Exercises in Algebraic Topology (master)

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Exercise sheet no 10 due: 26th of June 2019

1 (Orientation covering) Let M be an m-dimensional connected topological manifold.
a) Prove that there is an oriented manifold $\hat{M}$ and a 2-fold covering $p\colon \hat{M} \to M$ called the orientation covering.
b) Are the following statements equivalent?

(1) M is orientable.

(2) The orientation covering is a trivial covering, *i.e.*, $\hat{M} \cong M \times \mathbb{Z}/2\mathbb{Z}$ as spaces over M.

c) Assume that M is finite dimensional, path connected with $\pi_1(M) = 1$. Is M orientable?
d) What is the orientation covering of $\mathbb{R}P^n$ for even n? What about the Klein bottle and the open Möbius strip?

2 (Non-orientable surfaces) You know the spaces N_g from Exercise 6.4. We called N_g the non-orientable surface of genus g. Justify that name.

3 (R-orientations) Let R be a commutative ring with unit and let M be a connected m-dimensional manifold together with an R-orientation. Show that the group of units of R, $R^\times$, acts free and transitively on the set of all R-orientations of M. For $R = \mathbb{Z}$ this should look familiar.

4 (Manifolds with boundary) Let

$$\mathbb{R}^m_- := \{(x_1, \ldots, x_m), x_i \in \mathbb{R}, x_1 \leqslant 0\}$$

be an m-dimensional half-space. Its topological boundary is

$$\partial\mathbb{R}^m_- = \{(x_1, \ldots, x_m), x_i \in \mathbb{R}, x_1 = 0\}.$$

An m-dimensional topological manifold with boundary, M with ∂M, is a Hausdorff space with a countable basis of its topology together with homeomorphisms $h_i\colon U_i \to V_i$. Here $U_i \subset M$ and $V_i \subset \mathbb{R}^m_-$ are open and the U_i's cover M.

An $x \in M$ is a boundary point of M if there is a homeomorphism $h\colon U \to V$ with U open in M, V open in $\mathbb{R}^m_-$, $x \in U$ and $h(x)$ in $\partial\mathbb{R}^m_-$. The set of boundary points of M is denoted by ∂M.

What is ∂M in the following examples:
a) $\partial([0,1])$,
b) $\partial(\mathbb{D}^2 \times \mathbb{S}^1)$,
c) $\partial(\mathbb{D}^2 \times \mathbb{D}^2)$, and
d) $\partial((\mathbb{S}^1 \times \mathbb{S}^1)\backslash\mathring{\mathbb{D}}^2_\epsilon)$, where $\mathring{\mathbb{D}}^2_\epsilon$ is a small open 2-disk, that is suitably embedded into the torus.
e) What is a general formula for $\partial(M \times N)$ if M and N are manifolds with boundary?