TOWARDS TOPOLOGICAL HOCHSCHILD HOMOLOGY OF JOHNSON-WILSON SPECTRA

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ABSTRACT. We offer a complete description of $\mathrm{THH}(E(2))$ under the assumption that the Johnson-Wilson spectrum $E(2)$ at a chosen odd prime carries an E_∞-structure. We also place $\mathrm{THH}(E(2))$ in a cofiber sequence $E(2) \to \mathrm{THH}(E(2)) \to \overline{\mathrm{THH}}(E(2))$ and describe $\overline{\mathrm{THH}}(E(2))$ under the assumption that $E(2)$ is an E_3-ring spectrum. We state general results about the $K(i)$-local behaviour of $\mathrm{THH}(E(n))$ for all n and $0 \leqslant i \leqslant n$. In particular, we compute $K(i)_*\mathrm{THH}(E(n))$.

1. INTRODUCTION

The first Johnson-Wilson spectrum $E(1)$ at a prime p is the Adams summand of p-local periodic complex topological K-theory $KU_{(p)}$. It is known that it carries a unique E_∞-structure [MS93, BR05], thus $\mathrm{THH}(E(1))$ is a commutative $E(1)$-algebra spectrum. McClure and Staffeldt show that the unit map $E(1) \to \mathrm{THH}(E(1))$ is a $K(1)$-local equivalence, hence its cofiber $\overline{\mathrm{THH}}(E(1))$ is a rational spectrum. It is easy to calculate the rational homology of $\mathrm{THH}(E(1))$ as

$$H\mathbb{Q}_*\mathrm{THH}(E(1)) \cong \mathbb{Q}[v_1^{\pm 1}] \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(dv_1)$$

using the Bökstedt spectral sequence with E^2-term

$$E^2_{*,*} = \mathsf{HH}^{\mathbb{Q}}_{*,*}(\mathbb{Q}[v_1^{\pm 1}]).$$

There is a map

$$\Sigma^{2p-1}E(1) \to \mathrm{THH}(E(1)) \to \overline{\mathrm{THH}}(E(1))$$

that factors through $\Sigma^{2p-1}E(1)_{\mathbb{Q}} \to \overline{\mathrm{THH}}(E(1))$ since $\overline{\mathrm{THH}}(E(1))$ is rational, and that is defined such that the latter map is an equivalence detecting the $H\mathbb{Q}_*E(1)$-summand generated by dv_1. Since the unit map $E(1) \to \mathrm{THH}(E(1))$ splits, this yields a splitting [MS93, Theorem 8.1]

$$\mathrm{THH}(E(1)) \simeq E(1) \vee \Sigma^{2p-1}E(1)_{\mathbb{Q}}$$

as $E(1)$-modules. This computation was also carried out for $KU_{(p)}$ [Aus05], and pushed further to provide formulas for $\mathrm{THH}(KU)$ as a commutative KU-algebra by Stonek [Sto].

In this paper, we consider the higher Johnson-Wilson spectrum $E(n)$ with coefficient ring

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n, v_n^{-1}]$$

for an arbitrary value of $n \geqslant 1$ and p an odd prime. A main motivation here is to investigate whether the spectrum $\mathrm{THH}(E(n))$ also splits into copies of $E(n)$ and its lower chromatic localizations, generalizing McClure and Staffeldt's intriguing transchromatic result.

As a first step, we compute the Hochschild homology $\mathsf{HH}_*^{K(i)_*}(K(i)_*E(n))$ of $K(i)_*E(n)$, where $K(i)$ is the ith Morava K-theory, for $0 \leqslant i \leqslant n$, at an odd prime, see Theorem 3.4. We shy away from the prime 2 because Morava K-theory is not homotopy commutative at the prime 2. Theorem 3.4 yields a computation of $K(i)_*\mathrm{THH}(E(n))$ under the modest assumption that $E(n)$ admits an E_3-structure.

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We then focus on $E(2)$, and show in Theorem 5.4 that under the same commutativity assumption $\mathrm{THH}(E(2))$ sits in a cofiber sequence

$$E(2) \to \mathrm{THH}(E(2)) \to \Sigma^{2p-1} L_1 E(2) \vee \Sigma^{2p^2-1} E(2)_{\mathbb{Q}} \vee \Sigma^{2p^2+2p-2} E(2)_{\mathbb{Q}},$$

where $L_1 E(2)$ denotes the Bousfield localization of $E(2)$ with respect to $E(1)$. If the unit $E(2) \to \mathrm{THH}(E(2))$ splits, we then get a decomposition of $\mathrm{THH}(E(2))$ into four summands, a higher analogue of McClure-Staffeldt's formula for $\mathrm{THH}(E(1))$.

Remark 1.1. To study $\mathrm{THH}(E(n))$ by means of the Bökstedt spectral sequence, we need sufficient commutativity of $E(n)$. In this remark, we summarize what is known about multiplicative structures on $E(n)$ and related spectra. Basterra and Mandell showed [BM13] that the Brown-Peterson spectrum BP admits an E_4 structure. The Johnson-Wilson spectra $E(n)$ are built out of the $BP\langle n\rangle = BP/(v_i | i \geqslant n+1)$ by inverting v_n. In [Law18, Theorem 1.1.2] Tyler Lawson shows that the Brown-Peterson spectrum BP and the spectra $BP\langle n\rangle$ for $n \geqslant 4$ at the prime 2 do not possess an E_{12}-structure. Andrew Senger [Sen, Theorem 1.2] extends Lawson's result to odd primes p, and shows that BP and the $BP\langle n\rangle$'s (for $n \geqslant 4$) do not have an $E_{2(p^2+2)}$-structure. In particular, the $BP\langle n\rangle$'s are not E_∞-ring spectra at any prime for $n \geqslant 4$. Hence if $E(n)$ actually possesses an E_∞-structure for $n \geqslant 4$, then this structure does not come from one on the $BP\langle n\rangle$'s. In [Ric06, Proposition 8.2] it is proven that $E(n)$ at a prime p possesses at least a $(2p-1)$-stage structure. It is unclear how such a structure relates to the E_n-hierarchy, but Barwick conjectures [Bar18, p. 1948] that a $(2p-1)$-stage structure corresponds to an A_{2p}^{2p-1}-structure which in turn is a filtration piece of an E_{2p-1}-structure.

At the prime 2, Lawson and Naumann [LN12] show that there is an E_∞-model of $BP\langle 2\rangle$ and Hill and Lawson [HL10] prove that $BP\langle 2\rangle$ at the prime 3 possesses a model as an E_∞-ring spectrum. With [MNN15, Theorem A.1] this yields E_∞-structures on the corresponding Johnson-Wilson spectra $E(2)$ at these primes.

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2. Rationalized $E(n)$

For $n \geqslant 1$ the homotopy algebra of $L_{K(0)}E(n) = E(n)_{\mathbb{Q}}$ is $\mathbb{Q}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}]$ and its algebra of cooperations is

$$\pi_*(E(n)_{\mathbb{Q}} \wedge E(n)_{\mathbb{Q}}) \cong \pi_* E(n)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \pi_* E(n)_{\mathbb{Q}} \cong \mathbb{Q}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}, v_1', \ldots, v_{n-1}', {v_n'}^{\pm 1}].$$

This implies the following result.

Lemma 2.1. *There is a unique E_∞-ring structure on $E(n)_{\mathbb{Q}}$ for all $n \geqslant 1$.*

Proof. The obstruction groups for such an E_∞-ring structure on $E(n)_{\mathbb{Q}}$ are contained in the Gamma cohomology groups of $\pi_*(E(n)_{\mathbb{Q}} \wedge E(n)_{\mathbb{Q}})$ as a $\pi_* E(n)_{\mathbb{Q}}$-algebra [Rob03, Theorem 5.6]. As we work in characteristic zero, Gamma cohomology agrees with André-Quillen cohomology [RW02, Corollary 6.6]. The algebra $\mathbb{Q}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}, v_1', \ldots, v_{n-1}', {v_n'}^{\pm 1}]$ is smooth over $\mathbb{Q}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}]$ and therefore André-Quillen cohomology is concentrated in cohomological degree zero where it consists of derivations. The obstructions for existence and uniqueness of an E_∞-ring structure on $E(n)_{\mathbb{Q}}$ are concentrated in degrees bigger than zero. $\square$

As E_∞-ring structures can be rigidified to commutative ring structures (see *e.g.*, [EKMM97, II.3]), we pass to the world of commutative ring spectra from now on.

Topological Hochschild homology of a ring spectrum A can be modelled as the geometric realization of a simplicial spectrum. Using the inclusion of the 1-skeleton, McClure and Staffeldt [MS93, §3] construct a map

$$\sigma\colon \Sigma A \to \mathrm{THH}(A). \tag{2.1}$$

For a commutative ring spectrum A the multiplication maps from $A^{\wedge n+1}$ to A give rise to a map of commutative A-algebra spectra from $\mathrm{THH}(A)$ to A. Composing this map with the map $A \to \mathrm{THH}(A)$ gives the identity, hence we obtain a splitting of A-modules

$$\mathrm{THH}(A) \simeq A \vee \overline{\mathrm{THH}}(A)$$

where $\overline{\mathrm{THH}}(A)$ is the cofiber. The latter spectrum inherits the structure of a non-unital commutative A-algebra. In our case this implies the following result.

Corollary 2.2. *The topological Hochschild homology of $E(n)_\mathbb{Q}$ splits, as an $E(n)_\mathbb{Q}$-module, as*

$$\mathrm{THH}(E(n)_\mathbb{Q}) \simeq E(n)_\mathbb{Q} \vee \overline{\mathrm{THH}}(E(n))_\mathbb{Q}$$

where $\overline{\mathrm{THH}}(E(n))_\mathbb{Q}$ is the cofiber of the unit map $E(n)_\mathbb{Q} \to \mathrm{THH}(E(n)_\mathbb{Q}) \simeq \mathrm{THH}(E(n))_\mathbb{Q}$. Moreover, the spectrum $\overline{\mathrm{THH}}(E(n))_\mathbb{Q}$ is a non-unital commutative $E(n)_\mathbb{Q}$-algebra.

In the sequel, we follow Loday [Lod98, Definition E.1] for the definition of étale algebras. It is straightforward to calculate the topological Hochschild homology of $E(n)_\mathbb{Q}$.

Proposition 2.3.

$$\pi_*\mathrm{THH}(E(n))_\mathbb{Q} \cong \mathbb{Q}[v_1, \dots, v_{n-1}, v_n^{\pm 1}] \otimes \Lambda_\mathbb{Q}(dv_1, \dots, dv_n) \tag{2.2}$$

with $|dv_i| = 2p^i - 1$.

Proof. The Bökstedt spectral sequence for $\pi_*(\mathrm{THH}(E(n))_\mathbb{Q}) \cong H\mathbb{Q}_*\mathrm{THH}(E(n))$ is of the form

$$E^2_{*,*} = \mathsf{HH}^\mathbb{Q}_{*,*}(\pi_*E(n)_\mathbb{Q}) \Rightarrow \pi_*(\mathrm{THH}(E(n))_\mathbb{Q}).$$

As $\mathbb{Q}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ is étale over $\mathbb{Q}[v_1, \dots, v_{n-1}, v_n]$ and as $\mathbb{Q}[v_1, \dots, v_{n-1}, v_n]$ is smooth, we get

$$\mathsf{HH}^\mathbb{Q}_{*,*}(\pi_*E(n)_\mathbb{Q}) \cong \mathbb{Q}[v_1, \dots, v_{n-1}, v_n^{\pm 1}] \otimes \Lambda_\mathbb{Q}(dv_1, \dots, dv_n)$$

with dv_i having homological degree one and internal degree $2p^i - 2$. As the Bökstedt spectral sequence is multiplicative and as the algebra generator cannot support any differentials for degree reasons, the spectral sequence collapses at E^2. There are no multiplicative extensions and hence we get the result. □

Remark 2.4. As we work rationally, $\mathrm{THH}(E(n))_\mathbb{Q}$ is a commutative $H\mathbb{Q}$-algebra spectrum and hence corresponds to a commutative differential graded $\mathbb{Q}$-algebra (see [Shi07] or [RS17]).

3. $K(i)_*E(n)$ and $K(i)_*\mathrm{THH}(E(n))$

In the following we assume that p is an odd prime, and that n and i are integers with $1 \leqslant i \leqslant n$. The Hopf algebroid (BP_*, BP_*BP) represents the groupoid of strict isomorphisms of p-typical formal group laws [Lan75] (see also [Rav86, Theorem A2.1.27]). There are isomorphisms of graded $\mathbb{Z}_{(p)}$-algebras

$$BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots] \quad \text{and} \quad BP_*BP \cong BP_*[t_1, t_2, \dots],$$

where $|v_i| = |t_i| = 2(p^i - 1)$. By convention $v_0 = p$ and $t_0 = 1$. The ith Morava K-theory $K(i)$ is complex oriented, and its formal group law F_i (the Honda formal group law) corresponds to the map $BP_* \to K(i)_* = \mathbb{F}_p[v_i^\pm]$ sending v_i to v_i and v_k for $k \neq i$ to zero. The p-typical formal group law G_n over $E(n)_*$ comes from the map $BP_* \to E(n)_*$ that kills all v_i with $i > n$ and inverts v_n. Since $E(n)$ is a Landweber exact homology theory, we obtain an isomorphism

$$K(i)_*E(n) \cong K(i)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(n)_*. \tag{3.1}$$

Note that $K(i)_*E(n)$ is trivial for $i > n$ and that the Bousfield class of $E(n)$, $\langle E(n)\rangle$, is $\langle K(0) \vee \ldots \vee K(n)\rangle$.

We first treat the case $i = n$.

Proposition 3.1. *For all $n \geqslant 1$ the canonical map $E(n) \to \mathrm{THH}(E(n))$ is a $K(n)$-local equivalence.*

Proof. The algebra $K(n)_*E(n)$ is known as $\Sigma(n)$ and it is of the form

$$K(n)_*[t_1, t_2, \ldots]/(v_n t_i^{p^n} - v_n^{p^i} t_i, i \geqslant 1),$$

see [Rav86, 6.1.16]. If we set

$$C_*^{(k)} := K(n)_*[t_1, \ldots, t_k]/(v_n t_i^{p^n} - v_n^{p^i} t_i, 1 \leqslant i \leqslant k)$$

then $C_*^{(k)}$ is étale over $K(n)_*$ and $K(n)_*E(n)$ is the directed colimit of the $C_*^{(k)}$'s.

The $K(n)_*$-Bökstedt spectral sequence for $\mathrm{THH}(E(n))$ has as an E^2-term

$$\mathsf{HH}_*^{K(n)_*}(K(n)_*E(n)) \cong K(n)_*E(n)$$

concentrated in homological degree zero. Thus $K(n)_*\mathrm{THH}(E(n)) \cong K(n)_*E(n)$ and the isomorphism is induced by the map $E(n) \to \mathrm{THH}(E(n))$. Therefore, this map is a $K(n)$-equivalence and thus $K(n)$-locally $\mathrm{THH}(E(n))$ is equivalent to $E(n)$. □

We calculate $K(i)_*E(n)$ for $1 \leqslant i \leqslant n-1$ using the following description of morphisms of graded commutative BP_*-algebras from $K(i)_*E(n)$ to some graded commutative ring B_*. For $n = 2$ we had an argument that was rather involved and Paul Goerss suggested the following simpler proof.

We consider the map $g\colon BP_*BP \to K(i)_*E(n)$ of graded commutative $\mathbb{Z}_{(p)}$-algebras given by

$$BP_*BP \to K(i)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(n)_* \cong K(i)_*E(n)$$

which uses the canonical maps $BP_* \to K(i)_*$ and $BP_* \to E(n)_*$ and the isomorphism from (3.1). By [Rav86, Theorem A2.1.27] this map corresponds to a triple $((\eta_L)_*F_i, (\eta_R)_*G_n, f)$ where $\eta_L\colon K(i)_* \to K(i)_*E(n)$ is the left unit, $\eta_R\colon E(n)_* \to K(i)_*E(n)$ is the right unit and $(\eta_L)_*F_i$ and $(\eta_R)_*G_n$ are the p-typical formal group laws that are given by the corresponding change of coefficients. Here, f is a strict isomorphism between the p-typical formal group laws $(\eta_L)_*F_i$ and $(\eta_R)_*G_n$ over $K(i)_*E(n)$. By [Rav86, Lemma A2.1.26] such a strict isomorphism is always of the form

$$f(x) = \sum_j{}^{(\eta_R)_*G_n} t_j x^{p^j}.$$

The p-series of the Honda formal group law F_i is

$$[p]_{F_i}(x) = v_i x^{p^i}$$

and the same is true for $[p]_{(\eta_L)_*F_i}[x]$ because the left unit just embeds $K(i)_*$ into $K(i)_*E(n)$. The p-series of $(\eta_R)_*G_n$ is

$$[p]_{(\eta_R)_*G_n}(x) = w_1 x^p +_{(\eta_R)_*G_n} \cdots +_{(\eta_R)_*G_n} w_n x^{p^n}$$

for $w_i = \eta_R(v_i)$.

First, we state an elementary lemma about powers of p.

Lemma 3.2. *Let $m \geqslant 2$, let $r, \ell_1, \ldots, \ell_m$ be natural numbers bigger or equal to 1, and assume that $\ell_j \neq \ell_k$ for $j \neq k$. Then p^r cannot be written as a sum $p^{\ell_1} + \ldots + p^{\ell_m}$.*

Proof. Assume
$$p^r = p^{\ell_1} + \ldots + p^{\ell_m}.$$
Without loss of generality let ℓ_1 be minimal among the ℓ_j's. Then
$$p^r = p^{\ell_1}(1 + p^{\ell_2-\ell_1} + \ldots + p^{\ell_m-\ell_1}).$$
This is only possible if all the $\ell_j - \ell_1$ are equal to zero and if $m = p^{r-\ell_1}$. But $\ell_j - \ell_1 = 0$ for all $2 \leqslant j \leqslant m$ implies that all the ℓ_j's are equal to ℓ_1 and this contradicts our assumption. □

Proposition 3.3. *For all $1 \leqslant i \leqslant n$ $K(i)_*E(n)$ is a colimit of étale $K(i)_*[w_{i+1}, \ldots, w_n^{\pm 1}]$-algebras.*

Proof. In the following we fix i and n. We denote by $B(i,n)_*$ the graded commutative $K(i)_*$-algebra $K(i)_*[w_{i+1}, \ldots, w_{n-1}, w_n^{\pm 1}]$. For a given $m \geqslant 1$ consider the graded commutative BP_*-subalgebra $BP_*[t_1, \ldots, t_m]$ of BP_*BP and define
$$B_m = \mathrm{Image}\big(B(i,n)_*[t_1, \ldots, t_m] \to K(i)_*E(n)\big)\,.$$
Thus we can express B_m as $B(i,n)_*[t_1, \ldots, t_m]/\sim$ where $\sim$ denotes the quotient that arises from the relations that the t_r's and w_j's satisfy in $K(i)_*E(n)$. Note that B_{m+1} is free as a B_m-module for all $m \geqslant 1$. Indeed, in each step we adjoin a new polynomial generator x to a graded commutative ring R_* that satisfies relations of the form $x^{p^r} - ux - y$ with a unit $u \in R_*^\times$ and $y \in R_*$.

The strict isomorphism $f(x) = \sum_j^{(\eta_R)_*G_n} t_j x^{p^j}$ satisfies
$$[p]_{(\eta_R)_*G_n}(f(x)) = f([p]_{(\eta_L)_*F_i}(x))$$
and this yields the equality
$$w_1(f(x))^p +_{(\eta_R)_*G_n} \cdots +_{(\eta_R)_*G_n} w_n(f(x))^{p^n} = f(v_i x^{p^i}) = \sum_j{}^{(\eta_R)_*G_n} t_j (v_i x^{p^i})^{p^j}. \tag{3.2}$$

On the right hand side in $\sum_j^{(\eta_R)_*G_n} t_j v_i^{p^j} x^{p^{i+j}}$ the relations for the t_r are detected by the powers $x^{p^{i+r}}$. Lemma 3.2 ensures that for a given $x^{p^{i+r}}$ we only have to consider the coefficient $t_j v_i^{p^j}$ with $i + j = i + r$ coming from the linear term of the $(\eta_R)_*G_n$-sum $\sum_j^{(\eta_R)_*G_n} t_j v_i^{p^j} x^{p^{i+j}}$ and this is $t_r v_i^{p^r}$.

As the right hand side starts with x^{p^i}, it is a direct consequence that $w_1, \ldots, w_{i-1} = 0$ and from the coefficients of x^{p^i} we obtain that $w_i = v_i$ in $K(i)_*E(n)$.

We prove that B_1 is étale over $B(i,n)_*$ and that for every m, B_m is étale over B_{m-1}. It follows that the algebras B_m are étale over $B(i,n)_*$.

Thus we have to show that the modules of relative Kähler differentials $\Omega^1_{B_1|B(i,n)_*}$ and $\Omega^1_{B_m|B_{m-1}}$ are trivial for all $m \geqslant 2$.

For $m = 1$ we compare the coefficients of $x^{p^{i+1}}$ in (3.2). In this case only the linear terms of the $(\eta_R)_*G_n$-sums contribute something and we obtain
$$v_i t_1^{p^i} + w_{i+1} t_0 = t_1 v_i^p$$
and therefore $t_1 = v_i^{-p}(v_i t_1^{p^i} + w_{i+1})$. This gives a flat extension and the Kähler differential on t_1 is equal to
$$dt_1 = 0 + v_i^{-p} dw_{i+1}$$
and hence B_1 is étale over $B(i,n)_*$.

Consider B_m. Then the first relation for t_m is given by the relation of the coefficients for $x^{p^{i+m}}$.

We know that the formal group law $G_n(x,y)$ is of the form

$$G_n(x,y) = x + y + \sum_{i,j\geqslant 1} a_{i,j} x^i y^j$$

where the $a_{i,j} \in E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}]$. Equation (3.2) relates power series with coefficients in $K(i)_*E(n)$, hence the coefficients $\bar{a}_{i,j}$ of $(\eta_R)_*G_n$ are now considered in $K(i)_*E(n)$ and are elements of $\mathbb{F}_p[w_i, \ldots, w_{n-1}, w_n^{\pm 1}]$. On the left hand side of (3.2) we get coefficients that involve some polynomials of $\bar{a}_{i,j}$'s, some pth powers of t_j's and some expressions in w_k's. For $m + i \leqslant n$ we actually get a coefficient $w_{m+i} t_0^{p^{m+i+0}} = w_{i+m}$.

The $\bar{a}_{i,j}$'s are in $B(i,n)_*$, so they don't contribute anything to the relative Kähler differentials. The Kähler differentials on the $t_j^{p^k}$ are trivial because we are over $\mathbb{F}_p$. Hence we can express the Kähler differential dt_m up to a factor of $v_i^{p^m} = w_i^{p^m}$ via Kähler differentials in the w_k's. As $v_i^{p^m}$ is invertible in $B(i,n)_*$, the relative Kähler differentials $\Omega^1_{B_m|B_{m-1}}$ are trivial for all $m \geqslant 1$. □

Theorem 3.4. *For all $1 \leqslant i \leqslant n$ we have an isomorphism of $K(i)_*E(n)$-algebras*

$$\mathsf{HH}_*^{K(i)_*}(K(i)_*E(n)) \cong K(i)_*E(n) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \ldots, dw_n).$$

Proof. We have shown that $K(i)_*E(n)$ is the sequential colimit of the B_m's. As the $K(i)_*$-algebras B_m are étale over $B(i,n)_*$ and as Hochschild homology commutes with localization we can rewrite $\mathsf{HH}_*(B_m)$ as

$$\begin{aligned}\mathsf{HH}_*(B_m) &\cong B_m \otimes_{B(i,n)_*} \mathsf{HH}_*^{K(i)_*}(B(i,n)_*)\\ &\cong B_m \otimes_{B(i,n)_*} (B(i,n)_* \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \ldots, dw_n))\\ &\cong B_m \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \ldots, dw_n))\end{aligned}$$

using [WG91] and the Hochschild-Kostant-Rosenberg theorem. Hochschild homology commutes with colimits, hence we obtain

$$\mathsf{HH}_*^{K(i)_*}(K(i)_*E(n)) \cong \mathrm{colim}_m\, \mathsf{HH}_*^{K(i)_*}(B_m) \cong K(i)_*E(n) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \ldots, dw_n)\,.$$

□

Theorem 3.5. *Assume that p is an odd prime and that $E(n)$ is an E_3-ring spectrum. Then, for all $1 \leqslant i \leqslant n$, we have an isomorphism of $K(i)_*E(n)$-algebras*

$$K(i)_*\mathrm{THH}(E(n)) \cong K(i)_*E(n) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \ldots, dw_n).$$

Proof. We use the Bökstedt spectral sequence [Bök], [EKMM97, IX.2.9], with E^2-term

$$E^2_{r,s} = (\mathsf{HH}_r^{K(i)_*}(K(i)_*E(n)))_s\,,$$

where r denotes the homological and s the internal degree. By a result of Angeltveit and Rognes [AR05, Prop. 4.3], an E_3-structure on $E(n)$ implies that this spectral is one of commutative $K(i)_*E(n)$-algebras. The multiplicative generators dw_j for $i \leqslant j \leqslant n$ sit in bidegree $(1, 2p^j - 2)$ and hence they cannot carry any non-trivial differentials. Therefore the spectral sequence collapses at the E^2-term. As the abutment is a free graded commutative $K(i)_*E(n)$-algebra, there cannot be any multiplicative extensions. □

Remark 3.6. Note if $E(n)$ admits an E_2 structure, the Bökstedt spectral sequence is one of $K(i)_*$-algebras by [AR05, Prop. 4.3]. It therefore collapses since all $K(i)_*$-algebra generators lie in columns 0 and 1. This gives the same formula for $K(i)_*\mathrm{THH}(E(n))$ as a $K(i)_*$-module, but not as a $K(i)_*$-algebra, since there is now room for $K(i)_*$-algebra extensions.

4. Blue-shift for $\mathrm{THH}(E(n))$

If we assume that p is an odd prime and that $E(n)$ is an E_∞-ring spectrum, then $\mathrm{THH}(E(n))$ is a commutative $E(n)$-algebra spectrum and the cofiber of the unit map

$$\overline{\mathrm{THH}}(E(n)) = \mathrm{cofiber}\big(E(n) \to \mathrm{THH}(E(n))\big)$$

is a non-unital commutative $E(n)$-algebra spectrum. If $E(n)$ carries an E_3-structure, then by [BFV07, §3.3], [BM11] the morphism $E(n) \to \mathrm{THH}(E(n))$ is an E_2-map. This implies the following useful fact:

Lemma 4.1. *If $E(n)$ is an E_3-spectrum, then $\mathrm{THH}(E(n))$ is an $E(n)$-module spectrum and in particular, $\mathrm{THH}(E(n))$ is $E(n)$-local.*

Let L_n denote the localization at $E(n)$, and in particular L_0 is the rationalization. Recall that there is a well-known chromatic fracture square

$$\begin{tikzcd} L_nX \arrow[r] \arrow[d] & L_{K(n)}X \arrow[d] \\ L_{n-1}X \arrow[r] & L_{n-1}L_{K(n)}X. \end{tikzcd}$$

It is shown for instance in [ACB, Example 3.3] and [Bau14, Proposition 2.2] that the homotopy pullback of

$$\begin{tikzcd} & L_{K(n)}X \arrow[d] \\ L_{n-1}X \arrow[r] & L_{n-1}L_{K(n)}X. \end{tikzcd}$$

is an $E(n)$-localization of X. The statement in [Bau14, Proposition 2.2] is more general and [ACB] work out far more general local-to-global statements.

We always know from Proposition 3.1 that the unit map is a $K(n)$-local equivalence. The chromatic square for $\overline{\mathrm{THH}}(E(n))$ is:

$$\begin{tikzcd} \overline{\mathrm{THH}}(E(n)) = L_{K(n)\vee E(n-1)}\overline{\mathrm{THH}}(E(n)) \arrow[r] \arrow[d] & L_{K(n)}\overline{\mathrm{THH}}(E(n)) \arrow[d] \\ L_{E(n-1)}\overline{\mathrm{THH}}(E(n)) \arrow[r] & L_{E(n-1)}(L_{K(n)}\overline{\mathrm{THH}}(E(n)))\,. \end{tikzcd}$$

The $K(n)$-homology of $\overline{\mathrm{THH}}(E(n))$ is zero by Proposition 3.1. It follows that the localization $L_{K(n)}\overline{\mathrm{THH}}(E(n))$ is trivial, and hence $L_{E(n-1)}(L_{K(n)}\overline{\mathrm{THH}}(E(n)))$ is also trivial. Therefore the vertical map on the left hand side is an equivalence and we obtain a nice example of blue-shift:

Lemma 4.2. *If $E(n)$ is an E_3-spectrum, then the cofiber $\overline{\mathrm{THH}}(E(n))$ is $E(n-1)$-local.*

5. Topological Hochschild homology of $E(2)$

In this section, we discuss in more detail the topological Hochschild homology of $E(2)$, which we will denote by $E = E(2)$ to simplify the notation. As explained in the proof of Lemma 5.1, the computations of Theorem 3.5 for $E(2)$ can be expressed as follows:

$$K(0)_*\mathrm{THH}(E) \cong K(0)_*E \otimes \Lambda_{\mathbb{Q}}(dt_1, dt_2), \tag{5.1}$$

$$K(1)_*\mathrm{THH}(E) \cong K(1)_*E \otimes \Lambda_{\mathbb{F}_p}(dt_1), \tag{5.2}$$

$$K(2)_*\mathrm{THH}(E) \cong K(2)_*E. \tag{5.3}$$

Notice that these computations do not require the assumption that E is an E_3-ring spectrum: for the rational case we have a commutative structure anyhow, while in the $K(1)$ and $K(2)$

cases, the E^2 page of the Bökstedt spectral sequences is concentrated on columns 0 and 1 (respectively 0).

Lemma 5.1. *For $i = 1, 2$, there exist classes $\lambda_i \in \mathrm{THH}_{2p^i-1}(E)$ with the following properties. Under the Hurewicz homomorphism*

(a) *the class λ_i maps to $dt_i \in K(0)_{2p^i-1}\mathrm{THH}(E)$, for $i = 1, 2$;*
(b) *the class λ_1 maps to $dt_1 \in K(1)_{2p^2-1}\mathrm{THH}(E)$.*

Proof. We use McClure-Staffeldt's computation of $\mathrm{THH}_*(BP)$ in [MS93, Remark 4.3], which has been validated by the proof [BM13] that BP admits an E_4 structure. We briefly recall the computation. The integral, rational and mod p homology of BP are given as

$$H\mathbb{Z}_*BP \cong \mathbb{Z}_{(p)}[t_i \,|\, i \geqslant 1], \quad K(0)_*BP \cong \mathbb{Q}[t_i \,|\, i \geqslant 1] \quad \text{and} \quad H\mathbb{F}_{p*}BP \cong \mathbb{Z}[\bar{\xi}_i \,|\, i \geqslant 1],$$

where the class $t_i \in H\mathbb{Z}_{2p^i-1}BP$ maps to $\bar{\xi}_i$ under mod (p) reduction [Rav86, Proof of Theorem 5.2.8] and to the class with same name t_i under rationalization. The associated Bökstedt spectral sequences collapse, providing isomorphisms

$$\begin{aligned} H\mathbb{Z}_*\mathrm{THH}(BP) &\cong H\mathbb{Z}_*BP \otimes \Lambda_{\mathbb{Z}_{(p)}}(dt_i \,|\, i \geqslant 1), \\ K(0)_*\mathrm{THH}(BP) &\cong K(0)_*BP \otimes \Lambda_{\mathbb{Q}}(dt_i \,|\, i \geqslant 1) \ \text{ and} \\ H\mathbb{F}_{p*}\mathrm{THH}(BP) &\cong H\mathbb{F}_{p*}BP \otimes \Lambda_{\mathbb{F}_p}(d\bar{\xi}_i \,|\, i \geqslant 1), \end{aligned}$$

with $dx = \sigma_*(x)$, where $\sigma\colon \Sigma BP \to \mathrm{THH}(BP)$ is the map given in (2.1). There is an isomorphism

$$\mathrm{THH}_*(BP) \cong BP_* \otimes \Lambda_{\mathbb{Z}_{(p)}}(\lambda_i \,|\, i \geqslant 1),$$

and the Hurewicz homomorphism

$$\mathrm{THH}_*(BP) \to H\mathbb{Z}_*\mathrm{THH}(BP)$$

is an inclusion mapping λ_i to dt_i. In particular, the classes dt_i (integral and rational) and $d\bar{\xi}_i$ are spherical: they are the image of λ_i under the Hurewicz homomorphism mapping from $\mathrm{THH}_*(BP)$. For $i \geqslant 1$, let us define

$$\lambda_i \in \mathrm{THH}_{2p^i-1}(E)$$

as the image of the class with same name under the natural map

$$\mathrm{THH}_*(BP) \to \mathrm{THH}_*(E).$$

In the rational case, we have

$$\eta_R(v_i) \equiv \alpha_i t_i$$

modulo decomposables in $K(0)_*BP$, where $\alpha_i \in \mathbb{Q}$ is a unit. We deduce that

$$K(0)_*E \cong \mathbb{Q}[t_1, t_2][\eta_R(v_2)^{-1}]$$

and the Bökstedt spectral sequence recovers

$$K(0)_*\mathrm{THH}(E) \cong K(0)_*E \otimes \Lambda_{\mathbb{Q}}(dt_1, dt_2).$$

By naturality, comparing with the case of BP, we deduce that the Hurewicz homomorphism $\mathrm{THH}_*(E) \to K(0)_*\mathrm{THH}(E)$ maps λ_i to dt_i.

For $K(1)_*$-homology, we argue similarly, using the commutative square

$$\begin{tikzcd} \mathrm{THH}_*(BP) \arrow[r] \arrow[d] & K(1)_*\mathrm{THH}(BP) \arrow[d] \\ \mathrm{THH}_*(E) \arrow[r] & K(1)_*\mathrm{THH}(E). \end{tikzcd}$$

We have $K(1)_*BP \cong K(1)_*[t_i \,|\, i \geqslant 1]$, and the Bökstedt spectral sequence yields

$$K(1)_*\mathrm{THH}(BP) \cong K(1)_*BP \otimes \Lambda_{\mathbb{F}_p}(dt_i \,|\, i \geqslant 1).$$

Comparing the Bökstedt spectral sequences for $H\mathbb{Z}_*\mathrm{THH}(BP)$ and $K(1)_*\mathrm{THH}(BP)$, we deduce that the class $\lambda_1 \in \mathrm{THH}_*(BP)$ maps to $dt_1 \in K(1)_*\mathrm{THH}(BP)$. Recall that

$$K(1)_*E = K(1)_*[t_i \mid i \geqslant 1][\eta_R(v_2)^{-1}]/(\eta_R(v_j) \mid j \geqslant 3)$$

is a colimit of étale algebras over $K(1)_*[w_2, w_2^{-1}]$, where

$$w_2 = \eta_R(v_2) = v_1^p t_1 - v_1 t_1^p.$$

In particular $dw_2 = v_1^p dt_1$, and the Bökstedt spectral sequence provides the formula given above for $K(1)_*\mathrm{THH}(E)$. Now obviously $dt_1 \in K(1)_*\mathrm{THH}(BP)$ maps to $dt_1 \in K(1)_*\mathrm{THH}(E)$. This implies assertion (b) of the lemma. $\square$

Remark 5.2. Note that the above proof does not require the map $BP \to E(n)$ to be an E_3-map.

The class $\lambda_1 \in \mathrm{THH}_{2p-1}(E)$ of Lemma 5.1 corresponds to a map $\lambda_1 \colon S^{2p-1} \to \mathrm{THH}(E)$. Smashing with E, using the E-module structure of $\mathrm{THH}(E)$ (assuming an E_3 structure on E), and composing with the cofiber $\mathrm{THH}(E) \to \overline{\mathrm{THH}}(E)$ of the unit, we obtain a map

$$j_1 \colon \Sigma^{2p-1}E \cong E \wedge S^{2p-1} \to E \wedge \mathrm{THH}(E) \to \mathrm{THH}(E) \to \overline{\mathrm{THH}}(E).$$

In the same fashion, we obtain a map $j_2 \colon \Sigma^{2p^2-1}E \to \overline{\mathrm{THH}}(E)$ corresponding to the class λ_2.

Lemma 5.3. *The map j_1 factors through a map*

$$\bar{j}_1 \colon \Sigma^{2p-1}L_1E \to \overline{\mathrm{THH}}(E)$$

that is a $K(1)_*$-isomorphism, and whose cofiber $C(\bar{j}_1)$ is a rational spectrum.

Proof. Recall from Lemma 4.2 that the cofiber $\overline{\mathrm{THH}}(E)$ of the unit map is $E(1)$-local. In particular, the map j_1 factors through a map

$$\bar{j}_1 \colon \Sigma^{2p-1}L_1E \to \overline{\mathrm{THH}}(E).$$

The localization map $E \to L_1E$ is a $K(1)_*$-isomorphism, and therefore so are the induced maps $\ell \colon \mathrm{THH}(E) \to \mathrm{THH}(L_1E)$ and $\bar{\ell} \colon \overline{\mathrm{THH}}(E) \to \overline{\mathrm{THH}}(L_1E)$, by convergence of the $K(1)$-based Bökstedt spectral sequence. Hence, to prove the claim, it suffices to show that the composition

$$\Sigma^{2p-1}L_1E \xrightarrow{\bar{j}_1} \overline{\mathrm{THH}}(E) \xrightarrow{\bar{\ell}} \overline{\mathrm{THH}}(L_1E) \tag{5.4}$$

is a $K(1)_*$-isomorphism. The $K(1)$-based Bökstedt spectral sequence for L_1E is identical to the one of E, computed above as

$$E^2_{*,*} = K(1)_*E \otimes \Lambda_{\mathbb{F}_p}(dt_1) \Rightarrow K(1)_*\mathrm{THH}(E),$$

where $K(1)_*E$ is in filtration degree zero and $K(1)_*E\{dt_1\}$ is in filtration degree 1, and where all differentials are zero. By definition of the map j_1, if $1 \in K(1)_0E$ is the unit, then $j_{1*}(\Sigma^{2p-1}1)$ is represented modulo lower filtration by the permanent cycle dt_1 in $E^2_{1,*}$. Since this is a spectral sequence of $K(1)_*E$-modules, the composition (5.4) induces a map in $K(1)$ homology that is represented modulo lower filtration by the isomorphism $\Sigma^{2p-1}K(1)_*E \to E^2_{1,*} = K(1)_*E\{dt_1\}$ sending a class $\Sigma^{2p-1}w$ to wdt_1. It is therefore a $K(1)_*$-isomorphism, proving the claim.

Now we consider the cofiber $C(\bar{j}_1)$ of $\bar{j}_1$, sitting in an exact triangle

$$\Sigma^{2p-1}L_1E \xrightarrow{\bar{j}_1} \overline{\mathrm{THH}}(E) \xrightarrow{k} C(\bar{j}_1) \xrightarrow{\delta} \Sigma^{2p}L_1E. \tag{5.5}$$

Since $\bar{j}_1$ is a $K(1)_*$-isomorphism, we know that $K(1)_*C(\bar{j}_1) = 0$, and since $\overline{\mathrm{THH}}(E)$ and thus $C(\bar{j}_1)$ are $E(1)$-local, we deduce (as in Lemma 4.2) that $C(\bar{j}_1)$ is $E(0)$-local (*i.e.*, rational). $\square$

We now define a map $\lambda_{12} \colon L_0S^{2p^2-2p-2} \to C(\bar{j}_1)$ as a composition over the cofibers

$$L_0S^{2p^2-2p-2} \to L_0\mathrm{THH}(E) \to L_0\overline{\mathrm{THH}}(E) \to C(\bar{j}_1),$$

where the first map above realizes the class $dt_1 dt_2 \in K(0)_*\mathrm{THH}(E)$. Smashing λ_{12} with E and using the module structure we obtain a map

$$j_{12} \colon \Sigma^{2p^2-2p-2} L_0 E \to C(\bar{j}_1).$$

Similarly, λ_2 induces a map

$$j_2 \colon \Sigma^{2p^2-1} L_0 E \to C(\bar{j}_1).$$

Theorem 5.4. *Let p be an odd prime such that $E = E(2)$, the second Johnson-Wilson spectrum at p, is an E_3-ring spectrum. Then the map $j_2 \vee j_{12}$ lifts to a map*

$$\bar{j}_2 \vee \bar{j}_{12} \colon \Sigma^{2p^2-1} L_0 E \vee \Sigma^{2p^2-2p-2} L_0 E \to \overline{\mathrm{THH}}(E)$$

and the sum β of $\bar{j}_1$, $\bar{j}_2$ and $\bar{j}_{12}$ is a weak equivalence of E-modules

$$\beta \colon \Sigma^{2p-1} L_1 E \vee \Sigma^{2p^2-1} L_0 E \vee \Sigma^{2p^2+2p-2} L_0 E \to \overline{\mathrm{THH}}(E).$$

Proof. The composition $\delta \circ (j_2 \vee j_{12})$ is trivial, so that $j_2 \vee j_{12}$ lifts to a map $\bar{j}_2 \vee \bar{j}_{12}$:

$$\begin{tikzcd}
 & \Sigma^{2p^2-1} L_0 E \vee \Sigma^{2p^2+2p-2} L_0 E \arrow[dl, dotted, "\bar{j}_2 \vee \bar{j}_{12}"'] \arrow[d, "j_2 \vee j_{12}"] \arrow[dr, "\simeq *"] & \\
\overline{\mathrm{THH}}(E) \arrow[r, "k"] & C(\bar{j}_1) \arrow[r, "\delta"] & \Sigma^{2p} L_1 E.
\end{tikzcd}$$

Indeed, $\Sigma^{2p} L_1 E$ fits in the chromatic fracture pullback diagram

$$\begin{tikzcd}
\Sigma^{2p} L_1 E \arrow[r] \arrow[d] & \Sigma^{2p} L_{K(1)} E \arrow[d] \\
\Sigma^{2p} L_0 E \arrow[r] & \Sigma^{2p} L_0 (L_{K(1)} E).
\end{tikzcd}$$

The composition of $\delta \circ (j_2 \vee j_{12})$ with the left vertical map to $\Sigma^{2p} L_0 E$ is trivial, since it factors over the composition

$$L_0 \overline{\mathrm{THH}}(E) \to L_0 C(\bar{j}_1) \to \Sigma^{2p} L_0 E$$

of two consecutive maps in the ($E(0)$-localized) cofiber sequence (5.5). The composition of $\delta \circ (j_2 \vee j_{12})$ with the top map to $\Sigma^{2p} L_{K(1)} E$ is trivial as well; indeed, there is no non-trivial map from a $K(1)$-acyclic to a $K(1)$-local spectrum. This finishes the proof that $\delta \circ (j_2 \vee j_{12})$ is trivial and that the lift exists. We now define β as the sum

$$\beta = \bar{j}_1 \vee \bar{j}_2 \vee \bar{j}_{12} \colon \Sigma^{2p-1} L_1 E \vee \Sigma^{2p^2-1} L_0 E \vee \Sigma^{2p^2+2p-2} L_0 E \to \overline{\mathrm{THH}}(E).$$

Finally, we claim that β is a $K(0)_*$-isomorphism: this is analogous to the proof above that $\bar{j}_1$ is a $K(1)_*$-isomorphism, working this time with the $K(0)$-based Bökstedt spectral sequence. Since β is a $K(0)_*$- and a $K(1)_*$-isomorphism of $E(1)$-local spectra, it is a weak equivalence. □

Assume now that in addition to E being an E_3-ring spectrum, the unit map $E \to \mathrm{THH}(E)$ splits in the homotopy category (this holds for example if E is an E_∞-ring spectrum). We then have a weak equivalence of E-modules $E \vee \overline{\mathrm{THH}}(E) \to \mathrm{THH}(E)$. On the other hand, summing β with the identity of E gives a weak equivalence

$$\mathrm{id} \vee \beta \colon E \vee \Sigma^{2p-1} L_1 E \vee \Sigma^{2p^2-1} L_0 E \vee \Sigma^{2p^2+2p-2} L_0 E \to E \vee \overline{\mathrm{THH}}(E).$$

This implies the following corollary of Theorem 5.4.

Corollary 5.5. *Assume that p is an odd prime, and that the second Johnson-Wilson spectrum $E = E(2)$ admits an E_3-structure. If the unit map $E \to \mathrm{THH}(E)$ splits in the homotopy category, then the maps above provide a weak equivalence of E-modules*

$$E \vee \Sigma^{2p-1} L_1 E \vee \Sigma^{2p^2-1} L_0 E \vee \Sigma^{2p^2+2p-2} L_0 E \to \mathrm{THH}(E).$$

Remark 5.6. Corollary 5.5 implies that

- the 2^0 summand of $K(2)_*E$ in $K(2)_*\mathrm{THH}(E)$ indexed by 1,
- the 2^1 summands of $K(1)_*E$ in $K(1)_*\mathrm{THH}(E)$ indexed by 1 and dt_1,
- the 2^2 summands of $K(0)_*E$ in $K(0)_*\mathrm{THH}(E)$ indexed by 1, dt_1, dt_2 and dt_1dt_2

assemble, in $\mathrm{THH}(E)$, into

- the 2^0 summand E indexed by 1 and detected by $K(0)_*$, $K(1)_*$ and $K(2)_*$,
- the $2^1 - 2^0$ summand L_1E indexed by dt_1 and detected by $K(0)_*$ and $K(1)_*$, and
- the $2^2 - 2^1$ summands L_0E indexed by dt_2 and dt_1dt_2 and detected by $K(0)_*$.

Notice that Bruner and Rognes [BR] obtain very similar computations for $K(i)_*\mathrm{THH}(\mathrm{tmf})$ for $i = 0, 1, 2$, where tmf denotes the connective spectrum of topological modular form.

We can picture the summands of $\mathrm{THH}(E)$ in a 2-dimensional cube of local pieces (up to suspensions, where $E = L_2E$):

	1	dt_1
1	E	L_1E
dt_2	L_0E	L_0E

We conjecture that this picture extends to describe a decomposition of $\mathrm{THH}(E(n))$ into 2^n summands, with summands placed in an n-dimensional cube, where the ith edge has two coordinates 1 and dt_i. We formulate this as follows.

Conjecture 5.7. If p is an odd prime such that $E(n)$ is a sufficiently commutative S-algebra, then $\mathrm{THH}(E(n))$ decomposes as a sum of 2^n factors, namely 2^{n-i-1} suspended copies of $L_iE(n)$ for each $0 \leqslant i \leqslant n-1$, plus one copy of $E(n)$. More precisely, the $L_iE(n)$ summands are indexed by the 2^{n-i-1} monomial generators

$$\omega \in \Lambda_{\mathbb{Q}}(dt_1, \ldots, dt_{n-i-1})\{dt_{n-i}\} \subset K(0)_*\mathrm{THH}(E(n)),$$

and the summand corresponding to such a monomial ω is $\Sigma^{|\omega|}L_iE(n)$.

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