Exercises in Algebraic Topology (master)

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Exercise sheet no 3 due: 19th of April 2024, 11:45h in H2

1 (Linear algebra) (3 + 1 points)
Compare the homology groups of $GL_n(\mathbb{R})$ and $O(n)$. What about $GL_n(\mathbb{C})$ and $U(n)$?

2 (The 3×3=9-Lemma) (3 + 1 points)
Assume that the following diagram of abelian groups commutes and that all three columns are short exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\
0 & \longrightarrow & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & & \\
0 & \longrightarrow & C_1 & \xrightarrow{\gamma_1} & C_2 & \xrightarrow{\gamma_2} & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Prove that the third row is short exact if the first two rows are short exact (and dually, that the first row is short exact if the last two rows are short exact).

3 (Snake Lemma) (2 points)
Prove the famous Snake Lemma: If

$$\begin{array}{ccccccccc}
 & & A' & \xrightarrow{\alpha} & A & \xrightarrow{\beta} & A'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & B' & \xrightarrow{\alpha'} & B & \xrightarrow{\beta'} & B'' & &
\end{array}$$

is a commutative diagram with exact rows, then there is an exact sequence

$$\ker(f') \longrightarrow \ker(f) \longrightarrow \ker(f'') \xrightarrow{\delta} \mathrm{coker}(f') \longrightarrow \mathrm{coker}(f) \longrightarrow \mathrm{coker}(f'').$$

Define δ explicitly in this case. Why "snake"?
(For an alternative: `http://www.youtube.com/watch?v=etbcKWEKnvg`)

4 (Exactness and homomorphisms) (2 + 2 points)
Let M be an abelian group and let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence of abelian groups.

(a) What can you say about the exactness of the sequence

$$0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{\alpha_*} \mathrm{Hom}(M, B) \xrightarrow{\beta_*} \mathrm{Hom}(M, C) \longrightarrow 0?$$

(b) What happens if M is free abelian?