Exercises in Algebraic Topology (master)

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Exercise sheet no 6 due: 22nd of May 2019

1 (Euler characteristic) Let X be a finite CW complex. The *Euler characteristic* of X, $\chi(X)$, is then defined as

$$\chi(X) := \sum_{n \geqslant 0} (-1)^n \mathrm{rk}(H_n(X;\mathbb{Z})).$$

Here, rk denotes the rank of a finitely generated abelian group, *i.e.*, the number of its free summands.

a) Why is $\chi(X)$ well-defined?

b) Let $c_n(X)$ denote the number of n-cells of X. Prove that

$$\chi(X) = \sum_{n \geqslant 0} (-1)^n c_n(X).$$

c) What is $\chi(X)$ for a torus, a sphere or a general oriented compact closed surface of genus g, F_g?

d) What can you say about $\chi(X \sqcup Y)$ for two finite CW complexes X and Y? What about $\chi(X \cup Y)$ if X and Y are not necessarily disjoint? Assume that $X \cap Y$ is a subcomplex of X and Y.

e) For finite CW complexes X and Y, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

f) Assume that X is a finite CW complex and that $p\colon \tilde{X} \to X$ is an n-sheeted covering. Prove that $\chi(\tilde{X}) = n\chi(X)$.

2 (Products)

(1) Give a CW model of $\mathbb{S}^n \times \mathbb{S}^m$.

(2) If you start with $\mathbb{S}^1 \vee \mathbb{S}^1$ how many 2-cells do you have to glue in to get the torus $\mathbb{S}^1 \times \mathbb{S}^1$?

(3) Compute the homology groups of $\mathbb{S}^1 \times \mathbb{S}^1$ and of $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$. Are they isomorphic? What about the homology groups of their universal covering spaces?

3 (Moore spaces) Let G be an arbitrary finitely generated abelian group.

a) Construct a CW complex $M(G,n)$ whose reduced homology is concentrated in degree n with $\tilde{H}_n(M(G,n)) \cong G$. Such a space is called a *Moore space of type* (G,n).

b) How does $M(G,n)$ look like if G is a finite cyclic group?

c) What is $M(\mathbb{Z}/2\mathbb{Z},1)$?

4 (Non-orientable surfaces) For $g \geqslant 2$ consider a regular $2g$-gon $P_{2g} \subset \mathbb{R}^2$ with vertices $z_1, \ldots, z_{2g}$. We identify edges according to

$$(1-t)z_{2j-1} + tz_{2j} \sim tz_{2j+1} + (1-t)z_{2j}$$

(here the indices are to be read mod $2g$) and call the quotient $N_g = P_{2g}/\sim$ the *closed non-orientable surface of genus* g. What is N_2? Calculate the homology of N_{2g} using the cellular chain complex.