Exercises in Algebraic Topology (master)

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Exercise sheet no 7 For the exercise class on the 19th of June 2017

1 ((Co)Homology of real projective spaces)

a) Calculate $H^m(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$ and $H^m(\mathbb{R}P^n;\mathbb{Z})$ for all m and n.
b) Let $\alpha \in H^1(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z})$ and $a \in H_1(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z})$ be generators. What is $\alpha \cap a$?

2 (Relative variant of the cap-product) Let A and B be subspaces of a topological space X such that the inclusion $S^{\mathfrak{U}}_*(A \cup B) \hookrightarrow S_*(A \cup B)$ induces an isomorphism in homology (with $\mathfrak{U} = \{A, B\}$). Show that there is a variant of the cap-product

$$\cap\colon H^q(X, A) \otimes H_n(X, A \cup B) \to H_{n-q}(X, B).$$

3 (Splittings) Let C_* be a free chain complex and let G be an abelian group.
a) Show that the sequence

$$0 \to H_n(C_*) \otimes G \to H_n(C_* \otimes G) \to \mathrm{Tor}(H_{n-1}(C_*), G) \to 0$$

splits.
b) Prove the same for the cohomological version.