# Exercises in Algebraic Topology (master)

Prof. Dr. Birgit Richter

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**Exercise sheet no 4** For the exercise class on the 22nd of May 2017

**1** (Mapping degree)

(1) Show that the degree map $\deg\colon [\mathbb{S}^n, \mathbb{S}^n] \to \mathbb{Z}$ is surjective for all $n \geqslant 1$. (What about $n = 0$?)
(2) Prove the Brouwer fixed-point theorem: Let $X$ be a closed ball $B_R(x) \subset \mathbb{R}^n$ for $n \geqslant 1$ and let $f$ be a continuous map $f\colon B_R(x) \to B_R(x)$. Prove that $f$ has a fixed point.
(3) Use this to show that every $(a_{ij}) = A \in M(n \times n; \mathbb{R})$ with non-negative $a_{ij}$ must have an eigenvector with non-negative coordinates.

**2** (More linear algebra) Let $A \in O(n+1)$. Then multiplication by $A$ induces a continuous self-map on $\mathbb{S}^n$. What is the degree?

**3** (Euler characteristic) Let $X$ be a finite CW complex. The *Euler characteristic* of $X$, $\chi(X)$, is then defined as

$$\chi(X) := \sum_{n \geqslant 0} (-1)^n \mathrm{rk}(H_n(X; \mathbb{Z})).$$

Here, rk denotes the rank of a finitely generated abelian group, *i.e.*, the number of its free summands.

a) Why is $\chi(X)$ well-defined?

b) What is $\chi(X)$ for a torus, a sphere or a general oriented compact closed surface of genus $g$, $F_g$?

c) What can you say about $\chi(X \sqcup Y)$ for two finite CW complexes $X$ and $Y$? What about $\chi(X \cup Y)$ if $X$ and $Y$ are not necessarily disjoint? Assume that $X \cap Y$ is a subcomplex of $X$ and $Y$.

d) Let $c_n(X)$ denote the number of $n$-cells of $X$. Prove that

$$\chi(X) = \sum_{n \geqslant 0} (-1)^n c_n(X).$$

e) For finite CW complexes $X$ and $Y$, show that $\chi(X \times Y) = \chi(X)\chi(Y)$ using d).

**4** (Products)

(1) Give a CW model of $\mathbb{S}^n \times \mathbb{S}^m$.
(2) If you start with $\mathbb{S}^1 \vee \mathbb{S}^1$ how many 2-cells do you have to glue in to get the torus $\mathbb{S}^1 \times \mathbb{S}^1$?
(3) Compute the homology groups of $\mathbb{S}^1 \times \mathbb{S}^1$ and of $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$. Are they isomorphic? What about the homology groups of their universal covering spaces?