Exercises in Algebraic Topology (master)

Prof. Dr. Birgit Richter

Summer term 2019

Exercise sheet no 5 due: 15th of May 2019

1 (5-Lemma revisited) Consider the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 \end{array}$$

Check under which assumptions on f_1, f_2, f_4, f_5 we can deduce that the map f_3 is a monomorphism or an epimorphism.

2 (The $9 = 3 \times 3$-lemma) We consider the following commutative diagram with exact columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

(1) Prove that the top row is exact if the two bottom rows are exact and that the bottom row is exact if the two top rows are exact.
(2) What happens if the top and bottom rows are exact? Can you deduce that the middle row is exact or do you need an extra condition?

3 (Degrees)
(1) Prove the Brouwer fixed-point theorem: Let X be a closed ball $B_R(x) \subset \mathbb{R}^n$ for $n \geqslant 1$ and let f be a continuous map $f\colon B_R(x) \to B_R(x)$. Prove that f has a fixed point.
(2) Use this to show that every $(a_{ij}) = A \in M(n \times n; \mathbb{R})$ with non-negative a_{ij} must have an eigenvector with non-negative coordinates.

4 (More linear algebra) Let $A \in O(n+1)$. Then multiplication by A induces a continuous self-map on $\mathbb{S}^n$. What is its degree?