Exercises in Algebraic Topology (master)

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Exercise sheet no 11 due: 3rd of July 2019

1 (Degree again) Let M and N be connected, oriented compact m-manifolds without boundary.

(1) Consider the degree as a map from the set of homotopy classes $[M, \mathbb{S}^m]$ to $\mathbb{Z}$. Show that this map is surjective.
(2) Let $p\colon M \to N$ be an n-sheeted covering map. Show that p has degree $\pm n$.
(3) Assume that $f\colon M \to N$ is a map of degree one between connected compact oriented manifolds. Show that $\pi_1(f)\colon \pi_1(M) \to \pi_1(N)$ is surjective. Does this imply that $H_1(f)$ is surjective?
(4) Let g_1, g_2 be greater or equal to one. Prove that a degree one map $f\colon F_{g_1} \to F_{g_2}$ exists if and only if $g_1 \geqslant g_2$.

2 (Exactness of direct limits) Prove the remaining two bits that establish that direct limits map short exact sequences of directed systems of R-modules to short exact sequences of R-modules (proof of lemma 7.3).

3 (Compact support)
If X is a path-connected, non-compact space, what is $H^0_c(X)$?

4 (3-manifolds) Let M be a compact connected 3-manifold without boundary. Its first homology group is a finitely generated abelian group and is hence of the form

$$H_1(M) \cong \mathbb{Z}^n \oplus T$$

where T denotes the finite torsion part of $H_1(M)$.

a) Determine $H_2(M)$ if M is orientable.
b) Does $\pi_1(M)$ determine $H_*(M)$ in this case?
c) What happens if we drop the assumption that M is orientable? Can you still say something about $H_2(M)$?