Exercises in Algebraic Topology (master)

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Summer term 2017

Exercise sheet no 9 For the exercise class on the 3rd of July 2017

1 (Manifolds with boundary) Let

$$\mathbb{R}^m_- := \{(x_1, \ldots, x_m), x_i \in \mathbb{R}, x_1 \leqslant 0\}$$

be an m-dimensional half-space. Its topological boundary is

$$\partial\mathbb{R}^m_- = \{(x_1, \ldots, x_m), x_i \in \mathbb{R}, x_1 = 0\}.$$

An m-dimensional topological manifold with boundary, M with ∂M, is a Hausdorff space with a countable basis of its topology together with homeomorphisms $h_i\colon U_i \to V_i$. Here $U_i \subset M$ and $V_i \subset \mathbb{R}^m_-$ are open and the U_i's cover M.

An $x \in M$ is a boundary point of M if there is a homeomorphism $h\colon U \to V$ with U open in M, V open in $\mathbb{R}^m_-$, $x \in U$ and $h(x)$ in $\partial\mathbb{R}^m_-$. The set of boundary points of M is denoted by ∂M.

What is ∂M in the following examples:

a) $\partial(\mathbb{D}^2 \times \mathbb{S}^1)$,

b) $\partial(\mathbb{D}^2 \times \mathbb{D}^2)$,

c) $\partial([0, 1])$ and

d) $\partial((\mathbb{S}^1 \times \mathbb{S}^1)\backslash\mathring{\mathbb{D}}^2_\epsilon)$, where $\mathring{\mathbb{D}}^2_\epsilon$ is a small open 2-disk, that is suitably embedded into the torus.

e) Can you find a general formula for $\partial(M \times N)$ if M and N are manifolds with boundary?

2 (Exactness of direct limits) Prove the remaining two bits that establish that direct limits map short exact sequences of directed systems of R-modules to short exact sequences of R-modules (proof of lemma 7.3).

3 (Compact support)

If X is a path-connected, non-compact space, what is $H^0_c(X)$?

4 (3-manifolds) Let M be a compact connected 3-manifold without boundary. Its first homology group is a finitely generated abelian group and is hence of the form

$$H_1(M) \cong \mathbb{Z}^n \oplus T$$

where T denotes the torsion part of $H_1(M)$.

a) Determine $H_2(M)$ if M is orientable.

b) Does $\pi_1(M)$ determine $H_*(M)$ in this case?

c) What happens if we drop the assumption that M is orientable? What do you get for $H_2(M)$?