Exercises in Algebraic Topology (master)

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Summer term 2019

Exercise sheet no 12 due: 10th of July 2019

1 (Cup pairing)

a) What are the cup pairings on $\mathbb{S}^4$, $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{C}P^2$?

b) What can you say about the symmetry of the cup pairing if the dimension of the manifold is $4n$ or $4n+2$?

2 (Transfer or Umkehr maps) Assume that M is an m-dimensional and N is an n-dimensional topological manifold and both are connected, compact, closed and oriented with fundamental classes $[M]$ and $[N]$. Assume that $f\colon M \to N$ is continuous. Define $f^!\colon H^{m-p}(M) \to H^{n-p}(N)$ and $f_!\colon H_{n-p}(N) \to H_{m-p}(M)$ as

$$f^!(\alpha) = \mathsf{PD}_N^{-1} \circ H_*(f) \circ \mathsf{PD}_M(\alpha) \text{ and } f_!(a) = \mathsf{PD}_M(H^*(f)(\mathsf{PD}_N^{-1}(a)).$$

So these map go the wrong way – and this is why they are sometimes also called *wrong-way maps*.

a) Draw diagrams in order to understand how these maps work.

b) Show that $f^!(f^*(\alpha) \cup \beta) = \alpha \cup f^!(\beta)$ and also $f_!(\alpha \cap a) = f^*(\alpha) \cap f_!(a)$.

c) Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a map of degree $n > 1$. What is the effect of $f^!$ and $f_!$ in degree 1?

3 (Easy applications of duality)

(1) Prove that every homotopy equivalence $f\colon \mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$ must be orientation preserving.

(2) Let $n > m \geqslant 1$ and show that every continuous $f\colon \mathbb{R}P^n \to \mathbb{R}P^m$ induces $\pi_1(f) = 0$.

4 (Inverse limits)

a) Consider the short exact sequence of inverse systems

$$0 \to \{p^i\mathbb{Z}\} \to \{\mathbb{Z}\} \to \{\mathbb{Z}/p^i\mathbb{Z}\} \to 0.$$

Determine the inverse limits and the $\lim^1$-terms.

b) Let $\{A_i\}_{i\in\mathbb{N}_0}$ be an inverse system of abelian groups such that the structure maps $A_{i+1} \to A_i$ are monomorphisms. Define a topology on $A = A_0$ by declaring the sets $\{a + A_i\}$ to be open for $a \in A$ and $i \geqslant 0$. (The A_i are viewed as subsets of A via the monomorphisms.) Show that the inverse limit of the A_i is trivial if A is Hausdorff. When does the $\lim^1$-term vanish?

c) Let k be a commutative ring with unit. Show that the inverse limit of the inverse system $\{k[x]/x^n\}_{n\geqslant 1}$ is isomorphic to the formal power series ring $k[[x]]$.