Exercises in Algebraic Topology (master)

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Summer term 2019

Exercise sheet no 1 due: 10th of April 2019

As we have no-one who will correct the exercises, 'due' means that you should work on the exercises and should be able to present solutions during the exercise class on the 10th of April.

1 (Basics about abelian groups) Let A and B be two abelian groups. We denote by $\mathsf{Hom}(A, B)$ the set of group homomorphisms from A to B.

a) Show that $\mathsf{Hom}(A, B)$ is an abelian group.

b) Construct an explicit isomorphism $\varphi\colon \mathsf{Hom}(\mathbb{Z}, A) \cong A$ for all abelian groups A.

c) Let $n > 1$ be a natural number. Describe $\mathsf{Hom}(\mathbb{Z}/n\mathbb{Z}, A)$ as a subgroup of A. What is $\mathsf{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ or $\mathsf{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q})$?

2 (Disks and spheres)

Let $\mathbb{D}^n$ be the chain complex whose only non-trivial entries are in degrees n and $n-1$ with $\mathbb{D}^n_n = \mathbb{D}^n_{n-1} = \mathbb{Z}$. Its only non-trivial boundary operator is the identity.

Similarly, let $\mathbb{S}^n$ be the chain complex whose only non-trivial entry is in degree n with $\mathbb{S}^n_n = \mathbb{Z}$.

a) Assume that (C_*d) is an arbitrary chain complex. Describe the abelian group of chain maps from $\mathbb{D}^n$ to C_* and from $\mathbb{S}^n$ to C_* in terms of subobjects of C_n.

b) What is the homology of $\mathbb{D}^n$ and $\mathbb{S}^m$?

c) Let $f_*\colon C_* \to C'_*$ be a chain map and assume that f_n is a monomorphism for all n. Do we then know that $H_n(f_*)$ is also a monomorphism?

3 (Too much to ask for?)

a) What are the homology groups of the chain complex

$$C_* = (\ldots \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}/4\mathbb{Z} \to \ldots)?$$

b) Is there a chain homotopy from the identity of C_* to the zero map, *i.e.*, can there be maps $s_n\colon C_n \to C_{n+1}$ with $d \circ s + s \circ d = \mathrm{id}_{C_n}$ for all $n \in \mathbb{Z}$?

4 (Lego)

Let $(A_n)_{n\in\mathbb{Z}}$ be an arbitrary family of finitely generated abelian groups. Is there a chain complex F_* with F_n free abelian for all $n \in \mathbb{Z}$ and with $H_n(F_*) \cong A_n$? (Recall the structure theorem for finitely generated abelian groups for this.)