$ as $c = \sum_{i=1}^N g_i\sigma_i \in \bigoplus_{\sigma \text{ an } n\text{-cell}} G$ and let the boundary operator $\tilde{d}$ be defined by $\tilde{d}c = \sum_{i=1}^N g_i d(\sigma_i)$ where $d \colon C_n(X) \to C_{n-1}(X)$ is the boundary in the cellular chain complex of X.

We can transfer Theorem 12.6 to the case of homology with coefficients:

$$H_n(X;G) \cong H_n(C_*(X;G), \tilde{d})$$

for every CW complex X and therefore we denote the latter by $H_n(X;G)$ as well.

Note, that $H_n(X;\mathbb{Z}) = H_n(X)$ for every space X.

Example If we consider the case $X = \mathbb{R}P^2$, then we see that coefficients really make a difference.

Recall that for $G = \mathbb{Z}$ we had that $H_0(\mathbb{R}P^2) \cong \mathbb{Z}$, $H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_2(\mathbb{R}P^2) = 0$. However, for $G = \mathbb{Z}/2\mathbb{Z}$ the outcome differs drastically. The cellular chain complex looks as follows:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2=0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and therefore $H_i(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for $0 \leqslant i \leqslant 2$.

If we consider $H_*(\mathbb{R}P^2; \mathbb{Q})$ we obtain the cellular complex

$$0 \longrightarrow \mathbb{Q} \xrightarrow{\ 2\ } \mathbb{Q} \xrightarrow{\ 0\ } \mathbb{Q} \longrightarrow 0$$

But here, multiplication by 2 is an isomorphism and we get $H_0(\mathbb{R}P^2; \mathbb{Q}) = \mathbb{Q}$, $H_1(\mathbb{R}P^2; \mathbb{Q}) = \mathbb{Q}/2\mathbb{Q} = 0$ and $H_2(\mathbb{R}P^2; \mathbb{Q}) = 0$.

14. Tensor products and universal coefficient theorem

The question we want to pursue in this section is, whether $H_*(X, G)$ is computable from $H_*(X)$ and G. The general answer is 'Yes', but we need some basics from algebra to see that.

Let A and B be abelian groups.

Definition 14.1. The *tensor product of A and B*, $A \otimes B$, is the quotient of the free abelian group generated by $A \times B$ by the subgroup generated by

(a) $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$,
(b) $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$

for $a_1, a_1, a \in A$ and $b_1, b_2, b \in B$.

We denote an equivalence class of (a, b) in $A \otimes B$ by $a \otimes b$.

Note, that relations (a) and (b) imply that $\lambda(a \otimes b) = (\lambda a) \otimes b = a \otimes (\lambda b)$ for any integer $\lambda \in \mathbb{Z}$ and $a \in A$, $b \in B$. Elements in $A \otimes B$ are finite sums of equivalence classes $\sum_{i=1}^n \lambda_i a_i \otimes b_i$.

- Of course, $A \otimes B$ is generated by $a \otimes b$ with $a \in A$, $b \in B$.
- The tensor product is symmetric up to isomorphism and the isomorphism $A \otimes B \cong B \otimes A$ is given by
$$\sum_{i=1}^n \lambda_i a_i \otimes b_i \mapsto \sum_{i=1}^n \lambda_i b_i \otimes a_i.$$
- It is associative up to isomorphism:
$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$
for all abelian groups A, B, C.
- For homomorphisms $f\colon A \to A'$ and $g\colon B \to B'$ we get an induced homomorphism
$$f \otimes g\colon A \otimes B \to A' \otimes B'$$
which is given by $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$ on generators.
- The tensor product has the following universal property. For abelian groups A, B, C, the bilinear maps from $A \times B$ to C are in bijection with the linear maps from $A \otimes B$ to C.
- We've already seen tensor products: Note that $S_n(X) \otimes G$ is isomorphic to $S_n(X, G)$ and $C_n(X) \otimes G \cong C_n(X, G)$.

We collect the following properties of tensor products:

(a) For every abelian group A, we have
$$A \otimes \mathbb{Z} \cong A \cong \mathbb{Z} \otimes A.$$
(b) For every abelian group A, we have
$$A \otimes \mathbb{Z}/n\mathbb{Z} \cong A/nA.$$
Here, note that $nA = \{na | a \in A\}$ makes sense in any abelian group. The isomorphism above is given by
$$a \otimes \bar{i} \mapsto \bar{ia}$$
where $\bar{i}$ denotes an equivalence class of $i \in \mathbb{Z}$ in $\mathbb{Z}/n\mathbb{Z}$ and $\bar{ia}$ the class of $ia \in A$ in A/nA.

(c) If $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is a short exact sequence, then in general,

$$0 \longrightarrow A \otimes D \xrightarrow{\alpha\otimes\mathrm{id}} B \otimes D \xrightarrow{\beta\otimes\mathrm{id}} C \otimes \mathrm{id} \longrightarrow 0$$

is not exact for D abelian. For example,

$$0 \to \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \to 0$$

is exact, but

$$0 \to \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \to 0$$

isn't, because $\mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z} \cong 0$.

LEMMA 14.2. *For every abelian group D, $(-) \otimes D$ is right exact,* i.e., *if* $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *is a short exact sequence, then*

$$A \otimes D \xrightarrow{\alpha\otimes\mathrm{id}} B \otimes D \xrightarrow{\beta\otimes\mathrm{id}} C \otimes D \longrightarrow 0$$

is exact. If the exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *is a split short exact sequence, then*

$$0 \longrightarrow A \otimes D \xrightarrow{\alpha\otimes\mathrm{id}} B \otimes D \xrightarrow{\beta\otimes\mathrm{id}} C \otimes D \longrightarrow 0$$

is exact.

PROOF. Exercise. □

A consequence of the failure of the functor $(-) \otimes D$ to be exact on the left hand side has as a consequence that $H_n(X, G) = H_n(S_*(X) \otimes G)$ is not always isomorphic to $H_n(X) \otimes G = H_n(S_*(X)) \otimes G$.

DEFINITION 14.3. Let A be an abelian group. A short exact sequence $0 \to R \longrightarrow F \longrightarrow A \to 0$ with F a free abelian group is called a *free resolution of A*.

Note that in the situation above R is also free abelian because it can be identified with a subgroup of F.

Example For every $n \geqslant 1$, the sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$ is a free resolution of $\mathbb{Z}/n\mathbb{Z}$.

PROPOSITION 14.4. *Every abelian group possesses a free resolution.*

The resolution that we will construct in the proof is called the *standard resolution of A*.

PROOF. Let F be the free abelian group generated by the elements of the underlying set of A. We denote by y_a the basis element in F corresponding to $a \in A$. Define a homomorphism

$$p\colon F \to A,\ p(\sum_{a\in A} \lambda_a y_a) = \sum_{a \in A} \lambda_a a.$$

Here, $\lambda_a \in \mathbb{Z}$ and this integer is non-trivial for only finitely many $a \in A$. By construction, p is an epimorphism. We set R to be the kernel of p and in that way obtain the desired free resolution of A. □

DEFINITION 14.5. For two abelian groups A and B and for $0 \longrightarrow R \xrightarrow{i} F \longrightarrow A \longrightarrow 0$ the standard resolution of A we define

$$\mathrm{Tor}(A, B) := \ker(i \otimes \mathrm{id}\colon R \otimes B \to F \otimes B).$$

In general, $i \otimes \mathrm{id}$ doesn't have to be injective, thus $\mathrm{Tor}(A, B)$ won't be trivial. We will show that we can calculate $\mathrm{Tor}(A, B)$ via an arbitrary free resolution of A. To that end we prove the following result.

PROPOSITION 14.6. *For every homomorphism $f\colon A \to B$ and for free resolutions* $0 \longrightarrow R \xrightarrow{i} F \longrightarrow A \longrightarrow 0$ *and* $0 \longrightarrow R' \xrightarrow{i'} F' \longrightarrow B \longrightarrow 0$ *we have:*

(a) *There are homomorphisms $g\colon F \to F'$ and $h\colon R \to R'$, such that the diagram*

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "i"] \arrow[d, "h"] & F \arrow[r, "p"] \arrow[d, "g"] & A \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & R' \arrow[r, "i'"] & F' \arrow[r, "p'"] & B \arrow[r] & 0 \end{tikzcd}$$

commutes.

If g', h' are also homomorphisms with this property, then there is an $\alpha\colon F \to R'$ with $i' \circ \alpha = g - g'$ and $\alpha \circ i = h - h'$.

(b) *For every abelian group D the map $h \otimes \mathrm{id}\colon R \otimes D \to R' \otimes D$ maps the kernel of $i \otimes \mathrm{id}$ to the kernel of $i' \otimes \mathrm{id}$ and the restriction $h \otimes \mathrm{id}|_{\ker(i \otimes \mathrm{id})}$ is independent of the choice of g and h. We denote this map by $\varphi(f, R \to F, R' \to F')$.*

(c) *For a homomorphism $f'\colon B \to C$ the map $\varphi(f' \circ f, R \to F, R'' \to F'')$ is equal to the composition $\varphi(f', R' \to F', R'' \to F'') \circ \varphi(f, R \to F, R' \to F')$.*

Note that we can view the α above as a chain homotopy between the chain maps g, h and g', h'.

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "i"] \arrow[d, bend right, "h'"'] \arrow[d, bend left, "h"] & F \arrow[r] \arrow[dl, "\alpha"'] \arrow[d, bend right, "g'"'] \arrow[d, bend left, "g"] & 0 \\ 0 \arrow[r] & R' \arrow[r, "i'"] & F' \arrow[r] & 0 \end{tikzcd}$$

Proof. For (a) let $\{x_i\}$ be a basis of F and choose $y_i \in F'$ with $p'(y_i) = fp(x_i)$. We define $g\colon F \to F'$ via $g(x_i) = y_i$. Thus $p' \circ g(x_i) = p'(y_i) = fp(x_i)$. For every $r \in R$ we obtain $p' \circ g(i(r)) = f \circ p \circ i(r) = 0$ and therefore $g(i(r))$ is contained in the kernel of p' which is equal to the image of i'. In order to define h we use the injectivity of i', thus $h(r)$ is the unique preimage of $g(i(r))$ under i'.

For h, h' and g, g' as in (a) we get for $x \in F$ that $g(x) - g'(x)$ is in the kernel of p' which is the image of i'. Define α as $(i')^{-1}(g - g')$. Then by construction $i'\alpha = g - g'$ and

$$i'(h - h') = (g - g')i = i'\alpha i.$$

As i' is injective, this yields $h - h' = \alpha i$.

For (b) we consider an element z in the kernel of $i \otimes \mathrm{id}$. Note $\ker(i \otimes \mathrm{id}) \subset R \otimes D$. Then

$$(i' \otimes \mathrm{id}) \circ (h \otimes \mathrm{id})(z) = (g \otimes \mathrm{id}) \circ (i \otimes \mathrm{id})(z) = 0$$

and thus $(h \otimes \mathrm{id})(z)$ is in the kernel of $(i' \otimes \mathrm{id})$. If h' is any other map satisfying the properties, then

$$(h' \otimes \mathrm{id})(z) - (h \otimes \mathrm{id})(z) = ((h' - h) \otimes \mathrm{id})(z) = ((-\alpha \circ i) \otimes \mathrm{id})(z) = -(\alpha \otimes \mathrm{id})(i \otimes \mathrm{id})(z) = 0.$$

For (c) we note that the uniqueness in (b) implies (c). □

Corollary 14.7. *For every free resolution $0 \longrightarrow R' \xrightarrow{i'} F' \longrightarrow A \longrightarrow 0$ we get a unique isomorphism*

$$\varphi(\mathrm{id}_A, R' \to F', R \to F)\colon \ker(i' \otimes \mathrm{id}) \to \mathrm{Tor}(A, D).$$

Thus we can calculate $\mathrm{Tor}(A, D)$ with every free resolution of A.

Examples

(a) $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, D) \cong \{d \in D | nd = 0\}$ for all $n \geqslant 1$. That's why Tor is sometimes called torsion product. For the calculation we use the resolution $0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$. By definition and by Corollary 14.7 we have

$$\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, D) \cong \ker(n \otimes \mathrm{id}\colon \mathbb{Z} \otimes D \to \mathbb{Z} \otimes D).$$

As $\mathbb{Z} \otimes D \cong D$ and as $n \otimes \mathrm{id}$ induces the multiplication by n, we get the claim.

(b) From the first example we obtain $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z}$ because the n-torsion subgroup in $\mathbb{Z}/m\mathbb{Z}$ is $\mathbb{Z}/\gcd(m, n)\mathbb{Z}$.

(c) For A free abelian, $\mathrm{Tor}(A, D) \cong 0$ for arbitrary D. For this note that $0 \to 0 \to A = A \to 0$ is a free resolution of A and the kernel is a subgroup of $0 \otimes D = 0$ and hence trivial.

(d) For two abelian groups A_1, A_2, D there is an isomorphism

$$\mathrm{Tor}(A_1 \oplus A_2, D) \cong \mathrm{Tor}(A_1, D) \oplus \mathrm{Tor}(A_2, D).$$

Consider free resolutions

$$0 \to R_i \to F_i \to A_i \to 0, i = 1, 2.$$

Their direct sum

$$0 \to R_1 \oplus R_2 \to F_1 \oplus F_2 \to A_1 \oplus A_2 \to 0$$

is a free resolution of $A_1 \oplus A_2$ with

$$\ker((i_1 \oplus i_2) \otimes \mathrm{id}) = \ker(i_1 \otimes \mathrm{id}) \oplus \ker(i_2 \otimes \mathrm{id}).$$

We extend the definition of tensor products to chain complexes.

Definition 14.8. Are (C_*, d) and (C'_*, d') two chain complexes, then $(C_* \otimes C'_*, d_\otimes)$ is the chain complex with

$$(C_* \otimes C'_*)_n = \bigoplus_{p+q=n} C_p \otimes C'_q$$

and with $d_\otimes(c_p \otimes c'_q) = (dc_p) \otimes c'_q + (-1)^p c_p \otimes d'c'_q$.

Lemma 14.9. *The map $d_\otimes$ is a differential.*

Proof. The composition is

$$d_\otimes((dc_p) \otimes c'_q + (-1)^p c_p \otimes d'c'_q) = 0 + (-1)^{p-1}(dc_p) \otimes (d'c'_q) + (-1)^p (dc_p) \otimes (d'c'_q) + 0 = 0.$$

□

Example Let G be an abelian group, then let C_G be the chain complex with

$$(C_G)_n = \begin{cases} G, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Then for every chain complex (C_*, d)

$$(C_* \otimes C_G)_n = C_n \otimes G, \quad d_\otimes = d \otimes \mathrm{id}.$$

In particular, for every topological space X,

$$S_*(X) \otimes C_G \cong S_*(X) \otimes G = S_*(X, G).$$

Similarly, for a CW complex X we get $C_*(X; G) = C_*(X) \otimes C_G$.

For every pair of spaces (X, A) we set

$$S_*(X, A; G) := S_*(X, A) \otimes C_G.$$

A map $f\colon (C_*, d) \to (D_*, d_D)$ induces a map of chain complexes

$$f \otimes \mathrm{id}\colon C_* \otimes C'_* \to D_* \otimes C'_*.$$

In particular, for every continuous (cellular) map we get induced maps on singular (cellular) homology with coefficients.

Note, that $H_*(\mathrm{pt}; G) \cong \begin{cases} G, & * = 0 \\ 0, & * \neq 0. \end{cases}$

Definition 14.10. A chain complex C_* is called *free*, if C_n is a free abelian group for all $n \in \mathbb{Z}$.

Examples The complexes $S_*(X, A)$ and $C_*(X)$ are free.

Theorem 14.11. *(Universal coefficient theorem (algebraic version)) Let C_* be a free chain complex and G an abelian group, then for all $n \in \mathbb{Z}$ we have a split short exact sequence*

$$0 \to H_n(C_*) \otimes G \to H_n(C_* \otimes G) \to \mathrm{Tor}(H_{n-1}(C_*), G) \to 0,$$

in particular

$$H_n(C_* \otimes G) \cong H_n(C_*) \otimes G \oplus \mathrm{Tor}(H_{n-1}(C_*), G).$$

Theorem 14.12. *(Universal coefficient theorem (topological version)) For every space X there is a split short exact sequence*

$$0 \to H_n(X) \otimes G \to H_n(X; G) \to \mathrm{Tor}(H_{n-1}(X), G) \to 0,$$

and therefore we get an isomorphism

$$H_n(X; G) \cong H_n(X) \otimes G \oplus \mathrm{Tor}(H_{n-1}(X), G).$$

Example For $X = \mathbb{R}P^2$ we obtain

$$H_n(\mathbb{R}P^2; G) \cong H_n(\mathbb{R}P^2) \otimes G \oplus \mathrm{Tor}(H_{n-1}(\mathbb{R}P^2), G)$$

thus

$$H_0(\mathbb{R}P^2; G) \cong H_0(\mathbb{R}P^2) \otimes G \oplus \mathrm{Tor}(H_{-1}(\mathbb{R}P^2), G) \cong G,$$
$$H_1(\mathbb{R}P^2; G) \cong H_1(\mathbb{R}P^2) \otimes G \oplus \mathrm{Tor}(H_0(\mathbb{R}P^2), G) \cong G/2G \oplus 0 \cong G/2G,$$

and

$$H_2(\mathbb{R}P^2; G) \cong H_2(\mathbb{R}P^2) \otimes G \oplus \mathrm{Tor}(H_1(\mathbb{R}P^2), G) \cong \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, G).$$

The universal coefficient theorems are both corollaries of the following more general statement.

Theorem 14.13. *(Künneth formula) For a free chain complex C_* and a chain complex C'_* we have the following split exact sequence for every integer n*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \xrightarrow{\lambda} H_n(C_* \otimes C'_*) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(C'_*)) \longrightarrow 0,$$

i.e.,

$$H_n(C_* \otimes C'_*) \cong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \oplus \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(C'_*)).$$

The map $\lambda \colon \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \to H_n(C_* \otimes C'_*)$ in the theorem is given on the (p,q)-summand by

$$\lambda([c_p] \otimes [c'_q]) := [c_p \otimes c'_q]$$

for $c_p \in C_p$ and $c'_q \in C'_q$. By the definition of the tensor product of complexes, this map is well-defined.

Lemma 14.14. *For any free chain complex C_* with trivial differential and an arbitrary chain complex, C'_*, λ is an isomorphism*

$$\lambda \colon \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \cong H_n(C_* \otimes C'_*).$$

Proof. We abbreviate the subgroup of cycles in C'_q with Z'_q and the subgroup of boundaries in C'_q with B'_q and use analog abbreviations for C_*. By definition $0 \to Z'_q \longrightarrow C'_q \longrightarrow B'_{q-1} \to 0$ is a short exact sequence. By assumption Z_p is free because $Z_p = C_p$, in particular $Z_p \otimes (-)$ is exact and thus

$$0 \to Z_p \otimes Z'_q \longrightarrow Z_p \otimes C'_q \longrightarrow Z_p \otimes B'_{q-1} \to 0$$

is a short exact sequence and this implies that $Z_p \otimes Z'_q$ is the subgroup of cycles in $Z_p \otimes C'_q = C_p \otimes C'_q$. Summation over $p + q = n$ yields that the n-cycles in $C_* \otimes C'_*$ are

$$Z_n(C_* \otimes C'_*) = \bigoplus_{p+q=n} Z_p \otimes Z'_q$$

and the n-boundaries are given by

$$B_n(C_* \otimes C'_*) = \bigoplus_{p+q=n} Z_p \otimes B'_q.$$

The sequence

$$0 \to B'_q \longrightarrow Z'_q \longrightarrow H_q(C'_*) \to 0$$

is exact by definition. Tensoring with Z_p and summing over $p+q=n$ then yields due to the freeness of Z_p that

$$0 \to \bigoplus_{p+q=n} Z_p \otimes B'_q \longrightarrow \bigoplus_{p+q=n} Z_p \otimes Z'_q \longrightarrow \bigoplus_{p+q=n} Z_p \otimes H_q(C'_*) \to 0$$

is exact. Our identification of $Z_n(C_* \otimes C'_*)$ and $B_n(C_* \otimes C'_*)$ yields that the right-most term is isomorphic to the nth homology group of $C_* \otimes C'_*$ and therefore

$$H_n(C_* \otimes C'_*) \cong \bigoplus_{p+q=n} Z_p \otimes H_q(C'_*) = \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*).$$

□

Proof of Theorem 14.13. We consider again the short exact sequence $0 \to Z_p \longrightarrow C_p \longrightarrow B_{p-1} \to 0$ and tensor it with C'_q and sum over $p+q=n$. As B_{p-1} is free, the original sequence is split and hence the resulting sequence is exact.

We define two chain complexes Z_* and D_* via

$$(Z_*)_p = Z_p, (D_*)_p = B_{p-1}.$$

Then Z_* and D_* are free chain complexes with trivial differential and the exact sequence

$$0 \to \bigoplus_{p+q=n} Z_p \otimes C'_q \longrightarrow \bigoplus_{p+q=n} C_p \otimes C'_q \longrightarrow \bigoplus_{p+q=n} B_{p-1} \otimes C'_q \to 0$$

can be interpreted as a short exact sequence of complexes and this gives a long exact sequence

$$\ldots \longrightarrow H_{n+1}(D_* \otimes C'_*) \xrightarrow{\delta_{n+1}} H_n(Z_* \otimes C'_*) \longrightarrow H_n(C_* \otimes C'_*) \longrightarrow H_n(D_* \otimes C'_*) \xrightarrow{\delta_n} H_{n-1}(Z_* \otimes C'_*) \longrightarrow \ldots$$

Lemma 14.14 gives us a description of $H_*(D_* \otimes C'_*)$ and $H_*(Z_* \otimes C'_*)$ and therefore we can consider δ_{n+1} as a map

$$\begin{array}{c} \bigoplus_{p+q=n+1} H_p(D_*) \otimes H_q(C'_*) = \bigoplus_{p+q=n+1} B_{p-1} \otimes H_q(C'_*) \\ \big\downarrow {\scriptstyle j\otimes \mathrm{id}} \\ \bigoplus_{p+q=n} Z_p \otimes H_q(C'_*) = \bigoplus_{p+q=n} H_p(Z_*) \otimes H_q(C'_*) \end{array}$$

with $j\colon B_p \hookrightarrow Z_p$. We can cut the long exact sequence in homology into short exact pieces and obtain that

$$0 \to \mathrm{coker}(\delta_{n+1}) \longrightarrow H_n(C_* \otimes C'_*) \longrightarrow \ker(\delta_n) \to 0$$

is exact. The cokernel of δ_{n+1} is isomorphic to $\bigoplus_{p+q=n}(Z_p/B_p) \otimes H_q(C'_*)$ because the tensor functor is right exact, thus

$$\mathrm{coker}(\delta_{n+1}) \cong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*).$$

As $0 \to B_p \longrightarrow Z_p \longrightarrow H_p(C_*) \to 0$ is a free resolution of $H_p(C_*)$ we obtain that

$$\mathrm{Tor}(H_p(C_*), H_q(C'_*)) \cong \ker(j \otimes \mathrm{id}\colon B_p \otimes H_q(C'_*) \to Z_p \otimes H_q(C'_*))$$

and therefore

$$\bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(C'_*)) \cong \ker(\delta_n)$$

which proves the exactness of the Künneth sequence.

We will prove that the Künneth sequence is split in the case where both chain complexes, C_* and C'_*, are free. In that case the sequences

$$0 \to Z_p \to C_p \to B_{p-1} \to 0, \quad 0 \to Z'_q \to C'_q \to B'_{q-1} \to 0$$

are split and we denote by $r\colon C_p \to Z_p$ and $r'\colon C'_q \to Z'_q$ chosen retractions. Consider the two compositions

$$C_p \xrightarrow{r} Z_p \twoheadrightarrow H_p(C_*), \quad C'_q \xrightarrow{r'} Z'_q \twoheadrightarrow H_q(C'_*)$$

and view $H_*(C_*)$ and $H_*(C'_*)$ as chain complexes with trivial differential. Then these compositions yield a chain map

$$r \otimes r' \colon C_* \otimes C'_* \to H_*(C_*) \otimes H_*(C'_*)$$

which on homology is

$$H_n(C_* \otimes C'_*) \longrightarrow H_n(H_*(C_*) \otimes H_*(C'_*)) = \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*).$$

This map gives the desired splitting. $\square$

In the cases we are interested in (singular or cellular chains), the complexes will be free. Be careful! The splitting of the Künneth sequence is *not* natural. We have chosen a splitting of the short exact sequences in the proof and usually, there is no canonical choice possible.